JOURNÉES ÉQUATIONS AUX DÉRIVÉES PARTIELLES

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Journées Équations aux dérivées partielles (1997), p. 1-10

<http://www.numdam.org/item?id=JEDP_1997____A16_0>

CLR-ESTIMATE REVISITED: LIEB'S APPROACH WITH NO PATH INTEGRALS

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0. Suppose that a positive self-adjoint operator B in a Hilbert space is additively perturbed by a 'weaker' operator $-V \le 0$. The resulting operator $H = B - V$ may have non-empty negative spectrum. The question of estimating the number $N_-(H)$ of negative eigenvalues of H arises in many physical and mathematical applications. A typical example here is the Schrödinger operator $H = -\Delta - V$ in $L_2(\mathbb{R}^d)$, where $V \ge 0$ is a measurable function (the electric potential). Among other estimates for $N_-(-\Delta - V)$, the inequality

$$N_-(-\Delta - V) \le C(d) \int_{\mathbb{R}^d} V(x)^{\frac{d}{2}} dx, \qquad d \ge 3 \tag{CLR}$$

has certain distinguished features. It holds as long as the expression on the RHS is finite, and corresponds to the quasi-classical phase volume picture. Besides, it is sharp in the function classes for V, in the following sense. If in (CLR) we replace V by αV, then the RHS of the corresponding estimate for $N_-(-\Delta - \alpha V)$ involves the additional factor $\alpha^{\frac{d}{2}}$. It turns out that the asymptotic formula

$$N_-(-\Delta - \alpha V) \sim c(d)\alpha^{\frac{d}{2}} \int_{\mathbb{R}^d} V(x)^{\frac{d}{2}} dx, \qquad c(d) = (2\sqrt{\pi})^{-d}(\Gamma(1 + \frac{d}{2}))^{-1}$$

is correct under the same assumption $V \in L_{d/2}$. So, within the constant factor, $N_-(-\Delta - \alpha V)$ is estimated through its own asymptotics.

The above inequality is usually called Cwikel-Lieb-Rozenblum estimate (shortly, CLR-estimate), after the names of the authors of three earliest proofs, see [Cw, L1, Roz]. Two more proofs, in [LiY, C], appeared later. The proofs give different values of the constant $C(d)$ and the specifics of Euclidean space is used in them to a different extent. A closer analysis shows that two of the existing proofs, namely the ones of [L1] and [LiY], can be adapted to a much more general situation. In particular, in [LevS], by making abstract the approach of [LiY], a generalization of the CLR-estimate was obtained, with $\mathbb{R}^d$ replaced by an arbitrary space equipped with a σ-finite measure. The role of $-\Delta$ in [LevS] is played by any positive operator B, generating a Markov semigroup. The exponent in the estimate is determined by the exponent in the Sobolev-type embedding theorem for the domain of $B^{\frac{1}{2}}$.

Among different proofs of the CLR-estimate, Lieb's one, using the path integration formalism, gives the best known constant $C(d)$. Our goal here is to show that

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this approach can also be translated into the pure operator-theoretic language, so that no mentioning of path integration remains. Like in [LevS], instead of $\mathbb{R}^d$ one can take any space with σ-finite measure. The role of $-\Delta$ can be played by any operator A, generating a semigroup e^{-At} which is dominated by some positivity preserving semigroup, bounded as an operator from L_2 to L_∞. This class is much wider than the one covered by [LevS], so the results obtained contain the main theorem of [LevS] as a particular case. Such a generality enables one to treat, in a uniform way, various operators of Mathematical Physics (while earlier each new problem required a different version of path integration theory), and, in particular, to reproduce the best-known constant for the Schrödinger operator.

This investigation was motivated by discussions with E.Lieb, L.Saloff-Coste and H.Siedentop. We use this opportunity to express them our gratitude.

1. Let a selfadjoint operator B be a generator of positivity preserving (shortly, positive) semigroup e^{-tB} in $L_2 = L_2(\Omega, \mu)$, where (Ω, μ) is a measure space with σ-finite measure. Recall that positivity means that $e^{-tB}u \geq 0$ for any nonnegative $u \in L_2$.

