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Short Waves through Thin Interfaces and 2-microlocal Measures

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Abstract

We describe how short waves solutions to the Schrödinger equation propagate, in terms of their semiclassical measures, through a thin interface between two inhomogeneous media. We get matching conditions for the traces of the semiclassical measures from each side of the interface. When the thickness of the interface is smaller than the wavelength these conditions yield a microlocal Snell-Descartes law of refraction. When it is greater, they yield a classical scattering law. The methods also apply to the scalar wave equation.

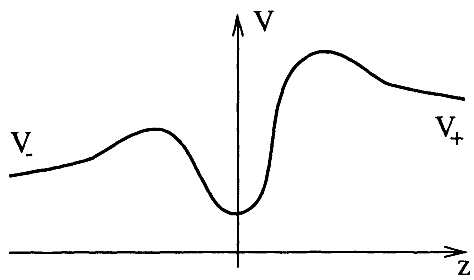
Introduction

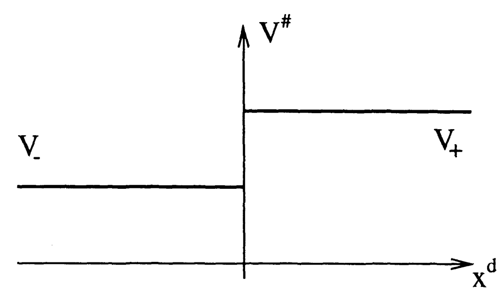
0.1 The problem

We consider the Cauchy problem for the semiclassical Schrödinger equation :

$$(1) \qquad \begin{cases} ih\partial_t u^{h,\varepsilon} &= -\frac{h^2}{2}\Delta u^{h,\varepsilon} + V(\frac{x^d}{\varepsilon}, x)u^{h,\varepsilon} & \text{in } \mathbb{R}_t \times \mathbb{R}^d_x \\ u^{h,\varepsilon}(t=0) &= \psi^h & \text{in } \mathbb{R}^d_x = \mathbb{R}^{d-1}_{x'} \times \mathbb{R}_{x^d} \,, \end{cases}$$

where the potential corresponds to a flat interface of codimension one and thickness $\varepsilon > 0$: $V \in C^\infty(\mathbb{R}^{d+1})$ is real and there exists $V_\pm \in C^\infty(\mathbb{R}^d)$ such that $\lim_{z\to\pm\infty} V(z,x) = V_\pm(x)$ for the C^∞-convergence on compact sets. In particular, the potential converges pointwise when $\varepsilon \to 0$ to : $V^\sharp(x) = V_-(x)\,\mathbf{1}_{\{x^d<0\}} + V_+(x)\,\mathbf{1}_{\{x^d\geqslant 0\}}$, which corresponds to a sharp interface along $\{x^d = 0\}$. To ensure that it defines a continuous unitary dynamic on the " energy " space $L^2(\mathbb{R}^d)$ for fixed ε, we make the additional assumption : there exists $\alpha, \beta > 0$ such that $V(z,x) > -\alpha|x|^2 - \beta$ for all z. We shall refer to the probability of presence $|u^{h,\varepsilon}(t,x)|^2\,dx$ of the waves at time t as their density. The dynamic conserves its total mass.





Given two sequences of positive real numbers $\mathbf{h} = (h_n)_{n\in\mathbb{N}}$ and $\varepsilon = (\varepsilon_n)_{n\in\mathbb{N}}$ which converge to 0, and a bounded sequence of data in L^2 which does not dissipate at infinity or through short oscillations with wavelength smaller than h (they would propagate at infinite speed), we are interested in the asymptotic evolution of this density. For notational convenience, we shall omit the index n and denote the asymptotic by $h \to 0$. The precise properties we demand of the data are usually called compacity at infinity and h-oscillation, and write : $\limsup_h \int_{|x|>R} |\psi^h|^2(x)\,dx$ and, for any localization function $\varphi \in C_o^\infty(\mathbb{R}^d)$, $\limsup_h \int_{|h\xi|>R} |\widehat{\varphi\psi^h}|^2(\xi)\,d\xi$ tend to 0 as R tends to $+\infty$. In particular, such data can be approximated in " energy " space by uniformly compactly supported and strongly h-oscillating data (i.e. : $\exists s > 0, \| \, |hD_x|^s \psi^h \|_{L^2} = O(1)$).

0.2 Semiclassical measures

In order to study how the wave density is refracted by the thin interface, we use the notion of semiclassical (or Wigner) measures (cf. the survey [9]) which allows, in particular, to establish for high-frequency " energy " density the analogue of the results of microlocal propagation of singularities (cf. [12] in general, and [22] for a transmission problem). We recall that a semiclassical measure of the sequence (ψ^h) bounded in $L^2_{loc}(\mathbb{R}^d)$ is a positive Radon measure m on the phase space $T^*\mathbb{R}^d = \mathbb{R}^d_x \times \mathbb{R}^d_\xi$ satisfying :

$$\forall a \in C^\infty_{comp}(\mathbb{R}^{2d}), \ \lim_h \left(a(x, hD_x)\,\psi^h, \psi^h\right)_{L^2} = \int a(x,\xi)\,dm. \tag{2}$$

This limit of observations (in the quantum mechanical sense) of the wave ψ^h by the semiclassical test operator $a(x, hD_x)$ measures its asymptotic microlocal density. Replacing $\mathbf{h}$ by a subsequence if needed, m always exist.

