Journées Équations aux dérivées partielles

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Journées Équations aux dérivées partielles (1992), p. 1-5
<http://www.numdam.org/item?id=JEDP_1992____A6_0>

Sharp bounds for the number of the scattering poles

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1 Introduction

In this talk we want to describe some recent author's results concerning upper bounds on the number of scattering poles for a large class compactly supported perturbations of the Laplacian. The details of the proofs can be found in [9], [10] and [11]. Our main result extends a recent result of Sjöstrand and Zworski [5] to the case of non-selfadjoint perturbations and in particular we give a simpler proof. As a consequence we obtain polynomial upper bounds on the number of the scattering poles for hypoelliptic operators. In the case of elliptic perturbations we get bounds $O(r^n)$ which are known to be sharp in some situations ([7], [12]).

Our second result concerns the distribution of scattering poles outside small conic neighbourhood of the real axis. We show that the growth of the number of the scattering poles in such regions does not depend on the regularity of the perturbed operator and is always $O(r^n)$. That is, in the case of hypoelliptic operators the main contribution to the total number comes from the poles close in some sense to the real axis. This observation combined with the recent result of Sjöstrand and Zworski [6], where they study the distribution of the scattering poles in conic neighbourhoods of the real axis with respect to both small angle and large radius, allow to obtain more precise upper bounds on the total number of scattering poles for a class of hypoelliptic operators.

2 Upper bounds for the total number

Let G be a linear unbounded closed operator in a complex Hilbert space $H = H' \oplus L^2(\mathbf{R}^n \setminus B_\rho)$, where $n \geq 3$ is odd, $B_\rho = \{x \in \mathbf{R}^n;\ |x| \leq \rho\}$; H' is a complex Hilbert space whose elements are supposed to satisfy the condition: $\chi f = 0$, $\forall f \in H'$, $\forall \chi \in C^\infty(\mathbf{R}^n)$, $\chi = 0$ on B_ρ. Denote by $D(G)$ the domain of G. The following assumption means that G is a compactly supported perturbation of the Laplacian.

(H1) $\forall \chi \in C^\infty(\mathbf{R}^n)$, $\chi = 0$ on B_ρ, $\chi = 1$ outside a compact domain, $\forall f \in D(G)$, we have $\chi f \in D(G) \cap H^2(\mathbf{R}^n \setminus B_\rho)$ and $G\chi f = -\Delta \chi f$, where Δ is the Laplacian in $\mathbf{R}^n$.

The following assumption means that the operator G is similar to the differential operators in sense that it preserves the supports.

(H2) $\forall \chi_1, \chi_2 \in C^\infty(\mathbf{R}^n)$ with $\operatorname{supp} \chi_1 \cap \operatorname{supp} \chi_2 = \emptyset$, we have $\chi_1 G \chi_2 f = 0$, $\forall f \in D(G)$.

To introduce the scattering poles (known also as resonances) associated to the operator G we also need the following assumptions.

(H3) The resolvent set of G is not empty.

It follows from this assumption that there exist $z_0 \in \mathbf{C}$ with $\Im z_0 > 0$ and an open neighborhood $\Lambda \subset \{z \in \mathbf{C};\ \Im z > 0\}$ of z_0 so that the resolvent $R(z) = (G - z^2)^{-1} \in \mathcal{L}(H, H)$ is well defined and holomorphic in Λ.

(H4) There exists a function $\chi \in C_0^\infty(\mathbf{R}^n)$, $\chi = 1$ on B_ρ, so that $\chi R(z_0)$ is a compact operator in $\mathcal{L}(H, H)$.

By definition, the scattering poles (or resonances) are the poles of the meromorphic continuation of the cutoff resolvent $R_\chi(z) = \chi R(z) \chi$ from Λ to the complex plane $\mathbf{C}$. If z_j is a pole of $R_\chi(z)$, its multiplicity is given by the rank of the operator

$$\int_{|z - z_j| = \varepsilon} z R_\chi(z)\, dz,$$

provided that $\varepsilon > 0$ is small enough. It is not hard to see that the scattering poles (with the multiplicities) are independent of the choice of χ, provided $\chi = 1$ on B_ρ. Let $\{z_j\}$ be the poles of $R_\chi(z)$, repeated according to multiplicity, and set $N(r) = \#\{z_j;\ |z_j| \leq r\}$. Our aim is to get upper bounds on $N(r)$ as $r \to +\infty$. Given any compact operator A, denote by $\mu_j(A)$ the characteristic values of A, repeated according to multiplicity, which are by definition the eigenvalues of $(A^*A)^{1/2}$. We need the following assumption:

(H5) There exists an increasing function $f(t) \in C[1, +\infty)$ such that $0 < f(t) \leq t^{1/n}$, $f(\theta t) \leq C'\theta^\delta f(t)$, $\forall t \geq 1$, $\forall \theta \geq 1$, with constants $C' > 0$, $0 < \delta < 1/2$; and $\mu_j(\chi R(z_0)) \leq Cf(j)^{-2}$, $\forall j$.

Define the function $\varphi \in C[1, +\infty)$ by $\varphi(f(t)) = t$. Our first result is:

Theorem 1 *Under the assumptions (H1)-(H5), the number $N(r)$ of the scattering poles associated to the operator G satisfies the bound*

$$N(r) \leq C\varphi(Cr), \tag{1}$$

with some constant $C > 0$.

As a consequence of this theorem we obtain the following bounds for hypoelliptic operators.

Corollary 1 *Let the operator G satisfy the assumptions (H1)-(H3) and the hypoelliptic estimates*

$$\|u\|_{s+2\varepsilon} \leq C_s\left(\|Gu\|_s + \|u\|_s\right), \qquad \forall s \geq 0,\ \forall u \in\ D(G),\ Gu \in H^s,$$

where $0 < \varepsilon \leq 1$, $\|\cdot\|_s$ denotes the norm in the Sobolev space H^s. Then

$$N(r) \leq Cr^{n/\varepsilon} + C. \tag{2}$$

It is easy to see that the above hypoelliptic estimates imply (H4) and (H5) with $f(t) = t^{1/p}$, where $p = n/\varepsilon$. Hence $\varphi(t) = t^p$ and (2) follows from (1) at once.

In the elliptic case ($\varepsilon = 1$) such a bound was first obtained by Melrose [4] in the case of obstacle scattering, and by Zworski [13] in the case of compactly supported potential. In the case of selfadjoint operators bounds equivalent to (1), (2) are proved by Sjöstrand and Zworski [5] by using the complex scaling method. Our proof of (1) avoids using this approach and is in the spirit of the earlier works on this problem (see [3], [4], [9], [13]).

3 Upper bounds for the number of the scattering poles outside a conic neighbourhood of the real axis

Let G be a second order differential operator in $\mathbf{R}^n$, which is a compactly supported perturbation of the Laplacian. Assume that G is a positively definite selfadjoint operator which satisfies the assumption (H4) above. Then it is well known that the scattering poles $\{z_j\}$ are all in $\{\Im z \leq 0\}$. Given any ε, $0 < \varepsilon \ll 1$, set $N(\varepsilon, r) = \#\{z_j;\ |z_j| \leq r,\ \varepsilon \leq \arg(-z_j) \leq \pi - \varepsilon\}$. Our second result is the following.

Theorem 2 *For any ε and r as above, $\exists C_\varepsilon > 0$ so that*

$$N(\varepsilon, r) \leq C_\varepsilon r^n + C_\varepsilon. \tag{3}$$

Note that this bound does not depend on the regularity of G. It shows that to study the total number $N(r)$ modulo $O(r^n)$ it suffices to study the number of the scattering poles near the real axis. In [6] Sjöstrand and Zworski used this observation to obtain quite precise upper bounds on the total number for a class of hypoelliptic operators.

The idea of the proof of (3) is to characterize the scattering poles in $\{\Im z < 0\}$, with multiplicity, as zeros of a function $h(z)$ defined and holomorphic in $\{\Im z < 0\}$, such that for any $\gamma > 0$, $\exists C_\gamma > 0$, satisfying the estimate

$$|h(z)| \leq C_\gamma \exp\left(C_\gamma |z|^n\right) \qquad \text{for } \Im z \leq -\gamma. \tag{4}$$

Then (3) is derived from (4) with the help of a classical result due to Carleman (see [8, Sec. 3]).

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