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THE SOLVABILITY OF NON L^2 SOLVABLE OPERATORS

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1. Introduction

Lerner proved in [4] that there are first order pseudodifferential operators of principal type satisfying condition (Ψ), that are not solvable in L^2 in any neighborhood of the origin. This was quite unexpected, since for first order differential operators of principal type, condition (Ψ) is equivalent to local L^2 solvability.

In this paper, we shall show that the counterexamples in [4] are locally solvable in C^∞, and that we lose at most one derivative in the estimate for the adjoint operators. In some cases we only lose ε derivatives in the estimate, for any $\varepsilon > 0$.

By local solvability in L^2 we mean that the equation $Pu = f$ has a local solution $u \in L^2(\mathbf{R}^n)$ for any $f \in L^2(\mathbf{R}^n)$ satisfying a finite number of compatibility conditions. We say that P is locally solvable in C^∞ if the equation has a solution $u \in \mathcal{D}'$ for any $f \in C^\infty$ satisfying a finite number of compatibility conditions. Recall that an operator is of principal type if the Hamilton field H_p of the principal symbol p is independent of the Liouville vector field.

Condition (Ψ) means that the imaginary part of the principal symbol does not change sign from $-$ to $+$ along the oriented bicharacteristics of the real part, see Definition 26.4.6 in [2]. This condition is invariant under multiplication of the principal symbol by non-vanishing factors.

It was conjectured by Nirenberg and Treves [5] that condition (Ψ) was equivalent to local solvability for operators of principal type, and they proved this in several cases. The necessity of (Ψ) for local solvability in the C^∞ category was proved by Moyers in two dimensions and by Hörmander in general, see Corollary 26.4.8 in [2]. In the analytic category, the sufficiency of condition (Ψ) for solvability of microdifferential operators acting on microfunctions was proved by Trépreau [6]. The sufficiency of (Ψ) for local L^2 solvability for first order pseudodifferential operators in two dimensions, was proved by Lerner [3].

For differential operators, condition (Ψ) is equivalent to condition (P), which rules out any sign changes of the imaginary part of the principal symbol along the bicharacteristics of the real part. The sufficiency of (P) for local L^2 solvability for first order pseudodifferential operators was proved by Nirenberg and Treves [5] in the case when the principal symbol is real analytic, and by Beals and Fefferman [1] in the general case.

2. Statement of results

We shall consider the following type of operators, which includes the operators Lerner used in his counter-examples. First, let $(t,x) \in \mathbf{R} \times \mathbf{R}^n$, $n \geq 2$, and

$$P = D_t + i \sum_{\nu \in \mathbf{Z}_+} Q_\nu(t, x_1, D_x) + R(t, x, D_x) \tag{2.1}$$

where $R(t,x,D_x) \in C^\infty(\mathbf{R}, \Psi^0_{1,0}(T^*\mathbf{R}^n))$ and $\sum_\nu Q_\nu(t,x_1,D_x) \in C^\infty(\mathbf{R}, \Psi^1_{1,0})$ is on the form

$$Q_\nu(t, x_1, D_x) = \alpha_\nu(t)(D_{x_1} + H(t)\nu^k W(\nu^k x_1))\Psi_\nu(D_x), \qquad \nu \in \mathbf{Z}_+. \tag{2.2}$$

Here $0 \le \alpha_\nu(t) \in C^\infty(\mathbf{R})$ uniformly, such that $0 \notin \operatorname{supp}\alpha_\nu$ and $\alpha_\nu(t)H(t)$ is non-decreasing with $H(t)$ the Heaviside function, $0 \le W(x_1) \in C^\infty(\mathbf{R})$ and $k > 0$. We also have $0 \le \Psi_\nu(\xi) \in S^0_{1,0}(T^*\mathbf{R}^n)$ uniformly, having non-overlapping interiors of the supports and $0 < c \le |\xi|2^{-\nu} \le C$ in $\operatorname{supp}\Psi_\nu$. Since $0 \notin \operatorname{supp}\alpha_\nu$ we may write $\alpha_\nu(t)H(t) \equiv \alpha_\nu(t)\beta_\nu(t)$, where $\beta_\nu(t) \in C^\infty$ (but not uniformly) such that $0 \le \beta_\nu(t) \le 1$ and $0 \le \partial_t\beta_\nu$. We find that $\sum_\nu \nu^k W(\nu^k x_1)\Psi_\nu(D_x) \in C^\infty(\mathbf{R}, \Psi^\varepsilon_{1,0})$, for any $\varepsilon > 0$. Since $0 \le \alpha_\nu(t)$ and $W(\nu^k x_1)\Psi_\nu(\xi) \ge 0$, it is clear that P satisfies condition (Ψ^*), i. e., the adjoint P^* satisfies condition (Ψ). In what follows, we shall suppress the t dependence and write S^m instead of $C^\infty(\mathbf{R}, S^m)$ for example. We shall use the classical calculus of pseudo-differential operators, but with the general metrics and weights of the Weyl calculus. For notation and calculus results, see chapter 18 in

We define the norms

$$\|u\|^2_{(s,k)} = \int |\hat{u}(\xi)|^2 \langle\xi\rangle^{2s}(\log\langle\xi\rangle + 1)^{2k}\, d\xi \qquad s,\ k \in \mathbf{R}, \tag{2.3}$$

where $\langle\xi\rangle^2 = 1 + |\xi|^2$. Then $\|u\|_{(s,0)} \cong \|u\|_{(s)}$, the usual Sobolev norm, and $\forall\, s,\ k \in \mathbf{R}$ we have

$$c_{k,\varepsilon}\|u\|_{(s-\varepsilon)} \le \|u\|_{(s,k)} \le C_{k,\varepsilon}\|u\|_{(s+\varepsilon)} \quad \forall\, \varepsilon > 0. \tag{2.4}$$

We find that $\|u\|_{(s,k)}$ is equivalent to $\sum_\nu \langle\xi_\nu\rangle^{2s}(\log\langle\xi_\nu\rangle + 1)^{2k}\|\psi_\nu(D_x)u\|^2$ if $\{\,\psi_\nu(\xi)\,\}_\nu$ is a partition of unity: $\sum_\nu |\psi_\nu|^2 = 1$ such that $\langle\xi\rangle \approx \langle\xi_\nu\rangle$ only varies with a fixed factor in $\operatorname{supp}\psi_\nu$.

