FANG HUA LIN

Rectifiability of defect measures, fundamental groups and density of Sobolev mappings

Journées Équations aux dérivées partielles (1996), p. 1-14

<http://www.numdam.org/item?id=JEDP_1996____A12_0>

Rectifiability of Defect Measures, Fundamental Groups and Density of Sobolev Mappings

Fang Hua Lin

Courant Institute of Mathematical Sciences

Abstract

Here we study all possible weak-limits of a minimizing sequence, for the p-energy functional, consisting of continuous maps between Riemann manifolds subject to a Dirichlet boundary condition. We show that if p is not an integer, then any such limit is, in particular, a stationary p-harmonic map which is $\mathbb{C}^{1,\alpha}$ continuous away from a closed subset of Hausdorff dimension $\leq n - [p] - 1$. If p is an integer, then any such limit is a weakly p-harmonic map along with a $(n - p)$-rectifiable Radon measure μ. Moreover, the limiting map is $\mathbb{C}^{1,\alpha}$ continuous away from a closed subset $\sum \equiv spt\mu \cup S$ with $H^{n-p}(S) = 0$. The same results valid for minimizing sequences in a homotopy class.Some density properties of smooth maps in Sobolev spaces of mappings are also implied by our measure theoretic results.

1 Introduction

Let M be a smooth, compact Riemannian manifold with smooth boundary ∂M, and let N be a smooth, compact Riemannian manifold without boundary. Suppose g: $\partial M \to N$ is a Lipschitz continuous map, and $1 < p < n$, $n = \dim M$, we consider the variational problem:

$$\min E_p(u) \equiv \int_M |\nabla u|^p dx \tag{1.1}$$

among maps u: $M \to N$ such that $u|_{\partial M} = g$.

The *first natural question* one has to address is whether or not the set,

$$W_g^{1,p}(M,N) = \{u\colon\ M \to N, u|_{\partial M} = g, \text{ and } \nabla u \in L^p(M)\} \tag{1.2}$$

is nonempty.

For this we have the following:

Theorem [HL] *If N is $[p-1]$-connected, then $W_g^{1,p}(M,N) \neq \emptyset$. In fact,*

$$\begin{aligned} &\inf\left\{\int_M |\nabla u|^p(x)dx\colon\ u \in W_g^{1,p}(M,N)\right\} \\ \leq\ & C_p(N)\inf\left\{\int_M |\nabla v|^p(x)dx\colon\ v \in W_g^{1,p}(M,\mathbb{R}^K)\right\}. \end{aligned} \tag{1.3}$$

Here we assume that N is isometrically embedded in $\mathbb{R}^K$, and $C_p(N)$ is a constant depending only on N and p. In particular, if g is a trace of $W^{1,p}(M,\mathbb{R}^K)$ map on ∂M, with $g(\partial M)$ contained in N, then $W_g^{1,p}(M,N) \neq \emptyset$.

In general, B. White [W] showed that $W_g^{1,p}(M,N) \neq \emptyset$ if and only if g has a continuous extension on $M \cup M^{[p]}$ where $M^{[p]}$ is a $[p]$-dimensional skeleton of M. The estimate (1.3) may not be valid, however, in this general situation.

If under certain topological conditions that the space $C_g^0(M,N)$ (continuous maps with trace g) is not empty, then one is interested in the following spaces:

$$\begin{aligned} H_s^{1,p}(M,N) &\equiv \text{ the strong closure of } C_g^0(M,N) \text{ in } W_g^{1,p}(M,N), \\ H_w^{1,p}(M,N) &\equiv \text{ the weak closure of } C_g^0(M,N) \text{ in } W_g^{1,p}(M,N). \end{aligned}$$

One deduces from the definitions above that

$$H_s^{1,p}(M,N) \subseteq H_w^{1,p}(M,N) \subseteq W_g^{1,p}(M,N). \tag{1.4}$$

We also define $R_g^{1,p}(M,N)$ to be the subset of $W_g^{1,p}(M,N)$ consisting of all such maps u that are smooth away from a $(n-[p]-1)$-dimensional skeleton of M. It follows from the proof of [HL, §6] that $R_g^{1,p}(M,N)$ is dense in $W_g^{1,p}(M,N)$ in the strong topology (cf. [CG]). Later, Bethuel [B] showed that $R_g^{1,p}(M,N)$ is always dense in $W_g^{1,p}(M,N)$ without the assumption of $[p-1]$-simply connectness of N. On the other hand, we have the following so-called "Gap-phenomena":

