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# On resonant scattering for time-periodic perturbations

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1. The energy of a quantum system desribed by a time-dependent Hamiltonian $H(t)$ is not conserved. However, if a dependence of $H(t)$ on $t$ is periodic, it can be changed only by some integer number. In other words, the quasi-energy, i.e. the energy defined up to an integer, is a conserved quantity.

Here we discuss scattering of a plane wave by a time-periodic potential. Due to the quasi-energy conservation such a process is desribed by a set of amplitudes $S_n(\lambda)$ where $\lambda$ is energy of an incident wave (in other terms, of a quantum particle) and $n$ is arbitary integer. We always decompose $\lambda$ as $\lambda = m+\theta$ where $m \in \mathbb{Z}$ is the entire part of $\lambda$ and $\theta \in [0,1]$. Each $S_n(\lambda)$ corresponds to a channel when energy is changed by $n-m$. Actually, amplitudes $S_n(\lambda)$ for $n \geqslant 0$ correspond to outgoing waves and amplitudes $S_n(\lambda)$ for $n<0$ correspond to exponentially decaying modes. In some sense these modes play the role of bound or quasi-bound states for time-independent Hamiltonians. It means that they represent states which can have long though finite time of life. Thus exponentially decaying modes are essential for a detailed picture of interaction of an incident wave with a quantum system but they do not contribute to the scattering matrix of this process. Our aim is to study the transformation of exponentially decaying modes into proper bound states as a time-periodic perturbation is switched off.

In fact, we shall consider the following situation. Suppose that $H(t) = H_1+\varepsilon V(t)$ where the Hamiltonian $H_1$ has a negative eigenvalue $\lambda_1$ and the coupling constant $\varepsilon$ is small. Physically, it is natural to conjecture that the bound state of the system with the Hamiltonian $H_1$ will give rise to some kind of long-living state for the family $H(t)$. Due to the quasi-energy conservation this state is insignificant if energy $\lambda$ of an incident particle and $\lambda_1$ do not coincide by modulus of $\mathbb{Z}$. However, if energy $\lambda$ is resonant, that is $\lambda - \lambda_1 = K \in \mathbb{Z}$, then an incident particle can strongly interact with this quasi-bound state. Therefore the corresponding amplitude $S_{m-K}(\lambda,\varepsilon)$ is expected to be very large for small $\varepsilon$. Below we will show at the example of zero-range potentials that this physical picture is correct.

The problem of resonances for time-periodic perturbations was studied earlier by K. Yajima [1] in a different, more mathematical, framework. Our approach is closer to physical papers [2]-[5]. In particular, in [5] an attempt was made to study the amplitudes $S_n$ for small time-periodic perturbations. However, the appearence of resonant energies seems to be neglected in this paper.

2. The Hamiltonian $H_1$ corresponding to a zero-range potential well of a "depth" $h_1$ is defined as $H_1 = -\frac{d^2}{dx^2}$, $x \in \mathbb{R}_+$, with the boundary condition $u'(o) = -h_1 u(o)$, $h_1 = \overline{h_1}$. The operator $H_1 > 0$, if $h_1 \leqslant 0$, and it has (exactly one) negative eigenvalue $\lambda_1 = -h_1^2$ with the eigenfunction $\exp(-h_1 x)$, if $h_1 > 0$. Let $H_0 = -d^2/dx^2$ with the boundary condition $u(o) = o$ be the "free" Hamiltonian. The scaltering matrix $S^{(1)}(\lambda)$ for the pair $H_0$, $H_1$ at energy $\lambda$ equals

$$S^{(1)}(\lambda) = (h_1 - i\lambda^{1/2})(h_1 + i\lambda^{1/2})^{-1}. \tag{1}$$

We shall consider zero-range potential well whose depth depends periodically on time. Mathematically this problem is governed by the equation

$$i\frac{\partial u}{\partial t} = -\frac{\partial^2 u}{\partial x^2}, \quad x \in \mathbb{R}_+, \tag{2}$$

with the time-dependent boundary condition

$$u'(0,t) = h(t)\,u(0,t), \quad \overline{h(t)} = h(t), \quad h(t+2\pi) = h(t) \tag{3}$$

We will look for solutions of equation (1) which have a representation of the form

$$u(x,t) = \sum_{n=-\infty}^{\infty} u_n(x)\, e^{-i(n+\theta)t} \tag{4}$$

where the parameter $\theta \in [0,1]$. Such solutions describe a stationary process in the sense that for any $\tau \in \mathbb{R}$

$$(2\pi)^{-1} \int_{\tau}^{\tau+2\pi} |u(x,t)|^2 \, dt = \sum_{n=-\infty}^{\infty} |u_n(x)|^2 \qquad (5)$$

Substituting (4) into (2) we find that $u_n(x)$ should satisfy the equations

$$-u_n^{n}(x) = (n+\theta)\, u_n(x), \qquad (6)$$

whose solutions are linear combinations of exponentials. In particular, the solution corresponding to the incoming wave $\exp(-i\lambda^{1/2}x)$, $\lambda = m+\theta$, $m \in \mathbb{Z}$, $\theta \in [0,1[$, has the form

$$u_n(x,\lambda) = S_{n\,m} \exp(-i\lambda^{1/2}x) - S_n(\lambda) \exp(i(\theta+n)^{1/2}x), \qquad (7)$$

where $S_{m\,m} = 1$, $S_{n\,m} = 0$, if $n \neq m$, and

$$i(\theta+n)^{1/2} = -|\theta+n|^{1/2}, \; n \leqslant -1.$$

The terms $S_n(\lambda) \exp(i(\theta+n)^{1/2}x)$ desribe out going waves, if $n \geqslant 0$, and they are exponentially decaying, if $n < 0$.