THEOREM 2.1. Let P be given by (2.1). Then, for any $s \in \mathbf{R}$ there exists positive T_s and C_s such that

$$\int \|u\|^2_{(s)}(t)\, dt \le C_s T^2 \int \|Pu\|^2_{(s,2k)}(t)\, dt \tag{2.5}$$

if $u \in \mathcal{S}$ has support where $|t| \le T \le T_s$.

Thus, we obtain for any $s \in \mathbf{R}$ that

$$\int \|u\|^2_{(s)}(t)\, dt \le C_{s,\varepsilon} T^2 \int \|Pu\|^2_{(s+\varepsilon)}\, ds \qquad \forall\, \varepsilon > 0 \tag{2.6}$$

if $u \in \mathcal{S}$ has support where $|t| \le T \le T_s$. This shows that P^* is locally solvable in C^∞, with loss of ε derivatives, $\forall\, \varepsilon > 0$.

We shall also consider the following operators, which includes the operators Lerner used in his counter-example with homogeneous symbols. Let

$$P = D_t + i\sum_{\nu\in J} Q_\nu(t,x,D_x) + R(t,x,D_x) \tag{2.7}$$

where J is a subset of $\mathbf{Z}_+$ and $\sum_\nu Q_\nu(t,x,D_x) \in \Psi^1_{1,0}$ is given by

$$Q_\nu(t,x,D_x) = \alpha_\nu(t)C(D_x)\chi_\nu(x_2)(D_{x_1} + H(t)\nu^k W(\nu^k x_1)2^{-\nu}D_{x_2}) \qquad \nu \in J. \tag{2.8}$$

Here we have the same conditions on α_ν, W and R as before. Also, $0 \le C(\xi)$ is homogeneous, supported where $|\xi_1| \le C\xi_2$ and $0 \le \chi_\nu(x_2) \in S(1, dx_2^2)$ uniformly with non-overlapping supports. In fact, there exists a function $\mu(\nu)$ on $\mathbf{Z}_+$ such that $\mu(\nu) \le C_N\nu^N$, for some $N > 0$, and there exists $\tilde{\chi}_\nu \in S(1, \mu^2(\nu)dx_2^2)$ uniformly, with disjoint supports such that $0 \le \tilde{\chi}_\nu(x_2) \le 1$ and $\tilde{\chi}_\nu = 1$ on $\operatorname{supp}\chi_\nu$. As before, we find that P satisfies condition (Ψ^*).

THEOREM 2.2. Let P be given in (2.7). Then, for every $s \in \mathbf{R}$ we find $T_s > 0$ and $C_s > 0$ such that

$$\int \|u\|^2_{(s)}(t)\,dt \le C_s T^2 \int \|Pu\|^2_{(s+1)}(t)\,dt \qquad \forall\, s \tag{2.9}$$

if $u \in \mathcal{S}$ has support where $|t| \le T \le T_s$.

Thus P^* is locally solvable in C^∞, with loss of one derivative. The theorems are going to be proved in the next sections.

3. Proof of Theorem 2.1

Clearly, by conjugating with $\langle D_x \rangle^s$ we may assume that $s = 0$, which only changes $R(t, x, D_x) \in \Psi^0_{1,0}$ (dependingly on s). Next, we shall eliminate $R(t, x, D_x)$. We choose $E_\pm(t, x, D_x) \in \Psi^0_{1,0}$ with principal symbols

$$e_\pm(t, x, \xi) = \exp(\pm \int_0^t iR(t, x, \xi)\,dt), \tag{3.1}$$

such that $E_- E_+ \cong E_+ E_- \cong \mathrm{Id}$ modulo $\Psi^{-\infty}$. Then by conjugating with $E_\pm$ we obtain $R \in \Psi^{-1}_{1,0}$, but this changes Q_ν into

$$Q_\nu(t, x, D_x) = \alpha_\nu(t)\left((D_{x_1} + H(t)\nu^k W(\nu^k x_1))\Psi_\nu(D_x) + \varrho_\nu(t, x, D_x)\right) \tag{3.2}$$

where $\{\, \varrho_\nu(t, x, \xi) \,\}_\nu \in S^0_{1,0}$. Since we may skip terms in Ψ^{-1} in P in the estimate (2.5), we may assume that $\operatorname{supp} \varrho_\nu \subseteq \operatorname{supp} \Psi_\nu$.

We shall localize in $S^0_{1/2,0}$ in order to separate the different Q_ν terms. Let $\{\, \phi_j(\xi) \,\}_j \in S^0_{1/2,0}$ be a partition of unity such that ϕ_j is supported where $|\xi - \xi_j| \le c\langle \xi_j \rangle^{1/2}$, and $\operatorname{supp} \phi_j$ is connected, $\forall\, j$. Let $J \subset \mathbf{Z}_+$ be the set of those j for which $\operatorname{supp} \phi_j$ intersects $\bigcap_\nu \complement \operatorname{supp} \Psi_\nu$. Since the principal symbol of $\sum_\nu Q_\nu \in \Psi^1_{1,0}$ vanishes of infinite order somewhere in $\operatorname{supp} \phi_j$ when $j \in J$, and $\phi_j(\xi) \in S^0_{1/2,0}$, we find that

$$\phi_j(D_x) Pu = \phi_j(D_x) D_t u + R_j(t, x, D_x) u \tag{3.3}$$

with $\{\, R_j \,\}_{j \in J} \in \Psi^0_{1,0}$ (with values in ℓ^2). We have

$$\begin{aligned} \int \|\phi_j(D_x)u\|^2(t)\,dt &\le CT^2 \int \|D_t \phi_j(D_x) u\|^2(t)\,dt \\ &\le CT^2 \int \|\phi_j(D_x)Pu\|^2(t) + \|R_j u\|^2(t)\,dt \end{aligned} \tag{3.4}$$

for $j \in J$. Since $\sum_{j \in J} \|R_j u\|^2 \le C\|u\|^2$, we get the result for small enough T, providing that we also have an estimate for the other terms.