Theorem [HL2] *There are smooth maps g from $\mathbb{S}^2 \equiv \partial B^3$ into $\mathbb{S}^2$ of degree zero such that*

$$\min_{W^{1,2}(B^3,\mathbb{S}^2)} \left\{ \int_{B^3} |\nabla u|^2(x)dx\colon\ u|_{\partial B^3} = g \right\}$$
$$< \inf_{H_s^{1,2}(B^3,\mathbb{S}^2)} \left\{ \int_{B^3} |\nabla u|^2(x)dx\colon\ u|_{\partial B^3} = g \right\}.$$

This gap-phenomena implies, in particular, that $H_s^{1,2}(B^3,\mathbb{S}^2) \neq W_g^{1,2}(B^3,\mathbb{S}^2)$. Some generalizations were made in [GMS2], but the optimal result was shown by Bethuel in [B].

Theorem [B] *$C_g^0(M,N)$ is dense in $W_g^{1,p}(M,N)$ with respect to the strong topology on $W_g^{1,p}(M,N)$ if and only if $\pi_{[p]}(N) = 0$.*

As an easy consequence, one deduces for p-energy functional that no gap-phenomena for any $g \in C^0(\partial M, N)$ if and only if $\pi_{[p]}(N) = 0$.

This leads us to ask the *second question*: Is

$$\inf \left\{ \int_M |\nabla u|^p(x)dx, u \in H_s^{1,p}(M,N) \text{ with } u|_{\partial M} = g \right\}$$

achieved? (cf. [HL2]).

Similar questions were also posed by Schoen-Uhlenbeck and Schoen-Yau before the work [HL2] for maps in a homotopy class. Indeed, in the case $\partial M = \emptyset$, it was shown in [SU] that any map $f \in W^{1,p}(M,N)$ induces $f^{\#}\colon\ H^k(N,R) \to H^k(M,R)$, for any $0 \leq k \leq [p-1]$, a homomorphism between cohomology classes. We also note that Burstall [Bu] proved $W^{1,2}(M,N)$ maps induce conjugate classes of homomorphism $\pi_1(M) \to \pi_1(N)$, and Schoen-Yau [SY] showed one can define conjugate classes of $\pi_{n-1}(M) \to \pi_{n-1}(N)$ for $W^{1,n}(M,N)$ maps. In [W], White established the following results:

Theorem [W].
(i) Let $d = \begin{cases} [p] & \text{if } p \text{ is not an integer} \\ p-1 & \text{if } p \text{ is an integer,} \end{cases}$ *then each* $f \in H^{1,p}_w(M,N)$ *has a well-defined d-homotopy type which is preserved under weak convergence;*

(ii) If $d = [p-1]$*, then each map* $f \in W^{1,p}(M,N)$ *has a well-defined d-homotopy type and it is preserved under weak-convergence of maps;*

(iii) Each map $f \in H^{1,p}_s(M,N)$ *has a well-defined* $[p]$*-homotopy type that is preserved under the strong convergence of maps.*

Examples show also that above results are optimal in general. This indicates also the answer to the second question may be rather difficult. Indeed there is very little progress being made toward the solution of the second question.

For the special case of maps from B^3 into $\mathbb{S}^2$. Bethuel-Brezis-Coron [BBC] introduced the so-called *relaxed-energy*:

$$F(u) \equiv E(u) + 8\pi L(u), \tag{1.5}$$

where $E(u) = \int_{B^3} |\nabla u|^2(x)dx$,

$$L(u) = \frac{1}{4\pi} \sup_{\substack{\xi:\, \mathbb{R}^3 \to R \\ |\nabla \xi| \le 1}} \left\{ \int_{B^3} D(u) \cdot \nabla \xi dx - \int_{\partial B^3} D(u) \cdot n \xi d\sigma \right\},$$

$D(u)$ is the dual vector of the 2-form $u^*\omega$, ω is the volume form on $\mathbb{S}^2$ (cf. [BBC] for a geometrical interpolation of $L(u)$).