Suppose also that the semigroup e^{-tB}, $t > 0$ acts continuously from L_2 to L_∞ and, by duality, also from L_1 to L_2. Then the semigroup property shows that e^{-tB} acts from L_1 to L_∞ and factorizes through L_2. It follows that e^{-tB} can be realized as an integral operator whose kernel, say $Q_B(t; x, y)$, is a function in $L_\infty(\Omega \times \Omega)$ for each $t > 0$. Moreover, the restriction of the kernel to the diagonal, $Q_B(t; x, x)$ is well defined as a nonnegative element of $L_\infty(\Omega)$. Denote

$$M_B(t) = \|Q_B(t; x, x)\|_{L_\infty(\Omega)},$$

then $M_B(t)$ is a nonincreasing function on $\mathbb{R}_+$. In this paper we require

$$\int_a^\infty M_B(t)dt < \infty, \qquad a > 0. \tag{1}$$

We will write $B \in \mathcal{P}$ if the selfadjoint operator B generates the semigroup which is both positive and $(2, \infty)$-bounded. For such B, the kernel $Q_B(t; x, y)$ is nonnegative.

2. Denote by $b[u]$ the quadratic form of the operator B, its domain is $\mathrm{Dom}(b) = \mathrm{Dom}(B^{1/2})$. Let $V \geq 0$ be a measurable function on Ω. Suppose that the quadratic form $\int_\Omega V|u|^2 d\mu$ is form-bounded with respect to b, with a bound less than 1. Then the selfadjoint bounded from below operator $B - V$ is well defined as a form-sum. Denote by $N_-(B - V)$ the number of its negative eigenvalues (counting multiplicities), with the usual convention $N_-(B - V) = \infty$ if there is some essential spectrum below zero.

Let $G(z)$ be a function on $[0,\infty)$, polynomially growing at infinity and such that $z^{-1}G(z)$ is integrable at zero. With any such G we associate another function $g = \mathfrak{L}(G)$:

$$g(\lambda) = \mathfrak{L}(G)(\lambda) := \int_0^\infty z^{-1}G(z)e^{-\frac{z}{\lambda}}dz. \tag{2}$$

The following statement is a generalized version of the CLR-estimate.

Theorem 1. *Let $B \in \mathcal{P}$ be such that $M_B(t)$ satisfies (1) and $M_B(t) = O(t^{-\alpha})$ at zero, with some $\alpha > 0$. Fix a nonnegative convex function G, polynomially growing at infinity and such that $G(z) = 0$ near $z = 0$. Put $g = \mathfrak{L}(G)$. Then*

$$N_-(B-V) \le \frac{1}{g(1)}\int_0^\infty \frac{dt}{t}\int_\Omega M_B(t)G(tV(x))dx, \tag{3}$$

as long as the expression on the right-hand side is finite.

REMARKS. 1. The finiteness of the last expression guarantees that the quadratic form of the operator $B - V$ is well defined as a form-sum, so the quantity $N_-(B-V)$ is also well defined.

2. It follows from convexity that $G(z)$ grows at infinity at least as az, $a > 0$. Therefore, the condition (1) is necessary in order that the estimate (3) be meaningful.

3. Function G is involved in the estimate (3) as a parameter. The idea of a "parametric" estimate (for the case of Laplacian) is due to Lieb [L1]. An appropriate choice of G allows one to optimize the estimate. To make it clear, suppose that $M_B(t) = ct^{-\alpha}$, $0 < t < \infty$, with some $\alpha > 1$; the last assumption is implied by (1). After the change of variables $s = tV(x)$, the inequality (3) turns into

$$N_-(B-V) \le C(G)\int_\Omega V(x)^\alpha dx, \qquad C(G) = cg(1)^{-1}\int_0^\infty G(t)t^{-\alpha-1}dt. \tag{4}$$

This shows that the choice of G affects only the value of the constant factor in the estimate.

3. For certain applications the assumption of positivity of e^{-tB}, required in Theorem 1, is too restrictive. A more general class of semigroups is singled out by the *positive domination* property.

We say that a semigroup $P(t) = e^{-tA}$ of selfadjoint contractions in L_2 is dominated by a positive semigroup $Q(t) = e^{-tB}$, if

$$|P(t)u| \le Q(t)|u| \text{ a.e. on } \Omega, \qquad \text{any } u \in L_2. \tag{5}$$

We also say that A is dominated by B and write $A \in \mathcal{PD}(B)$. In cases when B need not to be specified, we simply say that A generates a positively dominated semigroup.