Thus we only have to consider the case when $\operatorname{supp} \phi_j$ does not intersect $\bigcap_\nu \complement \operatorname{supp} \Psi_\nu$, i. e. $j \notin J$. Since $\operatorname{supp} \phi_j$ is connected, we find that $\operatorname{supp} \phi_j$ is contained in the interior of $\operatorname{supp} \Psi_\nu$ for some unique $\nu = \nu_j$ when $j \notin J$. Observe that this gives $|\xi_j| \approx 2^{\nu_j}$ in $\operatorname{supp} \phi_j$. Clearly, since $\operatorname{supp} Q_\nu \subseteq \operatorname{supp} \Psi_\nu$ we have $P\phi_j(D_x)u = P_{\nu_j}\phi_j(D_x)u$ where we define

$$P_\nu = D_t + iQ_\nu(t, x_1, D_x). \tag{3.5}$$

Now we use the following

Lemma 3.1. Let P_ν be given by (3.5). Then we find

$$\int \|u\|^2(t)(\nu^{2k}\alpha_\nu(t) + 1)\,dt \le CT^2\nu^{4k} \int \|P_\nu u\|^2(t)(\nu^{2k}\alpha_\nu(t) + 1)^{-1}\,dt \tag{3.6}$$

uniformly in ν, if $u \in \mathcal{S}$ has support in $|t| \le T$, for T small enough.

By substituting $\phi_j(D_x)u$, taking $\nu = \nu_j$ in (3.6), and replacing P_{ν_j} by P, we obtain for $j \notin J$ that

$$(3.7)\quad \int \|\phi_j(D_x)u\|^2(t)\,dt \le CT^2\nu_j^{4k}\int \|P\phi_j(D_x)u\|^2(t)\,dt \le CT^2\nu_j^{4k}\int \|\phi_j(D_x)Pu\|^2(t) + \|[P,\phi_j(D_x)]u\|^2(t)\,dt.$$

Now $\left\{\nu_j^{2k}[P,\phi_j(D_x)]\right\}_{j\notin J} \in \Psi^{\varepsilon-1/2}_{1/2,0}$ with values in ℓ^2, $\forall\,\varepsilon > 0$. In fact, we find that $\sum_\nu \nu^k W(\nu^k x_1)\Psi_\nu(D_x) \in C^\infty(\mathbf{R},\Psi^\varepsilon_{1,0})$ and $\left\{\nu_j^{2k}\phi_j(\xi)\right\}_{j\notin J} \in S^\varepsilon_{1/2,0}$, $\forall\,\varepsilon > 0$, since $\phi_j(\xi)$ is supported where $|\xi| \approx 2^{\nu_j}$ when $j \notin J$. Thus by summing up (3.4) and (3.7) we obtain (2.5) for $s = 0$ and small enough T. This completes the proof of Theorem 2.1.

Proof. [Proof of Lemma 3.1] We may assume ν is fixed in what follows. In the proof, we are going to localize in $|\xi_1| \gtrless \nu^{2k}$. For that purpose we use the metric

$$(3.8)\qquad g_\nu = \nu^{2k}|dx|^2 + |d\xi|^2/(\nu^{4k}+\xi_1^2) \qquad \nu \in \mathbf{Z}_+$$

which is uniformly slowly varying, σ temperate and

$$(3.9)\qquad g_\nu/g_\nu^\sigma = h_\nu^2 = \nu^{2k}/(\nu^{4k}+\xi_1^2)$$

which makes $h_\nu^{-2} = |\xi_1|^2\nu^{-2k} + \nu^{2k} \ge 2|\xi_1|$. We find that $Q_\nu \in \operatorname{Op} S(h_\nu^{-2}, g_\nu)$ but $\nu^k W(\nu^k x_1) \in S(h_\nu^{-1}, g_\nu)$ uniformly.

Now we localize with $\chi_0(\xi_1) = \chi(\xi_1\nu^{-2k}) \in S(1,g_\nu)$ where $\chi \in C_0^\infty$ is equal to 1 near 0, and with $\chi_\pm(\xi_1) = H(\pm\xi_1)(1-\chi_0(\xi_1)) \in S(1,g_\nu)$ which has support where $\pm\xi_1 > c\nu^{2k}$ so that $\chi_0 + \chi_+ + \chi_- \equiv 1$. We also choose non-negative $\widetilde{\chi}_\pm(\xi_1)$ and $\widetilde{\chi}_0(\xi_1) \in S(1,g_\nu)$ such $\widetilde{\chi}_\pm\chi_\pm = \chi_\pm$ and $\widetilde{\chi}_0\chi_0 = \chi_0$. This can be done so that $\widetilde{\chi}_\pm$ have support where $\pm\xi_1 > c_0\nu^{2k}$, $c_0 > 0$, and $\widetilde{\chi}_0$ has support where $|\xi_1| \le C\nu^{2k}$.