They showed that

a) $$|L(u) - L(v)| \le C_0 \|\nabla u - \nabla v\|_{L^2(B^3)} (\|\nabla u\|_{L^2(B^3)} + \|\nabla v\|_{L^2(B^3)}),$$

for all $u, v \in W^{1,2}_g(B^3, \mathbb{S}^2)$;

b) $\inf_{u \in H^{1,2}_s(B^3,\mathbb{S}^2)} \{E(u) \colon u|_{\partial B^3} = g\} = \inf_{u \in W^{1,2}_g(B^3,\mathbb{S}^2)} F(u)$, and $L(u) = 0$ if and only if $u \in H^{1,2}_s(B^3, \mathbb{S}^2)$;

c) $F(\cdot)$ is sequentially lower-semicontinuous with respect to weak convergence of maps in $W_g^{1,2}(B^2, \mathbb{S}^2)$.

¿From c) one sees that $\inf\{F(u)\colon u \in W_g^{1,2}(B^3, i\mathbb{S}^2)\}$ is achieved. Moreover, the resulting maps are not, in general, energy minimizing harmonic maps, (cf. [BBC]) though they will always be weakly harmonic maps.

In a series of very general works, Giaquinta-Modica-Souček studied an interesting object so-called *Cartesian current*. As an application, they deduced the above result of [BBC]. Moreover, they showed that if $u \in W_g^{1,2}(B^3, \mathbb{S}^2)$ minimizes $F(\cdot)$, then u is smooth away from a closed rectifiable subset of finite H^1-measure (see [GMS] for details). We should note that both arguments in [BBC] and in [GMS] seems to work only for the special case the target is $\mathbb{S}^2$ and the domain is B^3. The arguments in [GMS] may be generalizable to the case the target is 2-dimensional, but we do not know the exact statement. The *third natural question* is that: are minimizers of $F(u)$ continuous?

In [HLP], Hardt, Poon and the author showed that the answer to the third question is "No" for general maps in the axially symmetric class. Indeed, we showed when $F(\cdot)$ is restricted to the axially symmetric class of maps from B^3 into $\mathbb{S}^2$, the minimizers of $F(\cdot)$ may have isolated degree zero singularities. One does not know if this result remains true for minimizers of $F(\cdot)$ among all maps in $W_g^{1,2}(B^3, \mathbb{S}^2)$.

Another question closely related to our discussions is the following:

The Fourth Question: How to characterize those maps in $H_s^{1,p}(M, N)$?

Some partial progresses were made in [B2], [BCDH] and [Z].

Now we can state our main results

Theorem 1 *Suppose the space of continuous maps from M into N with trace g, $C_g^0(M, N)$, is not empty. Then any minimizing sequence for $E(v)$, $v \in C_g^0(M, N)$, $E(v) = \int_M |\nabla v|^2(x)dx$, contains a subsequence converging to a harmonic map u: $M \to N$, which is smooth away* *from a closed subset Σ of M with finite $(n-2)$-dimensional Hausdorff measure. Moreover, $\sum$ is rectifiable and the pair (ν, u) is stationary for the energy. Here ν is the corresponding defect measure lived on $\sum$. If, in addition, $\pi_2(N) = 0$, then $\nu \equiv 0$ and u is an absolutely energy minimizing map.*

Remark. The above theorem can be viewed as a generalization of the early stated results of [BBC] and [GMS]. The defect measure ν is defined in the next section and for the stationarity of the pair (ν, u) see [L]. With the same proof as that for Theorem 1, one can deduce the following results for $p =$ integer.