For example, the semiclassical measure of the coherent state $l^{-\frac{d}{2}} A(\frac{x}{l}) e^{ik.x/h}$ with amplitude $A \in L^2(\mathbb{R}^d)$ is $|A(x)|^2 dx \otimes \delta(\xi - k)$ if $l = 1$, and $\delta(x) \otimes |\hat{A}(\xi - k)|^2 \frac{d\xi}{(2\pi)^d}$ if $l = h$. More generally, the singular components of the measure represent the locus of positions and wave vectors where the density concentrates at the scale $\mathbf{h}$.

0.3 The results

We investigate how m determines the semiclassical measure μ of $(u^{h,\varepsilon})$ in time and space. The basic result concerns the case when there is no interface, i.e. V is independent of

z, and says that μ is invariant under the Hamiltonian flow like the classical phase-space distribution function, i.e. μ is the solution of the Vlasov equation (cf. [5] and [13]) :

$$
(3) \qquad \left\{ \begin{array}{rcl} \partial_t \mu & = & \nabla_x V(x).\nabla_\xi \mu - \xi.\nabla_x \mu \\ \mu(t=0) & = & m \otimes \delta(\tau + \frac{|\xi|^2}{2} + V(x)) \end{array} \right.
$$

In our setting, there are three main regimes depending on the limit ratio between the wavelength h and the thickness ε of the interface :

$\varepsilon \ll h$ Homogenization effects should prevail. The remnant of the interface at the limit should be some quantum effect. In fact (cf. sect. 0.4), under hypothesis (H1), it is easy to prove that the solutions can be approximated in the energy space by the solutions of the semiclassical Schrödinger equation with the discontinuous potential $V^\sharp$. Through an analysis in terms of semiclassical measures of the second order boundary value problems studied in [14] (cf. [8] and [2] for the Dirichlet problem, and [19] for some results on the Helmholtz equation), we have been able to prove a microlocal version of the Snell-Descartes law of refraction for such a sharp interface. This result includes the critical incidence and the diffractive grazing rays. A distinctive phenomenon is the bifurcation of the density when the incoming normal velocity is not too small compared to the jump of the potential at the interface. Therefore the evolution of μ is no longer classical : it is given by a Markov semigroup which reflects its wave-like probabilistic nature. The main drawback stems from the loss of coherence information in the semiclassical measures limit : the outcome of the interference on the interface of two waves coming from each side is not quantitatively determined by their microlocal density.

$\varepsilon \gg h$ Since the variation of the medium properties is small compared to the wavelength, classical mechanics (geometrical optics) should still be valid. Under the slightly stronger hypothesis (H2) (cf. sect. 0.4), we have proved that the incoming normal velocity ξ^d at the point x' of the interface determines whether the wave density is transmitted or reflected and the value of the outcomming velocity, according to the classical scattering properties of the one-dimensional potential $V(\cdot, x', 0)$, when ξ^d corresponds to a non-trapping energy. Our method rely on a 2-scales 2-microlocal refinement of semiclassical measures. Some 2-scales W.K.B. constructions for some less general potentials highlight another discrepancy with the previous regime : the wave is trapped inside the interface for a critical incoming normal velocity.

$\varepsilon \sim h$ The evolution of μ should involve the quantum scattering properties of $V(z, x', 0)$, for fixed x'. We refer to the results of Francis Nier for a potential $V(x) + U(\frac{x}{h})$ where U is short-range. In our setting, one should compare the evolution for the potential $V(z, x', 0)$ with the evolutions for the step potential $V_-(x', 0)\, \mathbf{1}_{\{z<0\}} + V_+(x', 0)\, \mathbf{1}_{\{z\geqslant 0\}}$ instead of the free evolution. We refer to [18] for the asymptotic completeness of step-like potentials (see also the references in [10]).

The methods we present here yield analogous results for the scalar wave equation :

$$
(4) \qquad \partial_t^2 w^h - \nabla_x.(c^2 \nabla_x) w^h = 0 \quad \text{in } \mathbb{R}_t \times \mathbb{R}_x^d.
$$

where we the celerity c has the same form and regularity as V, and satisfies additional assumptions for the dynamic to exist. The energy density of its solutions is : $|\partial_t w^h(t,x)|^2\,dx + |c(x)\nabla_x w^h(t,x)|^2\,dx$. We refer to [21], [4], and [7] for the definition of the corresponding microlocal measures.

0.4 Semiclassical measure traces at the interface

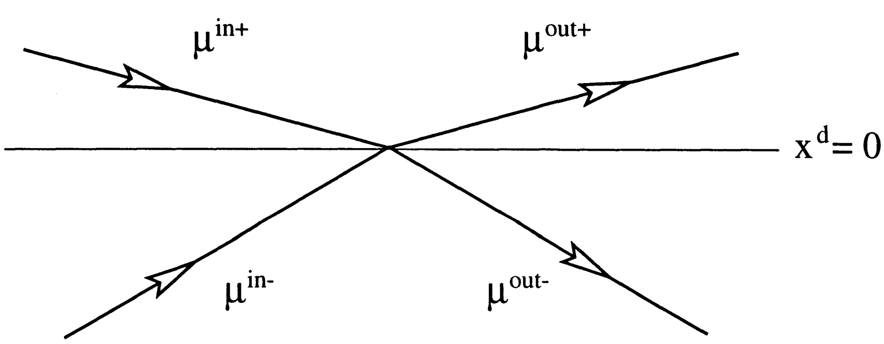
In this talk, we want to draw a parallel between the first (sect. 1) and second (sect. 2) regimes. The results we mention are proved in [16] in the context of two homogeneous media. The results of sect. 1 are announced in [15]. The complete proofs in the context of two inhomogeneous media are to appear.