If $Q(t)$ is $(2,\infty)$-bounded, (5) implies that $P(t)$ is $(2,\infty)$-bounded too. Such semigroup $P(t)$ consists, therefore, of integral operators. Denote the corresponding kernel by $P(t;x,y)$. The inequality (5), defining domination, is equivalent to

$$|P(t;x,y)| \leq Q(t;x,y) \text{ a.e. on } \mathbb{R}_+ \times \Omega \times \Omega.$$

The next statement extends Theorem 1 to the case of positively dominated semigroups.

Theorem 2. *Let $B \in \mathcal{P}$ and $A \in \mathcal{PD}(B)$. Suppose that $M_B(t)$ satisfies (1) and $M_B(t) = O(t^{-\alpha})$ at zero, with some $\alpha > 0$. Let G and g be the same functions as in Theorem 1. Then*

$$N_-(A - V) \leq \frac{1}{g(1)} \int_0^\infty \frac{dt}{t} \int_\Omega M_B(t) G(tV(x)) dx \tag{6}$$

as long as the expression on the right-hand side is finite.

Like in Theorem 1, the finiteness of the integral on the right-hand side of (6) guarantees that the operator $A - V$ is well defined as a form-sum.

Let us stress that the expression on the RHS of (6) involves information on the behaviour of the integral kernel of e^{-tB}, rather than of the one of e^{-tA}.

4. Here we present some applications of Theorems 1 and 2.

1°. Schrödinger operator. Let (Ω, μ) be $\mathbb{R}^d$ with Lebesgue measure, and $B = -\Delta$. Then $M_B(t) = (2\pi)^{-d/2} t^{-d/2}$ and after the change of variables (cf. (4)) the inequality (3) turns into the original CLR-estimate, in the form given by Lieb:

$$N_-(-\Delta - V) \leq C(G) \int_{\mathbb{R}^d} V(x)^{d/2} dx, \qquad d \geq 3. \tag{7}$$

The condition $d \geq 3$ in (7) is implied by (1). The optimal choice of G, which leads to the best known value of the constant in the CLR-estimate for $d = 3$, was pointed out by Lieb [L1].

2°. Magnetic Schrödinger operator. For a given magnetic vector potential $\mathbf{a}(x) = \{a_j(x)\}_{1 \leq j \leq d} \in L_{2,\mathrm{loc}}(\mathbb{R}^d)$, consider the operator $A = H_{\mathbf{a}} = -(\nabla - i\mathbf{a})^2$. This operator was studied, e.g. in [AHSim] and in [Sim], and it was shown that $H_{\mathbf{a}} \in \mathcal{PD}(-\Delta)$. Theorem 2 gives

$$N_-(H_{\mathbf{a}} - V) \leq C(G) \int_{\mathbb{R}^d} V(x)^{d/2} dx, \qquad d \geq 3,$$

with the same constant as in (7). The proof of the *magnetic CLR-estimate* (3.2), outlined in [Sim], uses the Itô stochastic integration. A more elementary proof in [MRoz] gives a somewhat worse constant than in the non-magnetic case.

3°. The relativistic Schrödinger operator. In the space $L_2(\mathbb{R}^d)$, $d \geq 3$, consider the operator $B = H_R = (-\Delta + 1)^{1/2} - 1$. This is a ΨDO of order 1, selfadjoint and nonnegative. It was studied, e.g., in [Ca] and [L2]. The semigroup $Q(t) = e^{-tB}$ is positive, its $(2,\infty)$-boundedness is implied by the explicit representation of the kernel on the diagonal

$$Q(t;x,x) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{-t((|\xi|^2+1)^{1/2}-1)} d\xi.$$

It follows that $M_B(t) \leq C(t^{-d/2} + t^{-d})$. Thus, Theorem 1 applies and leads to the estimate, first stated in [D]:

$$N_-(H_R - V) \leq C' \left(\int_{\mathbb{R}^d} V(x)^d dx + \int_{\mathbb{R}^d} V(x)^{d/2} dx \right), \ d \geq 3. \tag{8}$$

We do not attempt to optimize C' here.