First we estimate the $\chi_\pm(D_{x_1})u$ terms by Lemma 5.1 with the operator

$$(3.10)\qquad P_\pm = D_t + Q_\nu\widetilde{\chi}_\pm(D_{x_1}),$$

where

$$(3.11)\qquad \pm\operatorname{Re} Q_\nu\widetilde{\chi}_\pm(D_{x_1}) \ge \mp C \qquad \text{on } u \in \mathcal{S},$$

by the Fefferman–Phong inequality, where $\operatorname{Re} F = (F+F^*)/2$. In fact, the symbol of

$$(3.12)\qquad \pm\alpha_\nu(t)\operatorname{Re}\left(D_{x_1} + H(t)\nu^k W(\nu^k x_1)\right)\Psi_\nu(D_x)\widetilde{\chi}_\pm(D_{x_1})$$

is bounded from below, modulo terms in $S(1,g_\nu)$. Thus Lemma 5.1 gives (after changing t to $-t$ for P_-)

$$(3.13)\qquad \int \|u\|^2(t)\,dt \le CT^2\int \|P_\pm u\|^2(t)\,dt$$

if $u \in \mathcal{S}$ is supported where $|t| \le T$ and T is small enough. Now, by substituting $\chi_\pm(D_{x_1})u$ into (3.13) and using that $P_\pm\chi_\pm(D_{x_1}) = P_\nu\chi_\pm(D_{x_1})$ and that $[P_\nu,\chi_\pm(D_{x_1})] \in \operatorname{Op} S(1,g_\nu)$ is uniformly L^2 bounded, we find

$$(3.14)\qquad \int \|\chi_\pm(D_{x_1})u\|^2(t)\,dt \le C_0T^2\int \|P_\nu u\|^2(t) + \|u\|^2(t)\,dt$$

if $u \in \mathcal{S}$ is supported where $|t| \le T$ and T is small enough.

Next, we shall estimate $\|\chi_0(D_{x_1})u\|^2$. Let

(3.15)

$$B_\nu = D_{x_1}\Psi_\nu(D_x)\widetilde{\chi}_0(D_{x_1}) + \beta_\nu(t)\left(\nu^k W(\nu^k x_1)\Psi_\nu(D_x)\widetilde{\chi}_0(D_{x_1}) + \varrho\right) \in \operatorname{Op} S(h_\nu^{-1}, g_\nu),$$

where $\varrho > 0$. Here $\beta_\nu \in C^\infty$ such that $0 \le \beta_\nu(t) \le 1$, $0 \le \partial_t \beta_\nu$ and $\alpha_\nu(t)H(t) \equiv \alpha_\nu(t)\beta_\nu(t)$. Since $\nu^k W(\nu^k x_1)\Psi_\nu(D_x)\widetilde{\chi}_0(D_{x_1}) \in \operatorname{Op} S(h_\nu^{-1}, g_\nu)$ has positive principal symbol, we find

$$\partial_t B_\nu = \partial_t \beta_\nu(t)\left(\nu^k W(\nu^k x_1)\Psi_\nu(D_x)\widetilde{\chi}_0(D_{x_1}) + \varrho\right) \ge 0 \tag{3.16}$$

for large enough ϱ. We also find $B_\nu \in \operatorname{Op} S(\nu^{2k}, g_\nu)$ uniformly, thus $\|B_\nu\| \le C\nu^{2k}$. Applying Lemma 5.2 on $\chi_0(D_{x_1})u$, with $P_0 = D_t + \alpha_\nu(t)(B_\nu + r_\nu)$, $r_\nu = \varrho_\nu(t,x,D_x)\widetilde{\chi}_0(D_{x_1}) - \beta_\nu(t)\varrho$ and $M = C\nu^{2k}$, we find

(3.17)
$$\int \|\chi_0(D_{x_1})u\|^2(t)(\nu^{2k}\alpha_\nu(t)+1)\,dt \le C_1\nu^{4k}T^2 \int \|P_0\chi_0(D_{x_1})u\|^2(t)(\nu^{2k}\alpha_\nu(t)+1)^{-1}\,dt$$

if $u \in \mathcal{S}$ is supported where $|t| \le T$ and T is small enough. As before, we find $P_0\chi_0(D_{x_1}) = P_\nu\chi_0(D_{x_1})$ and we have $[P_\nu, \chi_0(D_{x_1})] = \alpha_\nu(t)f_\nu$, where $f_\nu \in \operatorname{Op} S(1, g_\nu)$ is uniformly L^2 bounded. Since

$$\nu^{4k}\alpha_\nu^2(t)/(\nu^{2k}\alpha_\nu(t)+1) \le \nu^{2k}\alpha_\nu(t) + 1, \tag{3.18}$$

we obtain

$$\begin{aligned}(3.19)\quad &\int \|\chi_0(D_x)u\|^2(t)(\nu^{2k}\alpha_\nu(t)+1)\,dt \\ &\le C_1T^2\left(\int \nu^{4k}\|P_\nu u\|^2(t)(\nu^{2k}\alpha_\nu(t)+1)^{-1}\,dt + \int \|u\|^2(t)(\nu^{2k}\alpha_\nu(t)+1)\,dt\right)\end{aligned}$$

if u is supported where $|t| \le T$ and T is small enough. Combining (3.14) and (3.19), we obtain (3.6) for small enough T. ∎

4. Proof of Theorem 2.2

First, we conjugate with $\langle D_x\rangle^{s+1/2}$ to reduce to the case $s = -1/2$ (this only changes $R(t,x,D_x)$ dependingly on s). We choose $E_\pm(t,x,D_x) \in \Psi^0_{1,0}$ with principal symbols

$$e_\pm(t,x,\xi) = \exp(\pm\int_0^t iR(t,x,\xi)\,dt), \tag{4.1}$$

such that $E_-E_+ \cong E_+E_- \cong \mathrm{Id}$ modulo $\Psi^{-\infty}$. As before, the calculus gives $R \in \Psi^{-1}_{1,0}$ for the new operator, but changes Q_ν into

(4.2)
$$Q_\nu(t,x,D_x) = \alpha_\nu(t)\left(C(D_x)\chi_\nu(x_2)(D_{x_1} + H(t)\nu^kW(\nu^kx_1)2^{-\nu}D_{x_2}) + \varrho_\nu(t,x,D_x)\right)$$

where $\varrho_\nu(t,x,\xi) \in S^0_{1,0}$ uniformly, with $\operatorname{supp}\varrho_\nu \subseteq \operatorname{supp}\chi_\nu$. Thus, we may assume $R \equiv 0$ since the term $CT\|Ru\|_{(1/2)}$ can be estimated by the left hand side of (2.9) for $s = -1/2$ and small enough T.