Let us first introduce the technical hypothesis that our methods require in each of these regimes :

(H1) $\varepsilon \ll h^3$ and $\int \sup_{x\in K} |V(z,x) - V^\sharp(x)|\,dz < +\infty$ for all compact set K.

(H2) $\varepsilon \gg h^{2/3}$, $lim_{z\to\pm\infty} z\partial_z V(z,x) = 0$ and $lim_{z\to\pm\infty}\partial_x^\alpha \partial_z^\gamma V(z,x) = 0$ uniformly on compact sets, for all multi-index α, for all $\gamma > 1$.

In both regimes, it is easy to deduce from these hypothesis that μ still satisfies Vlasov equations corresponding to $V_\pm$ outside the asymptotic interface $\{x^d = 0\}$. Introducing new coordinates $(y^0,\dots,y^d,\eta^0,\dots,\eta^d)$ on $T^*\mathbb{R}^{d+1}$ and the symbols $\omega_+(y,\eta') = -\tau - \frac{|\xi'|^2}{2} - V_+(x)$ on the half-space $Y^+ = \{y = (t,x)\,;x^d > 0\}$, and $\omega_-(y,\eta') = -\tau - \frac{|\xi'|^2}{2} - V_-(x', -x^d)$ on the half-space $Y^- = \{y = (t,x', -x^d)\,;x^d < 0\}$, these equations write : $H^\pm\mu = 0$ in $Y^\pm$, where $H^\pm = \nabla_y\omega_\pm.\nabla_\eta - \nabla_{\eta'}\omega_\pm.\nabla_{y'} + \eta^d\partial_{y^d}$. Since these Hamiltonian vector fields (transport operators) are transversal to $\{y^d = 0\}$ when $\eta^d \neq 0$ (non-grazing wave vectors), theorem 4.4.8' in [12], t. I, implies that μ is C^∞ up to the interface as a function of y^d taking its values in the space of distributions in the other variables. It allows us to define, in $\{\xi^d \neq 0\}$, its traces from the right and from the left $\mu^{0\pm}$ at $x^d = 0^\pm$. These distributions inherit the positivity of μ and consequently are positive measures satisfying the following " jump formula " : $\partial_{y^d}(\mathbf{1}_{\{y^d>0\}}\,\mu) = \mathbf{1}_{\{y^d>0\}}\,\partial_{y^d}\mu + \delta(y^d)\otimes\mu^{0\pm}$ in $Y^\pm \cap \{\eta^d \neq 0\}$. Since they also inherit the localization of the measure μ on the characteristic set, they can be decomposed as : $\mu^{0\pm} = \delta(\eta + \sqrt{2\omega_\pm(y',0,\eta')})\otimes\mu^{in\pm} + \delta(\eta - \sqrt{2\omega_\pm(y',0,\eta')})\otimes\mu^{out\pm}$, where the positive measures $\mu^{in\pm}$ and $\mu^{out\pm}$ correspond to the " incoming " and " outgoing " densities at $x^d = 0^\pm$.



Let $m_t^\pm$ be the push-forwards of the restrictions of m to $Y^\pm$ by the Hamiltonian flows of $H^\pm$ and $\mu_\theta = \mathbf{1}_{|t|<\theta} dt \otimes (m_t^+ + m_t^-)(dx, d\xi) \otimes \delta(\tau + \frac{|\xi|^2}{2} + V^\sharp(x))$. If the density μ does not reach the interface for $t \in [-\theta, \theta]$, then it is determined from the initial condition by the Vlasov transport equations on each side of the interface : $\mathbf{1}_{|t|<\theta}\mu = \mu_\theta$. The problem of studying how m determines μ (as long as it does not graze the interface) amounts to deriving appropriate matching conditions for the traces of μ on the interface. We obtain matching conditions for these four measures by means of the intermediary consideration of unknown measures on the interface : measures of the traces of the solution and its normal derivative on the interface in the " sharp interface " regime, and 2-scales 2-microlocal surface measures in the " classical interface " regime. In both regimes, we first get relations between all these measures on the interface thanks to symbolic calculus commutator expansions, and then we get rid of the unknowns through some positivity properties linked to Gårding's inequality.

1 Sharp interface : $\varepsilon \ll h \ll 1$

As explained in the introduction, under the hypothesis (H1), the waves $(u^{h,\varepsilon})$ can be approximated in L^2_{loc} by the solutions of the Cauchy problem for the semiclassical Schrödinger equation :

$$(5) \qquad \begin{cases} ih\partial_t u^h &= -\frac{h^2}{2}\Delta u^h + V^\sharp(x)u^h \quad \text{in } \mathbb{R}_t \times \mathbb{R}_x^d \\ u^h(t=0) &= \psi^h \quad \text{in } \mathbb{R}_x^d = \mathbb{R}_{x'}^{d-1} \times \mathbb{R}_{x^d}\,, \end{cases}$$

for the same data and a discontinuous potential which corresponds to a transmission problem across a flat interface of codimension one : $V^\sharp(x) = V_-(x)\,\mathbf{1}_{\{x^d<0\}} + V_+(x)\,\mathbf{1}_{\{x^d\geqslant 0\}}$.

If $V^\sharp$ is to represent a true interface, it is natural to assume (at least locally) that it satisfies a jump condition :

(H1') For all x', $V_+(x',0) - V_-(x',0) > 0$.

Under this jump condition, the multiplicators $\partial_{x^d} u^h$ and $V\partial_{x^d} u^h$, localized in time and space, yield that u^h and its semiclassical derivatives $hD_{t,x}u^h$ admit bounded traces at $x^d = 0^\pm$ in $L^2_{loc}(\mathbb{R}_t, L^2_{loc}(\mathbb{R}_{x'}^{d-1}))$.