Theorem 2 *Suppose $C_g^0(M, N) \neq \emptyset$, then any minimizing sequence for $E_p(\cdot)$ over the space $C_g^0(M, mN)$ contains a subsequence converging to a p-harmonic map u (p is an integer) which is $C^{1,\alpha}$ smooth away from a closed subset $\sum$ of M of finite $(n-p)$-dimensional Hausdorff measure. Moreover, $\sum$ is H^{n-p}-rectifiable and the pair (ν, u) is stationary for the p-energy. If, in addition, $\pi_p(N) = 0$, then the defect measure $\nu \equiv 0$, and u is an $E_p(\cdot)$ minimizing map.*

Theorem 3 *Suppose $\partial M = \emptyset$ and g: $M \to N$ is a map in $H_w^{1,p}(M, N)$, p, is an integer. Then there is a pair (ν, u) stationary for the p-energy $E_p(\cdot)$. Here u is a $C^{1,\alpha}$, p-harmonic map away from a closed, $(n-p)$-rectifiable set $\sum \subseteq M$, and ν is the defect measure supported in $\sum$. Moreover, u has the same $(p-1)$ homotopy type as g. If, in addition, $\pi_p(N) = 0$, then, for any map $g \in C^0(M, N)$, there is a energy minimizing p-harmonic map, u, in the p-homotopy class determined by g. Furthermore u is smooth away from a closed subset of M with Hausdorff dimension $\leq n - [p] - 1$.*

Next we consider the case that p is not an integer. We have somewhat better results.

Theorem 4 *Under the same assumption as in Theorem 1. Let u_p be a weak limit of a minimizing sequence for $\{E_p(v)\colon\ v \in C_g^0(M,N)\}$ with $p \neq$ integer. Then $u_p \in H_s^{1,p}(M,N)$, and hence u_p achieves the value* $\inf\{E_p(v)\colon\ v \in H_s^{;1p}(M,N)\}$. *In particular, u_p is a stationary map. Furthermore, u_p is $C^{1,\alpha}$ away from a closed subset of Hausdorff dimension $\leq n-[p]-1$.*

Remark. a) If for some $k \in \{2,3,\ldots,n-1\}$ that $p < k$ and $k-p$ sufficiently small, then the singular set of u_p has Hausdorff dimension $\leq n-[p]-2$. This follows from the global energy bound and [HLW].

b) Note that there is no defect measure in case p is not an integer. Indeed such minimizing maps u_p obtained in Theorem 2 above form a compact family whenever their energies staying bounded.This result is also closely related to the work [B] concerning the density of Sobolev maps and its relation to the fundamental groups of the target N.

c) The regularity of limiting maps obtained in both Theorem 1 and Theorem 4 was an open issue in the previous papers [B] and [W].

When $\partial M = \emptyset$, we have also the following.

Theorem 5 *Let $g\colon\ M \to N$ be a continuous map between compact Riemannian manifolds without boundary. Let $[g] \subset C^0(M,N)$ be the set of all maps homotopic to g. Then for any minimizing sequence of maps in $[g]$ for the p-energy $E_p(\cdot)$, for some noninteger $p \in (1,n)$, there is a converging subsequence such that the limiting map u_p is a stationary p-harmonic map. The map u_p is $C^{1,\alpha}$ away from a closed subset $\sum$ with Hausdorff dimension $\leq n-[p]-1$. Moreover, the map u_p has the sum $[p]$-homotopy type as g.*

Finally the following results may be known to several experts, but no proof has been given. It is an easy consequence of Theorem 4.

Corollary. Let $g\colon\ \partial B^4 \equiv \mathbb{S}^3 \to \mathbb{S}^2$ be a smooth map with Hopf-invariant of g, $H(g) = 0$. There the $\inf\{E_p(u)\colon\ u \in C_g^0(B^4,\mathbb{S}^2)\}$ is achieved by a map $u_p \in H_s^{1,p}(B^4,\mathbb{S}^2)$, for $3 < p < 4$.

Moreover, u_p is a stationary p-harmonic map which is smooth except at most finitely many points $\{x_1, \ldots, x_N\} \subset B^4$ such that $H(u_p|_{\partial B_r(x_i)}) = 0$, for $i = 1, \ldots, N$, and for all sufficiently small $r > 0$.