4°. The relativistic magnetic Schrödinger operator. Consider the operator $A = H_{\mathbf{a},R} = (H_{\mathbf{a}} + 1)^{1/2} - 1$, where $H_{\mathbf{a}}$ is the magnetic Schrödinger operator considered in **3°**. It can be easily shown that the semigroup $e^{-tH_{\mathbf{a},R}}$ is dominated by e^{-tH_R}. Now, Theorem 2 gives for $N_-(H_{\mathbf{a},R} - V)$ the same estimate (8) as for the non-magnetic relativistic case. Such estimate seems to be new.

5°. Discrete Schrödinger operator. Let $\Omega = \mathbb{Z}^d$, with the counting measure. For a sequence $u \in l_2(\mathbb{Z}^d)$, we consider symmetrized difference operators $(\delta_j u)(k) = \frac{1}{2}(u(k+\ell_j) - u(k-\ell_j))$, where $j = 1, ..., d$, $k \in \mathbb{Z}^d$ and ℓ_j stands for the d-tuple with 1 in the position j and zeros in remaining positions. The corresponding discrete Laplace operator is defined as $\Delta = \sum_j \delta_j^2$, thus

$$(\Delta u)(k) = \frac{1}{4} \sum_j (u(k+2\ell_j) + u(k-2\ell_j) - 2u(k)).$$

The operator $-\Delta$ generates a positivity preserving semigroup in $l_2(\mathbb{Z}^d)$. The value on the diagonal of its integral kernel does not depend of k and is equal

$$Q(t;k,k) = (2\psi)^{-d/2} \int_{\mathbb{Q}^d} e^{-t\sum_j \sin^2 \xi_j} d\xi, \qquad k \in \mathbb{Z}^d.$$

It follows that $Q(t;.)$ is bounded at 0 and decays as $t^{-\frac{d}{2}}$ at infinity. Thus, for a discrete potential $V(k) \geq 0$, $k \in \mathbb{Z}^d$, we get the estimate

$$N_-(-\Delta - V) \leq C \sum_{k \in \mathbb{Z}^d} V(k)^{\frac{d}{2}}, \ d \geq 3. \tag{9}$$

6°. Discrete magnetic Schrödinger operator. When defining the magnetic discrete Schrödinger operator, we shall follow [Sh], where the case of a constant magnetic field is considered. So, let $\lambda_j(k)$, $j = 1, ..., d$ be real functions on $\mathbb{Z}^d$. We set

$$\Delta_\lambda = \sum_j e^{-i\lambda_j} \delta_j^2 e^{i\lambda_j} \tag{10}$$

thus

$$(Au)(k) = \frac{1}{4}\sum_j (e^{i(\lambda_j(k+2\ell_j)-\lambda_j(k))}u(k+2\ell_j)$$
$$+ e^{i(\lambda_j(k-2\ell_j)-\lambda_j(k))}u(k-2\ell_j) - 2u(k)), \ k \in \mathbb{Z}^d.$$

Now, according to the representation (10), the Trotter formula gives domination of the magnetic discrete Schrödinger semigroup by the nonmagnetic one. Therefore, the estimate (9) carries over to the magnetic case.

5. Evidently, Theorem 1 is a particular case of Theorem 2. A complete proof of the latter is given in the preprint [RozS]. In this paper we outline the proof of a statement, close to Theorem 2 but giving a bit weaker estimate. This happens because the function G will be specified, so the estimate we obtain is no more "parametric". Besides, the chosen G does not meet all the assumptions of Theorems 1 and 2. Remind that in the case of "pure powerlike" behaviour of $M_B(t)$ the choice of G affects only the value of the constant factor in the estimate, cf. (4).

Fix an integer $N \geq 1$ and set $F_N = (1-e^{-z})^N$, $G_N(z) = zF_N(z)$ and $g_N = \mathcal{L}(G_N)$. Denote by H_N a convex majorant for G_N, constructed in the following way. Let z_0 be the smallest positive number such that $G''_N(z_0) = 0$. Then we put $H_N(z) = G_N(z)$ for $z < z_0$ and $H_N(z) = G_N(z_0) + (z-z_0)G'_N(z_0)$ for $z \geq z_0$.