A large part of the analysis can now be done on each side of the interface separately. Moreover, the time variable does not play any particular role. Therefore it is convenient to make this analysis for more general boundary problems using simpler notations.

1.1 Geometry and semiclassical measures for boundary value problems

In this section, we shall analyze in terms of semiclassical measures the class of second order boundary value problems studied in [14] (cf. [8] and [2] for the Dirichlet problem, and [19] for some results on the Helmholtz equation) : Y is an open set in $\mathbb{R}^{d+1}$ with C^∞ boundary ∂Y, $P^h = p(y, hD_y)$ is a second order semiclassical differential operator with coefficients in $C^\infty(\overline{Y})$, real h-symbol p, and for which ∂Y is not characteristic. They can be reduced locally, in normal geodesic coordinates $(y^0, \cdots, y^d)$, to the following form (cf. [12], t. III, p. 424) : $Y = \{(y', y^d) \in \mathbb{R}^{d+1} \,; y^d > 0\}$, $\partial Y = \{(y', y^d) \in \mathbb{R}^{d+1} \,; y^d = 0\}$, and $p(y,\eta) = \frac{(\eta^d)^2}{2} - \omega(y,\eta')$, where $\omega \in C^\infty(\overline{Y} \times \mathbb{R}^d)$.

The cotangent bundle to the boundary $T^*\partial Y$ can be decomposed (intrinsically) as the disjoint union of the following regions depending on the signs of the functions $\omega_0 = \omega(y', 0, \eta')$ and $\dot\omega_0 = \partial_d\omega(y', 0, \eta')$ (cf. [12], t. III, pp. 430–432) : $\mathcal{E} = \{(y',\eta') \in T^*\partial Y \,; \omega_0(y',\eta') < 0\}$ (elliptic), $\mathcal{H} = \{\omega_0 > 0\}$ (hyperbolic), $\mathcal{G}_d = \{\omega_0 = 0,\, \dot\omega_0(y',\eta') > 0\}$ (diffractive), $\mathcal{G}_g = \{\omega_0 = 0,\, \dot\omega_0 < 0\}$ (gliding or pseudo-convex), and $\mathcal{G}_0 = \{\omega_0 = \dot\omega_0 = 0\}$ (gliding of higher order).

Let $(\mathbf{1}_{\{y^d>0\}}u^h)$ be a bounded family in $L^2_{loc}(\mathbb{R}^{d+1})$, strongly h-oscillating (cf. sect.0.1), satisfying the equation $P^h u^h = 0$ on Y, with semiclassical measure $\mu \in \mathcal{M}^+(T^*\overline{Y})$. From basic properties of semiclassical measures (cf. [9]), μ must be supported in the characteristic set $T^*\overline{Y} \cap \{p = 0\}$ and satisfy $H_p\mu = 0$ on Y, where $H_p = \eta^d\partial_{y^d} - H_\omega$ denote the Hamiltonian vector field associated to p. As in sect. 0.4, we introduce the decomposition of its trace at $y^d = 0^+$ on $\mathcal{H}$ into its " incoming " and " outgoing " parts : $\mu^0 = \delta(\eta + \sqrt{2\omega_0}) \otimes \mu^{in} + \delta(\eta - \sqrt{2\omega_0}) \otimes \mu^{out}$.

Let's also introduce the traces $v^h = u^h_{\rceil y^d=0}$ and $\dot v^h = hD_{y^d}u^h_{\rceil y^d=0}$ and assume they are bounded in $L^2_{loc}(\mathbb{R}^d_{y'})$, with diagonal semiclassical measures ν et $\dot\nu$, and joint semiclassical measure ν^j — a complex measure of the asymptotic correlation between the two traces (cf. [9] for the definition of matrix-valued semiclassical measures associated to vector-valued functions).

Applying to $\mathbf{1}_{\{y^d>0\}}u^h$ commutators of P^h with semiclassical test operators which are differential in η^d of order 0 or 1, with tangential pseudo-differential coefficients, we get the following transport equations where the " source " terms come from the boundary terms in the integrations by parts :

$$\partial_{y^d}\int(\eta^d)^2\mu(d\eta^d) - H_\omega\int\eta^d\mu(d\eta^d) - \partial_{y^d}\omega\int\mu(d\eta^d) = (\frac{1}{2}\dot\nu + \omega_0\nu)\otimes\delta(y^d) \tag{6}$$

$$\partial_{y^d}\int\eta^d\mu(d\eta^d) - H_\omega\int\mu(d\eta^d) = \operatorname{Re}\nu^j\otimes\delta(y^d). \tag{7}$$

By the " jump formula ", we let the traces of μ appear on the left hand side of these equations. When restricted to the interface, they yield relations between the traces of μ and the semiclassical measures of the traces of (u^h) in the hyperbolic region :

PROPOSITION 1.1 *i)* $2\omega_0(\mu^{out} + \mu^{in}) = \frac{1}{2}\dot\nu + \omega_0\nu$ *and* $\sqrt{2\omega_0}(\mu^{out} - \mu^{in}) = \operatorname{Re}\nu^j$ *on* $\mathcal{H}$.
ii) $\mu^{in} = 0 \Longrightarrow \mu^{out} = \nu = \frac{\dot\nu}{2\omega_0} = \frac{\operatorname{Re}\nu^j}{\sqrt{2\omega_0}}$ *on* $\mathcal{H}$.