Next, we localize in x_2 to separate the different Q_ν terms. By assumption there exists $\widetilde{\chi}_\nu(x_2) \in S(1, \mu^2(\nu)dx_2^2)$ uniformly when $\nu \in J$, with disjoint supports, such that $0 \le \widetilde{\chi}_\nu(x_2) \le 1$ and $\widetilde{\chi}_\nu\chi_\nu = \chi_\nu$. We also localize in ξ: let $\{\psi_j(\xi)\}_j$ and $\{\phi_j(\xi)\}_j \in S^0_{1,0}$ (with values in ℓ^2) such that $\sum_j \psi_j(\xi)^2 = 1$, $\phi_j(\xi)$ and $\psi_j(\xi)$ are non-negative, $\phi_j\psi_j = \psi_j$ and ψ_j, ϕ_j are supported where $0 < c \le |\xi|2^{-\nu} \le C$. We may also assume that for some fixed $N > 0$ we have $\sum_{|j-k|\le N}\psi_k^2(\xi) \equiv 1$ on $\operatorname{supp}\psi_j$, $\forall j$.

Since $\widetilde{\chi}_\nu \in S(1, \mu^2(\nu)dx_2^2)$ we find that $\{\psi_j(\xi)\widetilde{\chi}_\nu(x_2)\}_{\nu,j}$ is not in a good symbol class. Therefore, we put

$$\widetilde{\chi}_{0j}(x_2) = 1 - \sum_{\substack{0<\nu\leq j^2 \\ \nu\in J}} \widetilde{\chi}_\nu(x_2). \tag{4.3}$$

Since ψ_j is supported where $|\xi| \approx 2^j$ and $\mu(\nu) \leq C_N\nu^N$ for some $N > 0$, it is easy to see that $\{\widetilde{\chi}_\nu(x_2)\psi_j(\xi)\}_{\substack{J\ni\nu\leq j^2 \\ j}}$ and $\{\widetilde{\chi}_{0j}(x_2)\psi_j(\xi)\}_j \in \Psi^0_{1,\varepsilon}$, $\forall\varepsilon > 0$. Let

$$\alpha_{\nu j}(t) = \sqrt{\alpha_\nu(t) + 2^{-j}} \qquad \forall\, j \in J, \quad \forall\, \nu, \tag{4.4}$$

in what follows. Now, we are going to use the following

Lemma 4.1. We find that

$$\begin{aligned} \int \sum_{\substack{J\ni\nu\leq j^2 \\ j}} \|\alpha_{\nu j}(t)\widetilde{\chi}_\nu(x_2)\psi_j(D_x)u\|^2(t) + \sum_j \|\widetilde{\chi}_{0j}(x_2)\psi_j(D_x)u\|^2(t)\,dt \\ \leq CT \int \sum_{\substack{J\ni\nu\leq j^2 \\ j}} \|\alpha_{\nu j}^{-1}(t)\widetilde{\chi}_\nu(x_2)\psi_j(D_x)Pu\|^2(t) \\ + \sum_j \|\widetilde{\chi}_{0j}(x_2)\psi_j(D_x)Pu\|^2(t) + \|u\|^2_{(-1/2)}(t)\,dt. \end{aligned} \tag{4.5}$$

if $u \in \mathcal{S}$ has support in $|t| \leq T$ for T small enough.

Since $2^{-j/2} \leq \alpha_{\nu j}$, $|\xi| \approx 2^j$ in $\operatorname{supp}\psi_j$, the supports of $\widetilde{\chi}_\nu$ are disjoint and $\sum_{J\ni\nu\leq j^2}\widetilde{\chi}_\nu + \widetilde{\chi}_{0j} \equiv 1$, $\forall\, j$, it is easy to see that the left hand side of (4.5) is greater that $c\int \|u\|^2_{(-1/2)}(t)\,dt$ for some $c > 0$, and the right hand side is less that $CT\int \|Pu\|^2_{(1/2)}(t) + \|u\|^2_{(-1/2)}(t)\,dt$. Thus (4.5) implies (2.9) for the case $s = -1/2$ for small T, and completes the proof of Theorem 2.2.

Proof. [Proof of Lemma 4.1] Since $\psi_j(1-\phi_j) \equiv 0$ $\forall\, j$, the calculus gives that we may replace P by $P_j = D_t + i\sum_{\nu\in J} Q_\nu\phi_j(D_x)$ for the terms containing the factor $\psi_j(D_x)$ in (4.5).

For the terms $\|\widetilde{\chi}_{0j}(x_2)\psi_j(D_x)u\|^2$ we use the fact that $\nu^k W(\nu^k x_1)2^{-\nu}D_{x_2}\phi_j(D_x) \in \Psi^{-\infty}$ uniformly when $(\log|\xi|)^2 \approx j^2 < \nu$. Thus we use Nirenberg–Treves estimate in [2, Theorem 26.8.1] with $B = D_{x_1}\phi_j(D_x)$ bounded, and $0 \leq A \in \Psi^0_{1,0}$ such that

$$A \cong \sum_{J\ni\nu>j^2} \alpha_\nu(t)C(D_x)\chi_\nu(x_2) \qquad \text{mod } \Psi^{-1}_{1,0}. \tag{4.6}$$

By perturbing this estimate with L^2 bounded operators, and substituting the term $\widetilde{\chi}_{0j}(x_2)\psi_j(D_x)u$, we find for small enough T that

$$\int \|\widetilde{\chi}_{0j}(x_2)\psi_j(D_x)u\|^2(t)\,dt \leq CT^2 \int \|\widetilde{P}_j\widetilde{\chi}_{0j}(x_2)\psi_j(D_x)u\|^2(t)\,dt \qquad \forall\, j \tag{4.7}$$

when $|t| \leq T$ in $\operatorname{supp} u$. Here

$$\begin{aligned} \widetilde{P}_j = D_t + i \sum_{J\ni\nu>j^2} \alpha_\nu(t)\left(C(D_x)\chi_\nu(x_2)D_{x_1} + \varrho_\nu(t,x,D_x)\right)\phi_j(D_x) \\ \cong D_t + i\sum_{J\ni\nu>j^2} Q_\nu\phi_j(D_x) \qquad \text{modulo } \Psi^{-\infty}. \end{aligned} \tag{4.8}$$

Thus $\widetilde{P}_j$ satisfies condition (P), i. e., the imaginary part of the principal symbol has no sign changes for fixed (x,ξ).