2 The Ideas of Proofs

We shall sketch the proof of the Theorem 1. Theorem 2 can be shown in the same way, and Theorem 3 follows the corresponding statements in Theorem 2. For the proof of Theorem 4, we shall only point out why $p \neq$ integer is different from the case that p is an integer. In fact, the reset of arguments are again similar to that in the proof of Theorem 1.

To show Theorem 1, we let $u_i \in C_g^0(M, N)$ be a minimizing sequence for $E(\cdot)$ over $C_g^0(M, N)$. Consider a sequence of Radon measures

$$\mu_i = |\nabla u_i|^2(x)dx, \qquad i = 1, 2, \ldots .$$

We may assume that $u_i \rightharpoonup u$ in $W_g^{1,2}(M, N)$ weakly, and $\mu_i \rightharpoonup \mu = |\nabla u|^2 dx + \nu$ (as Radon measures), here $\nu \geq 0$ is also a Radon measure,which will be called defect measure.

Lemma 1 (Monotonicity) *For $x \in M$, $0 < r < d_x \equiv dist(x, \partial M)$, the function $\frac{\mu(B(x)r))}{r^{n-2}}$ is monotone nondecreasing in r.*

Remark. There is also a boundary version .

Proof.

$$\begin{aligned}
\frac{\mu(B_{\rho+\delta}(x)) - \mu(B_\rho(x))}{\delta} &= \lim_i \int_{B_{\rho+\delta}\setminus B_\rho(x)} |\nabla u_i|^2 dy \cdot \frac{1}{\delta} \\
&= \frac{1}{\delta} \lim_i \int_\rho^{\rho+\delta} \left(\int_{\partial B_r(x)} |\nabla u_i|^2 d\sigma \right) dr \\
&\leq \frac{1}{\delta} \frac{\lim}{i} \int_\rho^{\rho+\delta} \left(\frac{n-2}{r} \int_{B_r(x)} |\nabla u_i^*|^2 dy \right) dr
\end{aligned}$$

$$\leq \frac{n-2}{\rho+\delta}\int_{\rho}^{\rho+\delta}\left(\frac{\lim}{i}\int_{B_r(x)}|\nabla u_i^*|^2 dy\right)dr$$
$$\leq \frac{n-2}{\rho+\delta}\frac{1}{\delta}\int_{\rho}^{\rho+\delta}\mu(B_r)dr.$$

Here u^* is the homogeneous degree zero extension of u on $\partial B_r(x)$ into $B_r(x)$. The last inequality is valid because one can easily replace u_i^* by $\bar{u}_i$ which is equal to u_i^* on $B_r(x)\backslash B_{r/2}(x)$ and which is continuous inside $B_r(x)$. Moreover, one may choose $\bar{u}_i$ so that

$$\int_{B_r(x)}|\nabla \bar{u}_i|^2 dy \leq \frac{1}{i}+\int_{B_r(x)}|\nabla u_i^*|^2(y)dy, \qquad \text{for } i=1,2,\dots .$$

The conclusion of the Lemma 1 follows as usual.

Lemma 2 (small energy regularity) *There is an $\varepsilon_0>0$ such that $\frac{\mu(B_r(x))}{r^{n-2}}\leq \varepsilon_0$ implies that u is smooth in $B_{r/2}(x)$ and $\nu\equiv 0$ on $B_{r/2}(x)$.*

The proof of this lemma is somewhat long. It is done by an inductive argument. Starting with $n=\dim M=3$. Here one uses an argument of Almgren-Lieb [AL]. Indeed, by choosing a suitable $\rho\in\left(\frac{r}{2},r\right)$ so that

$$\int_{\partial B_\rho(x)}|\nabla u_i|^2 d\sigma \leq \varepsilon_0 \qquad \text{(and } \varepsilon_0 \text{ sufficiently small)},$$

then $\bar{u}_i$ a map that minimizes $E(\cdot)$ over $B_\rho(x)$ with $\bar{u}_i=u$ on $\partial B_\rho(x)$ is smooth inside $B_\rho(x)$ (cf. [AL]). Therefore,

$$\mu(B(x,\rho))=\lim_i\int_{B_\rho(x)}|\nabla u_i|^2 dy=\int_{B_\rho(x)}|\nabla \bar{u}|^2 dy.$$

Here $\bar{u}=\lim_i \bar{u}_i$ (one may assume this limit exists). Thus $\nu\equiv 0$ and $u=\bar{u}$ on B_ρ. This proves lemma for $n=3$.