The second point is crucial in the transmission problem, though it is an easy consequence of the first. In fact, when $\mu^{in} = 0$, the point i) yields the equality :

$$\frac{1}{2}\dot\nu + \omega_0\nu = \sqrt{2\omega_0}\operatorname{Re}\nu^j \leqslant 2(\frac{1}{2}\dot\nu)^{\frac{1}{2}}(\omega_0\nu)^{\frac{1}{2}}$$

where the inequality is the semiclassical measures version of the Cauchy-Schwarz inequality for (v^h) and $(\dot v^h)$. Since the sum of the squares of two terms is lower or equal to the double product of these terms if and only if these two terms are equal, the conclusion in the point ii) holds.

Equation (6), shows that $\partial_{y^d} \int (\eta^d)^2 \mu(d\eta^d)$ does not charge $\{y^d = 0\}$, which allows us to write : $\mu = \mathbf{1}_{\{y^d>0\}}\mu + \delta(y^d) \otimes \delta(\eta^d) \otimes \mu^\partial$, where μ^∂ is a positive measure supported in $\mathcal{G}$. From (6) and (7) we also obtain informations on the grazing region :

PROPOSITION 1.2 *i)* $-\partial_{y^d}\omega\mu^\partial = \frac{1}{2}\dot{\nu}$ *on* $\mathcal{G}$. *ii)* $\dot{\nu}(\mathcal{G} \setminus \mathcal{G}_g) = 0$ *and* $\operatorname{Re} \nu^j (\mathcal{E} \cup (\mathcal{G} \setminus \mathcal{G}_g)) = 0$.
iii) The restriction of μ *to the trajectories stemming from* $\mathcal{G}_d$ *is invariant under* F *in a neighbourhood of* $\mathcal{G}_d$.

Remarks. — For the Dirichlet boundary conditions $v^h = 0$, we have $\nu^j = 0$. Proposition 1.1 i) yields $\mu^{out} = \mu^{in}$ (total reflection) in the hyperbolic region. Proposition 1.2 iii) yields the propagation of μ in the diffractive region. The propagation of the support of μ in the grazing region with contact of finite order (which includes $\mathcal{G}_d$) is proved in [2].

1.2 Matching of the traces at the interface

The jump condition on the interface (H1') allows us to prove the estimates on the traces of the solution u^h of the transmission problem (5) that we need in order to apply the previous analysis on each side of the interface. Moreover, it yields the following partition of the hyperbolic region at $\{x^d = 0^-\}$: $\mathcal{H}^- = \mathcal{H}^+ \cup \mathcal{G}_d^+ \cup (\mathcal{H}^- \cap \mathcal{E}^+)$.

In [15], we define a semigroup S_s^* on $\mathcal{M}^+(T^*\mathbb{R}^{d+1})$ which describes the time invariance of μ as long as it satisfies the *non-interference* and *non-trapping* conditions : $\mu^{in+} \perp \mu^{in-}$ and $\mu((\mathcal{G}^+ \setminus \mathcal{G}_d^+) \cup (\mathcal{G}^- \setminus \mathcal{G}_d^-)) = 0$. We summarize its meaning :

i) The energy coming from $\{x^d < 0\}$ on $\mathcal{E}^+ \cup \mathcal{G}_d^+$ is reflected following the billard flow.

ii) The energy coming from $\{x^d < 0\}$ on $\mathcal{G}_d^-$ and from $\{x^d > 0\}$ on $\mathcal{G}_d^+$ is reflected along the diffractive bicharacteristic curves.

iii) On $\mathcal{H}^+$, the energy is both reflected and transmitted with proportions and directions which are explicit functions of the traces of the coefficients on the interface and the incoming wave vectors. These coefficients can be obtained from the usual plane waves computation.

Recall the definition $\mu_\theta = \mathbf{1}_{|t|<\theta} dt \otimes (m_t^+ + m_t^-)(dx, d\xi) \otimes \delta(\tau + \frac{|\xi|^2}{2} + V^\sharp(x))$ from sect. 0.4.

THEOREM 1 *Under hypothesis* (H1) *and* (H1'), *if* $supp(m) \cap \{x^d = 0\} = \emptyset$, *then there exists a time* $\theta > 0$ *such that* $\mathbf{1}_{|t|<\theta}\mu = \mu_\theta$. *Moreover, if* $S_s^*\mu_\theta$ *is without* interference *nor* trapping *for all* $s \in [0, T]$, *then* $\mathbf{1}_{|t-T|<\theta}\mu = S_T^*\mu_\theta$.

Remarks. — The non-interference condition avoids the phase coherence phenomena on $\mathcal{H}^+$ which semiclassical measures do not take into account. The non-trapping one stems from our present inability to prove that the density which propagates inside the non-diffractive grazing region $(\mathcal{G}^+ \setminus \mathcal{G}_d^+) \cup (\mathcal{G}^- \setminus \mathcal{G}_d^-)$ is progressively radiated.

Our approach also yields a microlocal version of Snell-Descartes's law of refraction for the scalar wave equation (4) where the discontinuous coefficient is the celerity c. The " ray picture " is recovered by applying this result to initial coherent states with Dirac semiclassical measures. Note that the asymptotic expansion of the solution for such data in the critical case ($\mathcal{G}_d^+$ in i)) seem to be already inextricable when $c_\pm$ are constants (cf. [11]). We refer to [22], sect. 1 : *transmission problem*, for propagation of singularities results in this context.

2 Thin classical interface : $h \ll \varepsilon \ll 1$

In this section, we explain how to match the traces of the semiclassical measure μ of the solutions of (1) under the hypothesis (H2) (cf. sect. 0.4). Our method rely on a quite general tool which we have called 2-scales 2-microlocal measures and the underlying 2-scales symbolic calculus.