Since $\alpha_\nu \le C\alpha_{\nu j}$ and $\operatorname{supp}\varrho_\nu \subseteq \operatorname{supp}\chi_\nu$, the calculus gives that

$$\left\{ [\tilde{P}_j, \tilde{\chi}_{0j}(x_2)\psi_j(D_x)] \right\}_j \cong \left\{ \sum_{\nu > j^2} \alpha_{\nu j}(t) f_{\nu j}(x, D_x) \right\}_j \mod \Psi^{-1/2}_{1,\varepsilon} \tag{4.9}$$

where $\{ f_{\nu j} \}_{\nu j} \in \Psi^0_{1,0}$ with values in ℓ^2, and $\operatorname{supp} f_{\nu j} \subseteq \operatorname{supp}\chi_\nu \psi_j$. In order to estimate these terms we need the following

Lemma 4.2. If $\{ f_{\nu j}(x, D_x) \}_{\nu j} \in \Psi^0_{1,0}$ with values in ℓ^2, and $\operatorname{supp} f_{\nu j} \subseteq \operatorname{supp}\chi_\nu\psi_j$, $\forall\, \nu\, j$, then

$$\sum_{\substack{\nu \in J \\ j}} \|\alpha_{\nu j}(t) f_{\nu j}(x, D_x)u\|^2 \le C\Big(\sum_{\nu \le j^2} \|\alpha_{\nu j}(t)\tilde{\chi}_j(x_2)\psi_j(D_x)u\|^2 + \sum_j \|\tilde{\chi}_{0j}(x_2)\psi_j(D_x)u\| + \|u\|^2_{(-1/2)}\Big) \tag{4.10}$$

for $u \in \mathcal{S}$.

Since $\tilde{\chi}_{0j} \equiv 0$ on $\operatorname{supp}\chi_\nu$ when $J \ni \nu \le j^2$, we find that $\left\{ \tilde{\chi}_{0j}(x_2)\psi_j(D_x)(\tilde{P}_j - P_j) \right\}_j \in \Psi^{-\infty}$, where as before $P_j = D_t + i\sum_{\nu\in J} Q_\nu \phi_j(D_x) \in \Psi^1_{1,0}$. Thus we find

$$\int \sum_j \|\tilde{\chi}_{0j}(x_2)\psi_j(D_x)\tilde{P}_j u\|^2(t)\, dt \le CT \int \sum_j \|\tilde{\chi}_{0j}(x_2)\psi_j(D_x)P_j u\|^2(t) + \|u\|^2_{(-1/2)}(t)\, dt. \tag{4.11}$$

This gives the estimate (4.5) for the terms $\|\tilde{\chi}_{0j}(x_2)\psi_j(D_x)u\|^2$ for small T, providing we can estimate the other terms.

As before, we are going to use Lemma 5.2 with $a(t) = \alpha_\nu(t)$ and

$$B_t = \operatorname{Re} C(D_x)\chi_\nu(x_2)\Big(D_{x_1}\phi_j(D_x) + \beta_\nu(t)\big(\nu^k W(\nu^k x_1)2^{-\nu} D_{x_2}\phi_j(D_x) + \varrho\big)\Big), \tag{4.12}$$

where $\varrho > 0$. Here $\beta_\nu \in C^\infty$ such that $0 \le \beta_\nu(t) \le 1$, $0 \le \partial_t\beta_\nu$ and $\alpha_\nu(t)H(t) \equiv \alpha_\nu(t)\beta_\nu(t)$. We have $\|B_t\| \le C2^j$, $\partial_t B_t \ge 0$ for large ϱ and $R_t \in \Psi^0$. By substituting $\tilde{\chi}_\nu(x_2)\psi_j(D_x)u$ in this Lemma, we find for small T that

$$\int \|\tilde{\chi}_\nu(x_2)\psi_j(D_x)u\|^2(t)(2^j\alpha_\nu(t) + 1)\, dt \le CT^2 2^{2j} \int \|(D_t + iQ_\nu\phi_j(D_x))\tilde{\chi}_\nu(x_2)\psi_j(D_x)u\|^2(t)(2^j\alpha_\nu(t) + 1)^{-1}\, dt \tag{4.13}$$

when $J \ni \nu \le j^2$, providing $|t| \le T$ in $\operatorname{supp} u$. This is equivalent to

$$\int \|\alpha_{\nu j}(t)\tilde{\chi}_\nu(x_2)\psi_j(D_x)u\|^2(t)\, dt \le CT^2 \int \|\alpha_{\nu j}^{-1}(t)(D_t + iQ_\nu\phi_j(D_x))\tilde{\chi}_\nu(x_2)\psi_j(D_x)u\|^2(t)\, dt. \tag{4.14}$$

