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A GENERALIZED COHERENT STATE APPROACH OF THE QUANTUM DYNAMICS FOR SUITABLE TIME-DEPENDENT HAMILTONIANS

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1. Introduction

The remarkable properties of inverse square potentials in the quantum mechanical treatment of the one- or many-body problem is known for a long time [1, 2]. Furthermore, the quantum dynamics for harmonic oscillators with a frequency which is variable in time is known to be exactly solvable in terms of the classical motion (see review [7] and references therein contained). The fact that these properties can be combined to solve the quantum dynamics for Hamiltonians of the form

$$H(t) = \frac{P^2 + Q^2 f(t)}{2} + \frac{g^2}{2Q^2} \qquad (1.1)$$

g being a constant, has also been discovered recently [3, 9], and exploited for example by myself for the study of the quantum dynamics of two or three ions in a quadrupole radio-frequency trap (also called Paul trap) [4, 5].

In this letter, I want to show that a very simple solution of the quantum mechanical dynamics for Hamiltonians (1.1) is obtained by the use of the so-called "Peremolov's generalized coherent states" of the Lie algebra of SU(1, 1) [9]. A privileged role is played by the classical solutions for the quadratic Hamiltonian

$$H_q(t) = \frac{P^2 + Q^2 f(t)}{2} \quad . \qquad (1.2)$$

We note in passing that these classical solutions also determine the classical trajectories for Hamiltonian (1.1).

2 - Quantum dynamics for Hamiltonians linear in the generators of SU(1, 1) Lie algebra

Let K_o and $K_\pm$ be generators of the Lie algebra of SU(1, 1). They satisfy the following commutation rules :

$$[K_o , K_\pm] = \pm K_\pm \qquad (2.1)$$

$$\left[K_- , K_+\right] = 2K_o \tag{2.2}$$

and in addition,

$$K_+ = K_-^* \quad . \tag{2.3}$$

Let H(t) be a quantum Hamiltonian given by

$$H(t) = \lambda(t)\, K_+ + \bar{\lambda}(t)\, K_- + \mu(t)\, K_o \tag{2.4}$$

where $t \rightarrow \lambda(t)$ is a complex valued function, whereas $t \rightarrow \mu(t)$ is real valued.

Let U(t, s) be the unitary evolution operator generated by H(t), namely solution of the Schrödinger equation

$$i \frac{d}{dt} U(t, s) = H(t)\, U(t, s) \tag{2.5}$$

with

$$U(t, t) = \mathbb{1} \qquad \text{(identity operator)}$$

for any t. Then we have the following result :

Proposition 1

For any complex number β with $|\beta| \leqslant 1$, define unitary operators $T(\beta)$ as :

$$T(\beta) = \exp\left(\delta K_+ - \bar{\delta} K_-\right) \tag{2.6}$$

with $$\delta = \frac{\beta}{|\beta|} \text{ Argth } |\beta| \quad . \tag{2.7}$$

Then we have

$$U(t, s) = T(\beta_t) \exp\left[i\left(\gamma_t - \gamma_s\right) K_o\right] T\left(-\beta_s\right) \tag{2.8}$$

where the complex function β and the real function γ satisfy the following differential equations :

$$i\dot{\beta} = \bar{\lambda}\,\beta^2 + \mu\beta + \lambda \qquad (2.9)$$

$$\dot{\gamma} = -\,\lambda\bar{\beta} - \bar{\lambda}\beta - \mu \quad . \qquad (2.10)$$

Remark 1

$T(\beta)$ is the generator of "generalized coherent states" introduced by Perelomov [9], and reduces to the generator of "squeezed states" in the harmonic oscillator's case where

$$K_o = \frac{1}{4}\left(a^+a + aa^+\right)$$
$$K_+ = \frac{1}{2}\,a^{+2}$$
$$K_- = \frac{1}{2}\,a^2$$

(see ref. [6.8]). Therefore theorem 1 is essentially given in Perelomov's book ([9], § 18.2). However we give here a simple proof, for the sake of completeness. It relies on the following lemma:

Lemma 1

$$i\frac{d}{dt}T(\beta_t) = \left(\alpha K_+ + \bar{\alpha}K_- + \rho K_o\right)T(\beta_t)$$

with

$$\begin{cases} \alpha = \dfrac{i\dot{\beta}}{1 - |\beta|^2} \\ \\ \rho = \dfrac{i(\beta\dot{\bar{\beta}} - \dot{\beta}\bar{\beta})}{1 - |\beta|^2} \end{cases} \qquad (2.11)$$

for the proof of lemma 1, see [6], lemma 4. Now, differentiating the RHS of (2.8) with respect to t (a dot denotes differentiation w.r. to t) we obtain :

$$i\frac{d}{dt}U(t, s) = \left[\alpha K_+ + \bar{\alpha}K_- + \rho K_0 - \dot{\gamma}\, T(\beta)\, K_0\, T(-\beta)\right] U(t, s)$$

and since

$$T(\beta)\, K_0\, T(-\beta) = -\, K_+ \frac{\beta}{1 - |\beta|^2} - K_- \frac{\bar{\beta}}{1 - |\beta|^2} + K_0 \frac{1 + |\beta|^2}{1 - |\beta|^2}$$

a necessary condition for (2.5) to hold, with H(t) given by (2.4) is :

$$\begin{cases} \alpha + \dfrac{\beta\dot{\gamma}}{1 - |\beta|^2} = \lambda \\ \rho - \dot{\gamma}\, \dfrac{1 + |\beta|^2}{1 - |\beta|^2} = \mu \end{cases} . \tag{2.12}$$

Now combining (2.11) and (2.12) we easily get (2.9, 10).

3. <u>The classical equations of motion for quadratic Hamiltonians</u>

A remarkable role for the solution of the problem of section 2, which is shown to reduce to solving differential equations (2.9, 10), is provided by the quadratic Hamiltonians, namely the time-dependent Hamiltonians that are quadratic in the quantum operators Q (= multiplication by x) and $P = -\, i \dfrac{\partial}{\partial x}$. In some sense, the quadratic Hamiltonians, as we shall see, are the paradigm of the most general case of generators of SU(1, 1) Lie algebra considered in section 3.

Assume K_0 and $K_\pm$ be the following operators in $L^2(\mathbb{R})$:

$$K_o = \frac{P^2 + Q^2}{4}$$

$$K_\pm = \frac{Q^2 - P^2}{4} \mp i \frac{QP + PQ}{4} \qquad (3.1)$$

so that H(t) given by (2.4) is the following quadratic Hamiltonian :

$$H_Q(t) = \frac{P^2}{2}\left(\frac{\mu}{2} - \text{Re}\,\lambda\right) + \frac{Q^2}{2}\left(\frac{\mu}{2} + \text{Re}\,\lambda\right) + \text{Im}\,\lambda\,\frac{QP + PQ}{2} \qquad (3.2)$$

The classical dynamics generated by (3.2) is given by the following Newton's equations, which are linear :

$$\begin{cases} \dot{q} = p\left(\frac{\mu}{2} - \text{Re}\,\lambda\right) + q\,\text{Im}\,\lambda \\ \dot{p} = -q\left(\frac{\mu}{2} + \text{Re}\,\lambda\right) - p\,\text{Im}\,\lambda \end{cases} \qquad . \qquad (3.3)$$

We shall see that such solutions generate the functions $\beta(t)$ and $\gamma(t)$ of section 2, which allowed to construct the exact quantum evolution operator (2.8) of the general case.

Proposition 2

Let (q, p) be the classical complex phase-space trajectory given by (3.3), with initial data (q_o, p_o). Define :

$$\begin{cases} \beta = \dfrac{q + ip}{q - ip} \\ \gamma = -\dfrac{1}{2}\text{Arg}\,(q - ip) \end{cases} \qquad . \qquad (3.4)$$

Then β_t and γ_t obey the differential equations (2.9, 10) respectively. Furthermore $|\beta_t| \leqslant 1$ is true provided the initial data (q_o, p_o) satisfy $|\beta_o| \leqslant 1$.

Proof :

$$|\beta|^2 = \frac{|q|^2 + |p|^2 + i(p\bar{q} - \bar{p}q)}{|q|^2 + |p|^2 - i(p\bar{q} - \bar{p}q)}$$

$W = i(p\bar{q} - \bar{p}q)$ is easily seen to be constant along any trajectory (3.3). Therefore, in order that $|\beta_o| \leqslant 1$, W must be negative real , which immediately implies that $|\beta_t| \leqslant 1$ for any t. Thus it is a good candidate for constructing the generator $T(\beta_t)$ of "generalized Perelomov's coherent states".

Now using (3.3) it is immediate to check that the differential equations (2.9, 10) are satisfied by the functions β and γ defined by (3.4).