We won't consider our results on wave trapping which are less complete and more technical.

2.1 2-scales 2-microlocal measures

Recall that semiclassical measures describe the asymptotic density in $T^*\mathbb{R}^d$ at some scale h. The aim of 2-scales 2-microlocal measures is to refine this description in relation to some involutive submanifold $\mathcal{I}$ at some coarser scale ε. In the present case, the involutive submanifold is simply the interface : $\mathcal{I} = \{x^d = 0\}$. The general idea is to blow-up the semiclassical measure with respect to $\mathcal{I}$ by the factor $\frac{1}{\varepsilon}$. This yields an " internal surface measure " on the normal fiber bundle $N\mathcal{I}$ which is like a semiclassical measure at the scale $\frac{h}{\varepsilon}$, and an " external surface measure " on the homogeneous normal fiber bundle $SN\mathcal{I} = N\mathcal{I}/\mathbb{R}_+^*$ which is like an H-measure corresponding to all the scales between h and $\frac{h}{\varepsilon}$. Thus the projection of the semiclassical measure on $\mathcal{I}$ is refined as a measure on the blow-up space of $\mathcal{I}$: $N\mathcal{I} \cup SN\mathcal{I}$. Since $\mathcal{I} = \{x^d = 0\}$ here, $SN\mathcal{I}$ has two connected components so that the " external surface measure " is further decomposed into measures on each side of the interface.

Remarks. — A similar notion of one-scale 2-microlocal measure has been introduced independently from us by Clotilde Fermanian-Kammerer in [3] and by Francis Nier in [17]. It is to 2-scales 2-microlocal measures, what semiclassical measures are to H-measures or Microlocal Defect Measures. It applies to the quantum interface regime $\varepsilon \sim h$ with the same limitations that Nier has pointed out in his work.

Since the operator of our problem (1) writes $P^{h,\varepsilon} = -ih\partial_t - \frac{h^2}{2}\Delta_x + V(\frac{x^d}{\varepsilon}, x)$, it is natural to quantify symbols of the form : $a(\frac{x^d}{\varepsilon}, x, h\xi)$. We shall use Weyl quantization rule, defined by

$$a^W(x,D)u(x) = \int_{\mathbb{R}^d} e^{i(x-y).\xi} a(\frac{x+y}{2}, \xi) u(y) \frac{d\xi}{(2\pi)^d},$$

so that our 2-scales quantization rule writes : $Op^W_{h,\varepsilon}(a) = a^W(\frac{x^d}{\varepsilon}, x, hD_x)$.

The class of test symbols Σ used to define 2-scales 2-microlocal measures should yield a nice symbolic calculus with $P^{h,\varepsilon}$ and be a separable subspace of $C^0_b(\mathbb{R}^{2d+1})$ to ensure the existence theorem for these measures.

DEFINITION 2.1 (TEST SYMBOLS) *The class Σ of test symbols is the the set of $a \in C^\infty(\mathbb{R}, C^\infty_o(\mathbb{R}^{2d}))$ such that : for all $\gamma \in \mathbb{N}$, $\partial_z^\gamma a(z,x,\xi)$ converge in $C^\infty_o(\mathbb{R}^{2d})$ when z tends to $+\infty$ or $-\infty$. We denote : $\lim_{z\to\pm\infty} a(z,x,\xi) = a_\pm(x,\xi)$.*

DEFINITION 2.2 (h-ε-2-MICROLOCAL MEASURES) *A 2-microlocal measure at the two scales $h \ll \varepsilon \ll 0$ with respect to $\mathcal{I} = \{x^d = 0\}$ of a bounded sequence $(u^{h,\varepsilon})$ of $L^2_{loc}(\mathbb{R}^d)$, is composed of three positive Radon measures :*

- *An internal surface measure : ν^0, on $N\mathcal{I} = \mathbb{R}_z \times \mathbb{R}^{d-1}_{x'} \times \mathbb{R}^d_\xi$.*
- *Two external surface measures : $\nu^\pm$, on the connected components of $SN\mathcal{I} = \{-1,+1\}_\omega \times \mathbb{R}^{d-1}_{x'} \times \mathbb{R}^d_{\xi'}$,*

satisfying, extracting a subsequence if necessary, the following oscillation-concentration property : $\forall a \in \Sigma$,

$$\lim_{h,\varepsilon} (a^W(\frac{x_d}{\varepsilon}, x, hD_x)\, u^{h,\varepsilon}, u^{h,\varepsilon})_{L^2} = \int_{\{x^d<0\}} a_-(x,\xi)\, d\mu + \int_{\{x^d>0\}} a_+(x,\xi)\, d\mu$$

$$+ \int a_-(x',0,\xi)\, d\nu^- + \int a_+(x',0,\xi)\, d\nu^+ + \int a(z,x',0,\xi)\, d\nu^0$$

In particular the Radon-Nikodym of the semiclassical measure with respect to the Dirac measure on $\{x^d = 0\}$ is :

$$\frac{d\mu}{d\delta(x^d)} = \nu^- + \nu^+ + \int_{\mathbb{R}} \nu^0(dz, x', \xi)\,.$$

Any bounded sequence of $L^2_{loc}(\mathbb{R}^d)$ has an h-ε-2-microlocal measure (cf. [16] for this existence theorem and the symplectic invariance properties).