Theorem 3. *Let A and B satisfy the conditions of Theorem 2, and let $N > \alpha - 1$. Then*

$$N_-(B-V) \leq \frac{1}{g_N(1)}\int_0^\infty \frac{dt}{t}\int_\Omega M_B(t)H_N(tV(x))dx. \tag{11}$$

Our approach is an adaptation of Lieb's proof of the estimate (7). The original Lieb's approach is based upon the path integrals technique. We do not use this formalism, though our main technical tool, the "suspended Trotter formula" (see Lemma 6 below), imitates path integrals. However, we make no use of any probabilistic technique. It is the decisive point: exactly this allows us to prove the results in such a general setting. A simple trace class analysis replaces convergence properties in infinite-dimensional integration and the only structure we need for our approach is the one of measure space.

We start with some necessary technical statements. In what follows, $\mathfrak{S}_2$ denotes the Hilbert-Schmidt class of compact operators in a Hilbert space, $\mathfrak{S}_1$ denotes the trace class.

Lemma 4. *Let $B \in \mathcal{P}$, $A \in \mathcal{PD}(B)$, and $V \geq 0$ be a measurable function on Ω. Then*

$$\|V^{\frac{1}{2}}e^{-t(A+V)}\|_{\mathfrak{S}_2} \leq M_B(2t)^{1/2}\|V^{1/2}\|_{L_1}.$$

Lemma 5. 1°. *Let $T \in \mathfrak{S}_1$, and $\{R_{1,n}\}$, $\{R_{2,n}\}$ be two sequences of bounded operators, converging to I strongly. Then $R_{1,n}TR^*_{2,n} \to T$ in $\mathfrak{S}_1$.*

2°. *Let T_n be a sequence of bounded operators in a Hilbert space, converging weakly to an operator T. If R_1, $R_2 \in \mathfrak{S}_2$, then $R_1T_nR_2 \to R_1TR_2$ in $\mathfrak{S}_1$.*

Lemma 6. *Let A be a nonnegative selfadjoint operator in L_2 and $0 \le V \in L_1 \cap L_\infty$. For any $t > 0$ and any $\delta > 0$ one has*

$$\int_{\delta t}^{(1-\delta)t} e^{-s(A+V)} V e^{-(t-s)(A+V)} ds$$
$$= (weak)\text{-} \lim_{n\to\infty} \frac{t}{n} \sum_{\delta < \frac{l}{n} \le 1-\delta} \left(e^{-\frac{tA}{n}} e^{-\frac{tV}{n}}\right)^l V \left(e^{-\frac{tA}{n}} e^{-\frac{tV}{n}}\right)^{n-l}.$$

6. What is given below, is basically the Lieb construction, as presented in [RSim]. Suppose first that $0 \le V \in L_1 \cap L_\infty$. Choose $r > 0$ and consider the Birman-Schwinger operator

$$K_r(V) = V^{\frac{1}{2}} A_r^{-1} V^{\frac{1}{2}} := V^{\frac{1}{2}} (A + r)^{-1} V^{\frac{1}{2}}.$$

The following equality will be used when deriving (13):

$$K_r(V)(1 + jK_r(V))^{-1} = V^{\frac{1}{2}}(A_r + jV)^{-1} V^{\frac{1}{2}}, \qquad j \ge 0.$$

The operator $K_r(V)$ is nonnegative and, under the above condition on V, compact. Let λ_k be its eigenvalues and $n(\lambda, K_r(V)) = \#\{k : \lambda_k > \lambda\}$ - their distribution function. By the Birman-Schwinger principle, one has $N_-(A_r - V) = n(1, K_r(V))$ and therefore,

$$N_-(A - V) = \lim_{r\to 0+} n(1, K_r(V)). \tag{12}$$

The function g_N is continuous, nonnegative and nondecreasing on $\mathbb{R}_+$. It can be expressed as

$$g_N(\lambda) = \lambda \sum_{j=0}^{N} c_{N,j} (1 + j\lambda)^{-1}; \qquad c_{N,j} = (-1)^N \binom{N}{j}$$

which implies

$$g_N(K_r(V)) = \sum_{j=0}^{N} c_{N,j} V^{\frac{1}{2}} (A_r + jV)^{-1} V^{\frac{1}{2}} = \int_0^\infty \sum_{j=0}^{N} c_{N,j} V^{\frac{1}{2}} e^{-t(A_r + jV)} V^{\frac{1}{2}} dt. \tag{13}$$