When $n\geq 4$ one needs the following:

Sublemma (Schoen-Uhlenbeck type Lemma)

$$\frac{\mu(B(x,r))}{r^{n-2}} \leq \theta \frac{\mu(B(x,2r))}{(2r)^{n-2}} + C(\theta) \frac{\int_{B_{2r}(x)} |u - \bar{u}_{2r}|^2 dy}{r^n},$$

for any $\theta \in (0, 1/2)$, whenever

$$\frac{\mu(B(x,2r))}{(2r)^{n-2}} \leq \varepsilon_1, \text{ for some positive } \varepsilon_1.$$

The proof of above lemma follows from a modified construction of Schoen-Uhlenbeck (cf. [SU, §4]). This lemma combines with usual arguments yield the following important estimate:

$$(*) \qquad E(u,r) \equiv \frac{\int_{B_r(x)} |\nabla u|^2 dy}{r^{n-2}} \leq C \fint_{B_{2r}(x)} |u - \bar{u}_{2r}|^2 dy.$$

Now the conclusions of Lemma 2 follows from $(*)$ and a blow-up argument in the usual regularity theory for harmonic maps.

In order to complete the proof of Theorem 1, we need the following.

Lemma 3 (Rectifibility Lemma) *Let* $\sum = \{x \in M\colon\ \Theta^{n-2}(\mu, x) \geq \varepsilon_1 > 0\}$, *then* $\sum$ *is rectifiable with respect to* H^{n-2}*-measure.*

Here $\Theta^{n-2}(\mu,x) = \lim\limits_{r \downarrow 0} \frac{\mu(B(x,r))}{r^{n-2}}$. By the monotonicity lemma we know $\Theta^{n-2}(\mu, x)$ exists for all $x \in M$ and is an upper-semicontinuous function of $x \in M$. Moreover, whenever $\Theta^{n-2}(\mu,x) > 0$, $\Theta^{n-2}(\mu,x) \geq \varepsilon_0$, via the small energy regularity lemma. Here we may assume $\varepsilon_0 > \varepsilon_1 > 0$. We note that the conclusion of Lemma 3 follows from a very deep theorem of D. Preiss [P]. The latter also based on earlier works of J. Marstrand and P. Mattila (see references in [P]).

But here we can give a direct, simple, self-contained proof of this result. It can be divided into three steps.

Step I. Existence of weak-tangent $(n-2)$-planes for ν - a.e. x.

That is for ν - a.e. $x \in M$, and for any $\delta > 0$ there is a $\delta_x > 0$, such that, if $0 < r < \delta_x$, then there is $(n-2)$-plane V_r passes through x such that

$$\nu(B(x,r) - V_r^\delta) \leq \varepsilon(r) \to 0, \text{ as } r \to 0^+.$$

Here V_r^δ is the δ-neighborhood of V_r. The proof of this statement is to make a good use of the monotonicity lemma.

Step II. Suppose $E \subset \sum$ is purely unrectifiable, then $H^{n-2}(P_V(E)) = 0$ for any $V \in GL(n, n-2)$. Here P_V denotes the orthogonal projection onto the $(n-2)$-plane V in $\mathbb{R}^n$.

The proof of above statement is to use the weak-tangent property of $\sum$ and a covering argument which is somewhat standard in the geometric measure theory.

Step III. $\overline{\lim\limits_{r \downarrow 0}} \sup\limits_{v \in GL(n,n-2)} \frac{H^{n-2}(P_V(E \cap B(x,r)))}{r^{n-2}} > 0$ forν - a.e. $x \in E$.

The conclusion of Lemma 3 follows form Step II and III combined with the structure theorem [F].