Corollary 1

Let $$H(t) = \frac{P^2 + Q^2 f(t)}{2} + \frac{g^2}{2Q^2}$$

where f(t) is any real function, and g a real constant. Let ξ be a complex function solution of :

$$\ddot{\xi} + f\xi = 0 \quad . \tag{3.5}$$

Then the quantum evolution operator U(t, s) generated by H(t) is

$$U(t, s) = T(\beta_t) \exp\left(i\, K_o(\gamma_t - \gamma_s)\right) T(-\beta_s) \tag{3.6}$$

where

$$\beta_t = \frac{\xi + i\dot{\xi}}{\xi - i\dot{\xi}} \tag{3.7}$$

and

$$\gamma_t = -\frac{1}{2} \operatorname{Arg}(\xi - i\dot{\xi}) \tag{3.8}$$

where K_o, and $K_\pm$ defining $T(\beta)$ are the following operators

$$\begin{cases} K_o = \dfrac{P^2 + Q^2}{4} + \dfrac{g^2}{4Q^2} \\ K_\pm = \dfrac{Q^2 - P^2}{4} - \dfrac{g^2}{4Q^2} \mp i \dfrac{QP + PQ}{4} \end{cases} \tag{3.9}$$

with natural domains.

Proof :

Clearly, the operators K_o and $K_\pm$ defined by (3.9) obey the commutations relations (2.1, 2). Furthermore, H(t) is of the form (2.4) with real functions λ and μ :

$$\mu = 1 + f(t)$$
$$\lambda = \frac{1}{2}\left(f(t) - 1\right) \quad .$$

Therefore the Newton's equations (3.3) reduce to equation (3.5), and (3.7, 8) are nothing but (3.4). Corollary 1 is thus an immediate consequence of propositions 1 and 2.

Remark 2

We note also that solutions of (3.5), namely of the classical equations of motion for $H_Q(t) = \dfrac{P^2 + Q^2 f(t)}{2}$ provide a solution of the classical equation of motion for H(t) : assume ξ be a complex solution of (3.5) with initial conditions

$$\xi(0) = 1$$
$$\ddot{\xi}(0) = ig$$

and let $\xi(t) = u(t)\, e^{i\theta(t)}$ be the polar decomposition of ξ. Then u(t) obeys

$$\ddot{u} + fu = \frac{g^2}{u^3}$$

with $u(0) = 1$ and $\ddot{u}(0) = 0$, and is therefore a real classical trajectory of Hamiltonian $H(t) = \dfrac{P^2 + Q^2 f(t)}{2} + \dfrac{g^2}{2Q^2}$. Namely we have

$$2i\theta = \mathrm{Log}\,\frac{\xi}{\bar{\xi}}\ , \qquad \text{and therefore} \qquad 2i\dot{\theta} = \frac{\dot{\xi}\,\bar{\xi} - \xi\,\dot{\bar{\xi}}}{|\xi|^2}\ .$$

But $\dot{\xi}\,\bar{\xi} - \xi\,\dot{\bar{\xi}}$ is constant and equals $2ig$. Therefore $\dot{\theta} = gu^{-2}$. This implies

$$\dot{\xi} = e^{i\theta}\left(\dot{u} + ig\, u^{-1}\right)$$

and therefore

$$\ddot{\xi} = e^{i\theta}\left(\ddot{u} - g^2\, u^{-3}\right) = -f\, e^{i\theta}\, u$$

which was the claim.

4. The quantum N-body problem with inverse square interactions

We now turn to the following N-body Hamiltonian in dimension one :

$$\tilde{H}(t) = -\frac{1}{2}\sum_{j=1}^{N}\frac{\partial^2}{\partial x_j^2} + \frac{1}{2N} f(t) \sum_{j<\ell}\left(x_j - x_\ell\right)^2$$
$$+ \frac{g^2}{2}\sum_{j\neq k}\left(x_j - x_k\right)^{-2}\ .$$

As already suggested in ref. [9], it can be exactly solved along the lines described above.

If $X = N^{-1} \sum_{j=1}^{N} x_j$ is the center-of-mass coordinate, we define N (non-independent) variables ξ_j as

$$\xi_j = x_j - X \tag{4.1}$$

and formal ξ_j derivatives :

$$\frac{\partial}{\partial \xi_j} = \frac{\partial}{\partial x_j} - \frac{\partial}{\partial X} \ . \tag{4.2}$$

where $\frac{\partial}{\partial X} = N^{-1} \sum_{1}^{N} \frac{\partial}{\partial x_k}$.