The integrand on the right-hand side of (13) belongs to $\mathfrak{S}_1$ and, moreover, for $N > \alpha - 1$ the integral converges in the $\mathfrak{S}_1$-norm. This is easy but not quite trivial, because the last assertion fails for each single term of the integrand. Anyhow, we obtain $g_N(K_r(V)) \in \mathfrak{S}_1$. This and (12) yield

$$N_-(A - V) \le g_N(1)^{-1} \lim_{r\to 0+} \operatorname{Tr} g_N(K_r(V)). \tag{14}$$

The next important relation is

$$\mathrm{Tr}\ g_N(K_r(V)) = \int_0^\infty \mathrm{Tr}\ W_p(t)dt \tag{15}$$

where

$$W_p(t) = \frac{p}{t(p-2)} \sum_{j=0}^{J} c_{N,j} \int\limits_{tp^{-1}}^{t(1-p^{-1})} e^{-s(A_r+jV)} V e^{-(t-s)(A_r+jV)} ds. \tag{16}$$

In (16) p is an integer, $p > 2$. We would like to stress that the equality (15)-(16) is valid just for traces, the similar equality for the operators fails.

Let us comment the equality (15)-(16). We first note that in view of Lemma 4 the integrands in (16) belong to $\mathfrak{S}_1$, and their $\mathfrak{S}_1$-norms are controlled by the expression $\sqrt{M_B(2s)M_B(2t-2s)}\|V\|_{L_1}$. This shows that the integrals in (16) converge in $\mathfrak{S}_1$, so $W_p(t) \in \mathfrak{S}_1$. Now it is clear that (15) follows from the cyclicity of the trace. Note that the inclusion $W_p(t) \in \mathfrak{S}_1$ is wrong for $p = \infty$, that is if one integrates over $(0, t)$.

Further, choose an expanding (as $\varepsilon \to 0$) family of subsets $\Omega_\varepsilon \subset \Omega$ of finite measure, such that their union is Ω. Introduce a family of regularizers $R_\varepsilon = \chi_\varepsilon e^{-\varepsilon B_r/2}$ where $B_r = B + r$ and χ_ε is the indicator function of Ω_ε. By Lemma 4, $R_\varepsilon \in \mathfrak{S}_2$ and besides, $R_\varepsilon \to I$ strongly as $\varepsilon \to 0$. Then $R_\varepsilon W_p(t) R_\varepsilon^* \to W_p(t)$ in $\mathfrak{S}_1$ by Lemma 5(1°) and therefore,

$$\mathrm{Tr}\ W_p(t) = \lim_{\varepsilon\to 0} \mathrm{Tr}\ R_\varepsilon W_p(t) R_\varepsilon^*. \tag{17}$$

To evaluate $\mathrm{Tr}\ R_\varepsilon W_p(t) R_\varepsilon^*$, we apply Lemma 6, substituting $A_r = A + r$ for A and setting $\delta = p^{-1}$. It is convenient to take $n = mp$, $m \in \mathbb{N}$. The limit in this Lemma exists in weak sense, however according to Lemma 5(2°), after multiplying from both sides by operators from $\mathfrak{S}_2$ we get trace class convergence for products. Therefore,

$$\frac{p-2}{p} \mathrm{Tr}\ R_\varepsilon W_p(t) R_\varepsilon^* =$$

$$\lim_{\substack{n\to\infty \\ n=mp}} \mathrm{Tr} \left(\frac{1}{n} \sum_{j=0}^{N} c_{N,j} \sum_{l=m+1}^{n-m} R_\varepsilon \left(e^{-\frac{t}{n}A_r} e^{-\frac{t}{n}jV}\right)^l V \left(e^{-\frac{t}{n}A_r} e^{-\frac{t}{n}jV}\right)^{n-l} R_\varepsilon^* \right).$$