The proof of Step III is based on energy comparison argument. Here one needs to rule out the so-called "Bar-Bell" picture in the construction of [HL] for the "gap-phenomena". It is the same construction but for exactly opposite purpose as we are now looking for minimizers among continuous maps.

Finally we want to apply the following result of Marstrand in order to show $\nu \equiv 0$ in the statements of Theorem 4. Also the compactness of such p-energy minimizers follows.

Lemma 4 *If $\mu \geq 0$ is a Radon measure with $\frac{\mu(B(x,r))}{r^{n-p}}$ monotone nondecreasing in r, and if for μ - a.e. $x \in spt\mu$, $\Theta^{n-p}(\mu, x) \geq \varepsilon_0$ for a positive number ε_0, then $n-p$ is an integer. We refer the readers to [L] for more detailed proofs.*

Acknowledgement

The author wishes to thank Professor P. Bolley and Professor F. Nicoleau for the invitation to the Saint Jean de Monts 1996, and for their warm hospitality. The research is also partially supported by a NSF grant.

References

[AL] F. Almgren, and E. Lieb, *Singularities of energy minimizing maps from the ball to the sphere: examples, counterexamples, and bounds.* Annals of Math. (2) **128**, 1988, no. 3, 483–530.

[B] F. Bethuel, *The approximation problem for Sobolev maps between two manifolds*, Acta Math. **167**, 1991, 153–206.

[B2] ______________, *A Characterization of maps in $H^1(B^3, \mathbb{S}^2)$ which can be approximated by smooth maps*, Ann. Inst. H. Poincaré, Anal. Nonlineaire **7**, 1990, 4, 269–286.

[BCDH] F. Bethuel, J. M. Coron, F. Demengel, and F. Helein, *A cohomological criterion for density of smooth maps in Sobolev spaces between two manifolds*, *Nematics* ______________, ed. by J. M. Coron et al., NATO ASI Series, Kluwer Academic Publishers, 1990.

[Bu] F. Burstal, *Harmonic maps of finite energy from noncompact manifolds*, J. London Math. Soc. **30**, 1984, 361–370.

[BBC] F. Bethuel, H. Brezis, and J. M. Coron, *Relaxed energies for harmonic maps*, *Variational Methods*, 1988, 37–52, Birkhauser, Boston, MA, 1990.

[CG] J. M. Coron, and R. Gulliver, *Minimizing p-harmonic maps into spheres*, J. Reine Angrew Math. **401**, 1989, 82–100.

[GMS] M. Giaquinta, L. Modica, and J. Souček, *The Dirichlet energy of mappings with values into the sphere*, Manuscripta Math. **65**, 1989, 4, 489–507.

[GMS2] ___________, *The gap phenomenon for variational integrals in Sobolev spaces*, preprint 1991.

[HL] R. Hardt, and F. H. Lin, *Mapping minimizing the L^p-norm of the gradient*, Comm. Pure and Appl. Math. **40**, 1987, 555-588.

[HL2] ___________, *A remark on H^1-mappings*, Manuscripta Math. **56**, 1986, no. 1, 1–10.

[HLP] R. Hardt, F. H. Lin, and C. C. Poon, *Axially symmetric harmonic maps minimizing a relaxed energy*, Comm. Pure and Appl. Math. **45**, 1992, 417–459.

[L] F. H. Lin, *Rectifiability of defect measures, fundamental groups and density of Sobolev maps*, preprint.

[P] D. Preiss, *Geometry of measures in $\mathbb{R}^n$: distribution, rectifiability, and densities*, Ann. of Math. (2) **125**, 1987, 537–643.

[SU] R. Schoen, and K. Uhlenbeck, *A regularity theory for harmonic maps.* J. Diff. Geom. **17**, 1982, 307–335.

[SU2] ___________, *Approximation theorems for Sobolev mappings*, preprint, 1984.

[SY] R. Schoen, and S. T. Yau, *The existence of incompressible minimal surfaces and topology of three-dimensional manifolds with non-negative scalar curvature*, Ann. of Math. **110**, 1979, 127–142.

[W] B. White, *Homotopy classes in Sobolev spaces and existence of energy mminimizing maps*, Acta Math. **160**, 1988, 1–17.