Let us illustrate the meaning of each of the three components with an example for which the other components are null. The h-ε-2-microlocal measure of $\frac{1}{\sqrt{\varepsilon}} A(x', \frac{x^d - l}{\varepsilon}) e^{ix.\frac{k}{h}}$, where $A \in L^2(\mathbb{R}^d)$, is $\nu^\pm = ||A(x')||^2_{L^2(\mathbb{R}_{x^d})}\, dx' \otimes \delta(\xi - k)$ if $l = \pm\sqrt{\varepsilon}$, and $\nu^0 = |A(x',z)|^2\, dz \otimes dx' \otimes \delta(\xi - k)$ if $l = \varepsilon^2$.

The symbolic calculus which suits the quantization $Op^W_{h,\varepsilon}$ naturally is the Weyl-Hörmander calculus (c.f. sect. 18.5 in [12], t. III) associated to the following Hörmander metric :

$$g_{h,\varepsilon} = |dx'|^2 + \left|\frac{dx^d}{\varepsilon}\right|^2 + \frac{|hd\xi|^2}{\langle h\xi\rangle^2}\,, \quad \text{with : } \langle\xi\rangle = (1+|\xi|^2)^{\frac{1}{2}}\,.$$

Its uncertainty function is : $H_{h,\varepsilon} = \frac{h}{\varepsilon}\langle h\xi\rangle^{-1} \leqslant \frac{h}{\varepsilon} \leqslant 1$. We can take the constants of slowness and temperance (in the terminology of [1]) equal to 1, uniformly with respect to h and ε. This metric is well adapted to the 2-scales quantization of the classical classes of symbols $S^m = S^m(\langle\xi\rangle^m, g)$, where $g = |dz|^2 + |dx|^2 + \langle\xi\rangle^{-2}|d\xi|^2$. In fact, defining the k^{th} seminorm by : $||a||_{k,S(M,g)} = \sup_{l\leqslant k\ ;X\in\mathbb{R}^{2d+1}\ ;g_X(T_j)\leqslant 1} M(X)^{-1}|\partial_{T_1}\ldots\partial_{T_l}a(X)|$, we have : $||a(\frac{x^d}{\varepsilon}, x, h\xi)||_{k,S(\langle h\xi\rangle^m, g_{h,\varepsilon})} \leqslant 2^k||a||_{k,S(\langle\xi\rangle^m,g)}$, since $g_{h,\varepsilon}(\frac{x^d}{\varepsilon}, x', 0, \xi) = g_{h,\varepsilon}(0,x,\xi) = g(x,\xi)$. Therefore : if $a \in S^m$ then $a(\frac{x^d}{\varepsilon}, x, h\xi)$ is uniformly bounded in $S(\langle h\xi\rangle^m, g_{h,\varepsilon})$.

The h-ε-symbol of $P^{h,\varepsilon}$ pertains to the class $\Sigma^{[m]}_{loc}$ of polynomial symbols in ξ with coefficients like V with the hypothesis (H2). Note that : if $p \in \Sigma^{[m]}_{loc}$ then, for all $\varphi \in C^\infty_o(\mathbb{R}^d)$, $\varphi(x)p \in S^m$. In order to derive properties of the h-ε-2-microlocal measures from $P^{h,\varepsilon}u^{h,\varepsilon} = 0$, we shall need the following product and commutation estimates, which are a consequence of theorem 18.5.4 in [12], t. III, since $\Sigma \subset S^{-\infty}$:

LEMMA 2.1 *The following estimates are uniform with respect to to the seminorms of a in Σ, p in $\Sigma^{[m]}_{loc}$ and φ in $C^\infty_o(\mathbb{R}^d)$:*

$$\begin{aligned}\varphi\, Op^W_{h,\varepsilon}(a) Op^W_{h,\varepsilon}(p)\varphi &= \varphi\, Op^W_{h,\varepsilon}(ap)\varphi + O_{\mathcal{L}(L^2)}(\frac{h}{\varepsilon})\\ \varphi\,[Op^W_{h,\varepsilon}(a), Op^W_{h,\varepsilon}(p)]\,\varphi &= \frac{h}{\varepsilon}\,\varphi\, Op^W_{h,\varepsilon}(iL_p a)\varphi + h\,\varphi\, Op^W_{h,\varepsilon}(iH_p a)\varphi + O_{\mathcal{L}(L^2)}(\left(\frac{h}{\varepsilon}\right)^3)\end{aligned}$$

where $L_p = \partial_{\xi^d}p\partial_z - \partial_z p\partial_{\xi^d}$ and $H_p = \nabla_\xi p.\nabla_x - \nabla_x p.\nabla_\xi$.

The Gårding inequality, which relates the positivity of an operator to the positivity of its symbol, combined with the theorem of Schwartz which says that a positive distribution is a Radon measure, has been used as a means to prove the existence of microlocal measures (cf. [6] and [20]). Here, we could also use it for this purpose (cf. [16] for an alternative approach using wave-packets transforms). But we shall need its most sharpened form, the Fefferman-Phong inequality (cf. theorem 18.6.8 in [12], t. III), in the context of the $g_{h,\varepsilon}$ calculus, to derive some properties of the h-ε-2-microlocal measures :

LEMMA 2.2 *Uniformly with respect to the seminorms of a in S^0 :*

$$a \geqslant 0 \quad\Rightarrow\quad Op^W_{h,\varepsilon}(a) \geqslant -C^{h,\varepsilon}_a Id, \quad \textit{with} : 0 < C^{h,\varepsilon}_a = O(\left(\frac{h}{\varepsilon}\right)^2)$$

and therefore : $\forall a$, $||Op^W_{h,\varepsilon}(a)||_{\mathcal{L}(L^2)} \leqslant ||a||_{L^\infty} + O((\frac{h}{\varepsilon})^2)$.