Denoting by $P_r(t;x,y)$ and $Q_r(t;x,y)$ the kernels of the semigroups e^{-tA_r} and e^{-tB_r}, write down the trace in the last expression as a multiple integral:

$$\frac{1}{n} \sum_{j=0}^{N} c_{N,j} \sum_{l=m+1}^{n-m} \int\limits_{\Omega\times\cdots\times\Omega} Q_r(\frac{\varepsilon}{2}; y, x_0) P_r(\frac{t}{n}; x_0, x_1) e^{\frac{t}{n}jV(x_1)} \ldots P_r(\frac{t}{n}; x_{l-1}, x_l)$$

$$\times e^{\frac{t}{n}jV(x_l)} V(x_l) \ldots P_r(\frac{t}{n}; x_{n-1}, x_n) e^{-\frac{t}{n}jV(x_n)} Q_r(\frac{\varepsilon}{2}; x_n, y) \chi_\varepsilon(y) dy dx_0 \ldots dx_n.$$

Rearranging the terms, we come to

$$\frac{1}{n}\int\limits_{\Omega\times\cdots\times\Omega}\chi_\varepsilon(y)Q_r(\frac{\varepsilon}{2};y,x_0)\prod_{k=1}^{n}P_r(\frac{t}{n},x_{k-1},x_k)$$

$$\times\sum_{j=0}^{N}c_{N,j}e^{-\frac{t}{n}j\sum\limits_{\nu=1}^{n}V(x_\nu)}\sum_{l=m+1}^{n-m}V(x_l)\,Q_r(\frac{\varepsilon}{2};x_n,y)dydx_0\ldots dx_n. \quad (18)$$

The sum over j is equal to $F_N(\frac{t}{n}\sum_{\nu=1}^{n}V(x_\nu))$ and therefore nonnegative. Thus we can majorize the last integral by replacing P_r by Q_r. After this, all the terms in (18) become nonnegative; we make them even bigger by replacing χ_ε by 1 and extending summation over l to all l, $1\le l\le n$. Then we integrate in y, using the semigroup property. According to (17), we arrive to

$$\operatorname{Tr}W_p(t)\le\limsup_{\varepsilon\to0}\limsup_{n\to\infty}\frac{1}{t}\frac{p}{p-2}\int\limits_{\Omega\times\cdots\times\Omega}Q_r(\varepsilon;x_n,x_0)\prod_{k=1}^{n}Q_r(\frac{t}{n};x_{k-1},x_k)$$

$$\times\, G\left(\frac{t}{n}\sum_{\nu=1}^{n}V(x_\nu)\right)dx_0...dx_n,$$

Now we apply the equality (15). Since its LHS does not depend on p, we can drop the factor $\frac{p}{p-2}$ in the last inequality. This leads to an important "intermediate" estimate:

$$\operatorname{Tr}\,g(K_r(V))\le\int\limits_0^\infty\frac{dt}{t}\limsup_{\varepsilon\to0+}\limsup_{n\to\infty}\int\limits_{\Omega\times\cdots\times\Omega}Q_r(\varepsilon;x_n,x_0)$$

$$\times\prod_{k=1}^{n}Q_r(\frac{t}{n};x_{k-1},x_k)\;G\left(\frac{t}{n}\sum_{\nu=1}^{n}V(x_\nu)\right)dx_0...dx_n. \quad (19)$$

Finally, for n,ε and t fixed, consider the multiple integral in (19). Replacing G_N by its convex majorant H_N and using Jensen's inequality, we majorize this integral by

$$\frac{1}{n}\sum_{\nu=1}^{n}\int\limits_{\Omega\times\cdots\times\Omega}Q_r(\varepsilon;x_n,x_0)\prod_{k=1}^{n}Q_r(\frac{t}{n};x_{k-1},x_k)H_N(tV(x_\nu))dx_0\ldots dx_n. \quad (20)$$

Here, for ν fixed, we integrate over all x_k with $k\ne\nu$, using the semigroup property. All the resulting integrals are identical and thus (20) is equal to

$$\int_\Omega Q_r(t+\varepsilon;x,x)H_N(tV(x))dx,$$

which does not depend on n. Taking into account that

$$Q_r(t+\varepsilon;x,x)\le e^{-rt}M_B(t+\varepsilon)\le M_B(t),$$

we see that this integral does not exceed $M_B(t)\int_\Omega H_N(tV(x))dx$. In view of (14), integration in t gives (11).

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