Now, it follows from the asymptotic expansion that

$$\{ [Q_\nu\phi_j(D_x), \tilde{\chi}_\nu(x_2)\psi_j(D_x)] \}_{\substack{J\ni\nu\le j^2 \\ j}} \cong \left\{ \alpha_\nu(t)\tilde{f}_{\nu j}(t, x, D_x) \right\}_{\substack{J\ni\nu\le j^2 \\ j}} \tag{4.15}$$

modulo $\Psi^{-1/2}_{1,\varepsilon}$, where $\{ \tilde{f}_{\nu j}(t,x,D_x) \}_{\nu j} \in \Psi^0_{1,0}$ with values in ℓ^2, $\operatorname{supp} \tilde{f}_{\nu j} \subseteq \operatorname{supp} \chi_\nu \psi_j$, $\forall t$. Thus, we may estimate the commutator terms by Lemma 4.2:

$$\sum_{\substack{J \ni \nu \le j^2 \\ j}} \|\alpha_{\nu j}(t) \tilde{f}_{\nu j}(t,x,D_x) u\|^2 \le C \left(\sum_{\nu \le j^2} \|\alpha_{\nu j} \tilde{\chi}_j \psi_j u\|^2 + \sum_j \|\tilde{\chi}_{0j} \psi_j u\|^2 + \|u\|^2_{(-1/2)} \right) \qquad \forall t. \tag{4.16}$$

Since the supports of $\tilde{\chi}_\nu$ are disjoint, and $\sum_{J \ni \mu \ne \nu} Q_\mu \phi_j(D_x) \in \Psi^1_{1,0}$ uniformly, we obtain that

$$\left\{ \tilde{\chi}_\nu(x_2) \psi_j(D_x) \sum_{J \ni \mu \ne \nu} Q_\mu \phi_j(D_x) \right\}_{\substack{J \ni \nu \le j^2 \\ j}} \in \Psi^{-\infty} \tag{4.17}$$

with values in ℓ^2. Thus we may replace $D_t + iQ_\nu \phi_j(D_x)$ by P_j in the estimate, which proves (4.5). ■

Proof. [Proof of Lemma 4.2] Since $\sum_{|j-k| \le N} \psi_k^2(\xi) \equiv 1$ on $\operatorname{supp} f_{\nu j}$ and $\{ f_{\nu j} \}_{\nu j} \in S^0_{1,0}$, we may use the calculus to write

$$\sum_{\nu,j} \|\alpha_{\nu j}(t) f_{\nu j}(x,D_x) u\|^2 \le \sum_{\substack{\nu,j \\ |k-j| \le N}} \|\alpha_{\nu j}(t) e_{\nu jk}(x,D_x) \psi_k(D_x) u\|^2 + C\|u\|^2_{(-1)}, \tag{4.18}$$

where $\{ e_{\nu jk} \}_{\nu jk} \in \Psi^0_{1,0}$ with values in ℓ^2, and $\operatorname{supp} e_{\nu jk} \subseteq \operatorname{supp} f_{\nu j} \psi_k$. Since $\tilde{\chi}_{0k} + \sum_{\mu \le k^2} \tilde{\chi}_\mu \equiv 1$, we find

$$\sum_{\substack{\nu,j \\ |k-j| \le N}} \|\alpha_{\nu j}(t) e_{\nu jk}(x,D_x) \psi_k(D_x) u\|^2 \le 2 \sum_{\substack{\nu,j \\ |k-j| \le N}} \|\alpha_{\nu j}(t) e_{\nu jk}(x,D_x) \tilde{\chi}_{0k}(x_2) \psi_k(D_x) u\|^2 + 2 \sum_{\substack{\nu,j \\ |k-j| \le N}} \left\| \alpha_{\nu j}(t) e_{\nu jk}(x,D_x) \sum_{\mu \le k^2} \tilde{\chi}_\mu(x_2) \psi_k(D_x) u \right\|^2 . \tag{4.19}$$

By summing up in j and ν we find

$$\sum_{\substack{\nu,j \\ |k-j| \le N}} \|\alpha_{\nu j}(t) e_{\nu jk}(x,D_x) \tilde{\chi}_{0k}(x_2) \psi_k(D_x) u\|^2 \le C_N \Big(\sum_k \|\tilde{\chi}_{0k}(x_2) \psi_k(D_x) u\|^2 + \|u\|^2_{(-1/2)} \Big), \tag{4.20}$$

since $\alpha_{\nu j} \le c$ and $\{ e_{\nu jk} \}_{\nu j} \in \Psi^0_{1,0}$ with values in ℓ^2, uniformly in k. Now $\alpha_{\nu j} \le C\alpha_{\nu k}$ when $|j-k| \le N$ which similarly gives by the calculus

$$\sum_{\substack{\nu,j \\ |k-j| \le N}} \left\| \alpha_{\nu j}(t) e_{\nu jk}(x,D_x) \sum_{\mu \le k^2} \tilde{\chi}_\mu(x_2) \psi_k(D_x) u \right\|^2 \le C \sum_{\mu \le k^2} \|\alpha_{\mu k}(t) \tilde{\chi}_\mu(x_2) \psi_k(D_x) u\|^2 + C\|u\|^2_{(-1/2)} \tag{4.21}$$

since $\operatorname{supp} e_{\mu jk} \subseteq \operatorname{supp} \chi_\mu \; \forall j, k$. ■

5. Some estimate lemmas

We assume that

$$P = D_t + iQ_t + R_t \tag{5.1}$$

where Q_t is a closed, densely defined operator on $L^2(\mathbf{R}^n)$ such that $\mathcal{S} \subset D(Q_t) \cap D(Q_t^*)$ $\forall\, t$, $t \mapsto \langle Q_t u, u\rangle$ is continous for $u \in \mathcal{S}$, and

$$\operatorname{Re} Q_t \geq -C_1 \qquad \text{on } \mathcal{S} \quad \forall\, t, \tag{5.2}$$

where $2 \operatorname{Re} Q_t = Q_t + Q_t^*$. We also assume that $\|R_t\| \leq C_0$ on $L^2(\mathbf{R}^n)$. Let $\|u\|$ be the L^2 norm of $u \in L^2(\mathbf{R}^n)$ and $\langle u, v\rangle$ the corresponding sesquilinear form.

Lemma 5.1. There exists $T_0 > 0$ and $C > 0$ such that

$$\int \|u\|^2(t) \leq CT^2 \int \|Pu\|^2(t)\, dt \tag{5.3}$$

if $u \in \mathcal{S}$ has support where $|t| \leq T \leq T_0$. Here T_0 and C only depend on C_0 and C_1.