2.2 Matching of the traces on the interface

Let us first introduce some objects related to the one dimensional classical scattering properties of the potential $U(z,x') = V(z,x',0)$, where x' plays the role of a parameter. We note $U_\pm(x') = V_\pm(x',0)$, $\Delta(x') = U_+(x') - U_-(x')$, $\bar{U}(x') = \sup_z U(z,x')$.

The point $(z,t,x',\tau,\xi') \in N\mathcal{I}$ is said to be almost bounded positively or negatively if $\frac{1}{2}|\xi^d|^2 + U(z,x') = U_\pm(x')$. Let $\mathcal{B} \subset N\mathcal{I}$ denote the union of trajectories which are positively or negatively bounded or almost bounded. The transmission and reflexion regions are defined by :

$$\mathcal{T}^{\pm} = \{(t,x',\tau,\xi) \in \mathcal{I}\,;\frac{1}{2}|\xi^d|^2 > \bar{U}(x') - U_{\pm}(x')\} \quad \text{and} \quad \mathcal{R}^{\pm} = \mathcal{I} \setminus (\mathcal{T}^{\pm} \cup \mathcal{C}^{\pm} \cup \{\xi^d = 0\})\,,$$

where the thresholds regions $\mathcal{C}^{\pm}$ are the set of critical values :

$$\mathcal{C}^{\pm} = \{(t,x',\tau,\xi') \in \mathcal{I}\,; \exists z, \partial_z U(z,x') = 0 \text{ and } \frac{1}{2}|\xi^d|^2 = U(z,x') - U_{\pm}(x')\}\,.$$

The transmission map $T\ : \mathcal{T}^- \to \mathcal{T}^+$ and the reflexion maps $R\ : \mathcal{I} \to \mathcal{I}$ are defined by : $T(t,x',\tau,\xi) = (t,x',\tau,\xi', sgn(\xi^d)\sqrt{(|\xi^d)^2 - 2\Delta(x')|}\,)$ and $R(t,x',\tau,\xi) = (t,x',\tau,\xi',-\xi^d)$.

PROPOSITION 2.1 *i)* $\nu^0(N\mathcal{I} \setminus \mathcal{B}) = 0$. *ii)* $\nu^{\pm}\left(\mathcal{I} \setminus (\mathcal{C}^{\pm} \cup \{\xi^d = 0\})\right) = 0$.
iii) $\xi^d \mu^{0-} = T^*\left(\xi^d \mu^{0+}\right)$ *on* $\mathcal{T}^-$. *iv)* $\mu^{0\pm} = R^*\left(\mu^{0\pm}\right)$ *on* $\mathcal{R}^{\pm}$.

The general idea is to " observe " $(u^{h,\varepsilon})$ through commutators of $P^{h,\varepsilon}$ with two-scales test operators. Expanding the commutator in $\frac{\varepsilon}{h} i(\,[Op^W_{h,\varepsilon}(a), P^{h,\varepsilon}\,]\,u^{h,\varepsilon}, u^{h,\varepsilon})_{L^2} = 0$ by lemma 2.1 and passing to the limit, we get : $L\nu^0 = 0$ with $L = \xi^d \partial_z - \partial_z U \partial_{\xi^d}$, and obtain the point i). If we do the same for $\frac{i}{h}(\,[Op^W_{h,\varepsilon}(a), P^{h,\varepsilon}\,]\,u^{h,\varepsilon}, u^{h,\varepsilon})_{L^2} = 0$ with test symbols a such that $La = 0$, we may prove that $\nu^{\pm}\left(\mathcal{T}^{\pm}\right) = 0$ which is enough to obtain the point iii) but not the point iv) (cf. [16]). To overcome this difficulty, we apply lemma 2.2 to the first term of the expansion of $\frac{i}{h}(\,[Op^W_{h,\varepsilon}(a), P^{h,\varepsilon}\,]\,u^{h,\varepsilon}, u^{h,\varepsilon})_{L^2} = 0$ with test symbols a such that $La > 0$. This weaker condition on a allows us get some other informations on $\nu^{\pm}$ in the form of inequalities and obtain the point ii). It results in the points iii) and iv) when test symbols a such that $La = 0$ are used. The transmission and reflexion maps arise for the following reason : $La = 0$ implies $a_+ \circ T = a_-$ on $\mathcal{T}^-$ and $a_{\pm} \circ R = a_{\pm}$ on $\mathcal{R}^{\pm}$.

From proposition 2.1, it is straightforward to define a broken flow C^s on $T^*\mathbb{R}^{d+1}$ which describes the time invariance of μ as long as it satisfies the *non-grazing* and *non-trapping* conditions : $Supp(\mu) \cap \mathcal{I} \cap \{\xi^d = 0\} = \emptyset$ and $Supp(\mu^{0\pm}) \cap \mathcal{C}^{\pm} = \emptyset$. Recall the definition $\mu_\theta = \mathbf{1}_{|t|<\theta} dt \otimes (m_t^+ + m_t^-)(dx, d\xi) \otimes \delta(\tau + \frac{|\xi|^2}{2} + V^\sharp(x))$ from sect. 0.4.

THEOREM 2 *Under hypothesis* (H2)*, if* $supp(m) \cap \{x^d = 0\} = \emptyset$*, then there exists a time* $\theta > 0$ *such that* $\mathbf{1}_{|t|<\theta}\mu = \mu_\theta$. *Moreover, if* $C^s_* \mu_\theta$ *is without* grazing nor trapping *for all* $s \in [0, T]$, *then* $\mathbf{1}_{|t-T|<\theta}\mu = C^T_* \mu_\theta$.

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