Then, clearly

$$N^{-1} \sum_{j<k} \left(x_j - x_k \right)^2 = \sum_{j=1}^{N} \xi_j^2$$

so that our Hamiltonian $\tilde{H}(t)$ can be rewritten as

$$\tilde{H}(t) = - \frac{N}{2} \frac{\partial^2}{\partial X^2} + H(t)$$

$$H(t) = - \frac{1}{2} \sum_{j=1}^{N} \frac{\partial^2}{\partial \xi_j^2} + \frac{f(t)}{2} \sum \xi_j^2 + \frac{g^2}{2} \sum_{j \neq \ell} \left(\xi_j - \xi_\ell \right)^{-2} \ . \tag{4.3}$$

Now defining for any j = 1, ... N :

$$\begin{cases} b_j^+ = \xi_j - \dfrac{\partial}{\partial \xi_j} \\ b_j = \xi_j + \dfrac{\partial}{\partial \xi_j} \end{cases} \tag{4.4}$$

it is clear that

$$\begin{cases} \left[b_j, b_k^+\right] = 2\left(\delta_{jk} - N^{-1}\right) \\ \left[b_j, b_k\right] = \left[b_j^+, b_k^+\right] = 0 \end{cases} \quad . \tag{4.5}$$

This allows us to define the following operators

$$\begin{cases} K_0 = \frac{1}{4}\left(\sum_{j=1}^{N} b_j^+ b_j + N - 1\right) + \frac{V}{2} \\ K_+ = \frac{1}{4} \sum_{j=1}^{N} b_j^{+2} - \frac{V}{2} \\ K_- = \frac{1}{4} \sum_{j=1}^{N} b_j^2 - \frac{V}{2} \end{cases} \tag{4.6}$$

with

$$V = \frac{g^2}{2} \sum_{j \neq k} \left(\xi_j - \xi_k\right)^{-2} \quad . \tag{4.7}$$

As for (3.9), it is not hard to check the commutation rules (2.1, 2) for K_0 and $K_\pm$ defined by (4.6, 7), using (4.5).

Furthermore H(t) is of the form

$$H(t) = \left[1 + f(t)\right] K_0 + \left(f(t) - 1\right) \frac{K_+ + K_-}{2} \tag{4.8}$$

since

$$K_0 = \frac{1}{4} \sum_{j=1}^{N} \left(\xi_j^2 - \frac{\partial^2}{\partial \xi_j^2}\right) + \frac{V}{2}$$

$$\frac{K_+ + K_-}{2} = \frac{1}{4} \sum_{j=1}^{N} \left(\xi_j^2 + \frac{\partial^2}{\partial \xi_j^2}\right) - \frac{V}{2} \quad .$$

Therefore omitting the trivial uniform center-of-mass motion we get the following result :

Proposition 3

Let ξ be a complex solution of equation (3.5), and let β and γ be defined by (3.7, 8) respectively. K_o and $K_\pm$ being defined by (4.6, 7) let $T(\beta)$ be

$$\exp\left(\delta K_+ - \bar{\delta} K_-\right)$$

with δ given by (2.7). Then the unitary evolution operator for the quantum Hamiltonian $H(t)$ is

$$U(t, s) = T(\beta_t) \exp\left[i(\gamma_t - \gamma_s)K_o\right] T(-\beta_s) \quad .$$

Proof :

Proposition 3 is an immediate consequence of proposition 1, given (4.8). Here, as in the one-body case of corollary 1, the solutions ξ of the classical equation of motion for the one-body quadratic Hamiltonian $H_q(t) = \dfrac{P^2 + f(t)\, Q^2}{2}$ determine exactly the quantum evolution.

5. Conclusion

We have seen that, in great generality, the generators $T(\beta)$ of the "Perelomov's generalized coherent states" of the SU(1, 1) Lie algebra allow to solve the time-dependent Schrödinger equation, when the Hamiltonian is a linear combination of the generators of this algebra, with time-dependent coefficients. Furthermore this approach outlines the prominent role of the solutions of classical linear equations of motion for constructing these generalized coherent states.

References

[1] F. Calogero, J. Math. Phys. 10 (1969) 2191-2196.

[2] F. Calogero, J. Math. Phys. 12 (1971) 419-436.

[3] S. M. Chumakov, V. V. Dodonov and V. I. Man'ko, J. Phys. A 19 (1986) 3229-3239.

[4] M. Combescure, Ann. Inst. H. Poincaré 44 (1986) 293-314.

[5] M. Combescure, Ann. Phys. 204 (1990) 113-123.

[6] M. Combescure, "The squeezed state approach of the semi-classical limit of the time-dependent Schrödinger equation", (Orsay, 1991).

[7] V. V. Dodonov and V. I. Man'ko, "Invariants and the evolution of nonstationary quantum systems", in : Proceedings of Lebedev Physics Institute, Acad. Sci. U. S. S. R., Vol. 183, ed. M. A. Markov (Nova Science Publishers, Commack N. Y. 1989).

[8] I. A. Pedrosa, Phys. Rev. D36 (1987) 1279-1280.

[9] A. Perelomov, "Generalized Coherent States and Their Applications", (Springer, Berlin 1986).