Proof. We only need to prove the estimate (5.1) for $R_t \equiv 0$, since we may perturb it with L^2 bounded terms for small T. We find

$$\langle Q_t u, u\rangle \geq -C_1 \|u\|^2 \qquad \forall\, t \tag{5.4}$$

when $u \in \mathcal{S}$. Since $iP = \partial_t - Q_t$, this gives

$$\begin{aligned} \|u\|^2(t) &= -\int_t^T 2 \operatorname{Re}\langle \partial_t u, u\rangle(t)\, dt \\ &= -\int_t^T 2 \operatorname{Re}\langle iPu, u\rangle(t) - \int_t^T 2 \operatorname{Re}\langle Q_t u, u\rangle(t)\, dt \\ &\leq -\int_t^T 2 \operatorname{Re}\langle iPu, u\rangle(t)\, dt + 2C_1 \int_t^T \|u\|^2(t)\, dt \end{aligned} \tag{5.5}$$

when $u \in \mathcal{S}$, and $u \equiv 0$ when $t \geq T$.

By integrating in t we find

$$\int_{-T}^T \|u\|^2(t)\, dt \leq 4T \int_{-T}^T \operatorname{Im}\langle Pu, u\rangle(t)\, dt + 4C_1 T \int_{-T}^T \|u\|^2(t)\, dt \tag{5.6}$$

By using the Cauchy–Schwarz inequality we obtain

$$2\langle Pu, u\rangle \leq \lambda \|u\|^2/T + \|Pu\|^2 T/\lambda \qquad \forall\, \lambda > 0. \tag{5.7}$$

This gives

$$(1 - 4CT - 2\lambda) \int \|u\|^2 \leq 2T^2/\lambda \int \|Pu\|^2\, dt, \tag{5.8}$$

which gives (5.3) when $T_0 \leq 1/16C$ and $\lambda \leq 1/4$. ■

The next case we shall consider is

$$P = D_t + ia(t)(B_t + R_t) \tag{5.9}$$

where $0 \leq a(t) \leq C_0$, B_t and $\partial_t B_t$ are self-adjoint and bounded, $\partial_t B_t \geq 0$ and $\|R_t\| \leq C_1$ on $L^2(\mathbf{R}^n)$. We also assume that there exists a constant $M > 0$ such that

$$\|B_t\| \leq M \qquad \forall\, t \tag{5.10}$$

$$\|[B_s, B_t]\| \leq M \qquad \forall\, s,\ t. \tag{5.11}$$

Lemma 5.2. There exists $T_0 > 0$ and $C > 0$ such that

$$\int \|u\|^2(t)(a(t)+M^{-1})\,dt \le CT^2 \int \|Pu\|^2(t)(a(t)+M^{-1})^{-1}\,dt \tag{5.12}$$

if $u \in \mathcal{S}$ has support where $|t| \le T \le T_0$. Here C_0 and T_0 are independent of M, and only depend on C_0 and C_1.

Proof. First we consider the case $a(t) \ge M^{-1} > 0$. Then (5.12) is equivalent to the estimate:

$$\int \|u\|^2(t)a(t)\,dt \le CT^2 \int \|Pu\|^2(t)\,dt/a(t) \tag{5.13}$$

if $u \in \mathcal{S}$ has support where $|t| \le T$ is small enough. Introducing $s = \int_0^t a(t)\,dt$ as a new time variable and $P_0 = D_s + iB_t$, we find that it suffices to prove

$$\int \|u\|^2(s)\,ds \le CT^2 \int \|P_0u\|^2(s)\,ds \tag{5.14}$$

if $u \in \mathcal{S}$ has support where $|t| \le T$, which implies $|s| \le CT$. In fact, we may then perturb the estimate with the L^2 bounded term iR_tu for small T.

Now $[P_0^*, P_0] = 2\partial_s B_t \ge 0$, which implies

$$\|P_0u\|^2 - \|P_0^*u\|^2 = \langle [P_0^*, P_0]u, u\rangle \ge 0. \tag{5.15}$$

Since $\|D_su\|^2 \le 2(\|P_0u\|^2 + \|P_0^*u\|^2)$, we find

$$\int \|u\|^2(s)\,ds \le C_0T^2 \int \|D_su\|^2(s)\,ds \le 4CT^2 \int \|P_0u\|^2(s)\,ds \tag{5.16}$$

if $u \in \mathcal{S}$ has support where $|s| \le CT$. This proves (5.13) in the case $a(t) \ge M^{-1}$.

Next we consider the case $a(t) \ge 0$. In order to reduce to the case $a \ge M^{-1}$ we conjugate with E_t solving

$$\begin{cases} \partial_t E_t = -E_tB_t/M \\ E_0 = \mathrm{Id}. \end{cases} \tag{5.17}$$

This gives bounds on $\|E_t\|$ and $\|E_t^{-1}\|$ when t is bounded (independently of M), and the conjugation transforms P into

$$\widetilde{P} = D_t + i(a(t)+M^{-1})B_t + a(t)\widetilde{R}_t = D_t + i(a(t)+M^{-1})(B_t+S_t) \tag{5.18}$$

where $\widetilde{R}_t = iE_t^{-1}[B_t + R_t, E_t] + iR_t$ and $S_t = a(t)\widetilde{R}_t/(a(t)+M^{-1})$ are uniformly bounded on $L^2(\mathbf{R}^n)$ for bounded t. In fact, if $F_r = [B_t, E_r]$, $\forall r$, then

$$\partial_r F_r = E_r[B_r, B_t]/M - F_rB_r/M \tag{5.19}$$

and $F_0 \equiv 0$, thus $F_t = [B_t, E_t]$ is bounded on $L^2(\mathbf{R}^n)$ for bounded t (independently of M). By using (5.13) with $\widetilde{P}$ and $a(t)+M^{-1}$, we obtain (5.12). ∎

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