Equations (6) are coupled by the boundary condition (3) which allows us to determine the amplitudes $S_n(\lambda)$. In fact, substituting (7) into (4) and then into (3) we obtain the equation

$$-i\lambda^{1/2} e^{-imt} - i \sum_{n=-\infty}^{\infty} (\theta+n)^{1/2} S_n(\lambda) e^{-int} =$$

$$h(t)\,(e^{-imt} - \sum_{n=-\infty}^{\infty} S_n(\lambda) e^{-int}). \qquad (8)$$

Explanding $h(t)$ in the Fourier series and comparing coefficients of $e^{-int}$ we arrive at an infinite set of algebraic equations for the amplitudes $S_n(\lambda)$.

Note that functions $S_n(\lambda)$ are continuous in $\lambda \in [m, m+1]$ for every $m = 0,1,2,\ldots$ Moreover, $S_n(m-0) = S_{n+1}(m+0)$ for all $n \in \mathbb{Z}$ and $m = 1,2,\ldots,$

3. Below we restrict ourselves to the consideration of the simplest case

$$h(t) - -h_1 + 2\varepsilon \cos t \qquad (9)$$

Then equation (8) is equivalent to the following system of equations

$$(i(\theta+n)^{1/2} + h_1)\, S_n - \varepsilon(S_{n+1} + S_{n-1}) = S_n^{(0)},\ n \in \mathbb{Z}, \tag{10}$$

where

$$S_m^{(0)}(\lambda) = h_1 - i\lambda^{1/2},\ S_{m-1}^{(0)}(\varepsilon) = S_{m+1}^{(0)}(\varepsilon) = -\varepsilon \tag{11}$$

and $S_n^{(0)} = 0$ for $|n-m| \geqslant 2$. We emphasize that the amplitudes $S_n = S_n(\lambda,\varepsilon)$ depend on energy $\lambda$ of incoming wave and on the parameter $\varepsilon$ in (9). It is convenient to rewrite the system (10) in vector notation. Set $s = \{S_n\}$, $s_0 = \{S_n^{(0)}\}$, $n \in \mathbb{Z}$, and

$$\Lambda = \mathrm{diag}\{i(\theta+n)^{1/2} + h_1\},\ K = \Gamma + \Gamma^*,$$

where $\Gamma$, $(\Gamma 6)_n = 6_{n+1}$, is the shift operator. Then (10) is equivalent to the equation

$$(\Lambda - \varepsilon K)\, s = s_0 \tag{12}$$

which can be considered, for example, in the space $\ell_2(\mathbb{Z})$.

In the case $\varepsilon = 0$ the function (9) does not depend on t so that equations (10) become independent and can be easily solved. In fact, $S_m(\lambda,0) = S^{(1)}(\lambda)$ and $S_n(\lambda) = 0$, if $n \neq m$, $n \geqslant 0$. For negative n the amplitude $S_n(\lambda,0) = 0$ in case

$$h_1 \neq |\theta+n|^{1/2} \tag{13}$$

and $S_n(\lambda,0)$ is arbitrary in case $h_1 = |\theta+n|^{1/2}$. The latter equality is possible only if $h_1 > 0$ and $\lambda - \lambda_1 \in \mathbb{Z}$. In this case the function (4) is given by the relation

$$u(x,t) = (\exp(-i\lambda^{1/2}x) - S^{(1)}(\lambda)\exp(i\lambda^{1/2}x))\exp(i\lambda t) + \gamma\exp(-h_1 x + i h_1^2 t) \tag{14}$$

with arbitrary $\gamma$. The last term in (14) disappears (i.e. $\gamma = 0$) if $h_1 \leqslant 0$ or $h_1 > 0$ and $\lambda - \lambda_1 \notin \mathbb{Z}$.

4. Our goal is to study the limit of the amplitudes $S_n(\lambda,\varepsilon)$ as $\varepsilon \to 0$. We first consider the non-resonant case when either $h_1 \leqslant 0$ or $h_1 > 0$ and $\lambda - \lambda_1 \notin \mathbb{Z}$. Then condition (13) holds for all $n = -1, -2, \ldots$ so that the operator $\Lambda$ is invertible and (10) is equivalent to the relation

$$(I - \varepsilon\Lambda^{-1}K)\, s = \Lambda^{-1} so$$

Since K is a bounded operator, for sufficiently small $\varepsilon$ this equation can be solved by iteration :

$$s(\varepsilon) = \sum_{p=0}^{\infty} \varepsilon^{p} (\Lambda^{-1} K)^{p} \Lambda^{-1} s_0(\varepsilon). \qquad (15)$$

Thus for non-resonant energies $\lambda, \lambda - \lambda_1 \notin \mathbb{Z}$, the asymptotic expansion of amplitudes is described by regular perturbation theory. In particular, (15) ensures that $S_n(\lambda,\varepsilon) = o(\varepsilon^{|n-m|})$ so that the probability of excitation of states with energies $\lambda + K$, $K \in \mathbb{Z}$, is proportional to $\varepsilon^{|K|}$. The amplitude $S_m(\lambda,\varepsilon)$ converges to the scattering matrix (1), i.e.

$$S_m(\lambda,\varepsilon) = (h_1 - i\lambda^{1/2})(h_1 + i\lambda^{1/2})^{-1} + o(\varepsilon^2). \qquad (16)$$

The leading term of the corrections to the case $\varepsilon = 0$ is determined by the amplitudes

$$S_{m\pm 1}(\lambda,\varepsilon) = -2i\varepsilon\lambda^{1/2}(h_1 + i(\lambda \pm 1)^{1/2})^{-1}(h_1 + i\lambda^{1/2})^{-1} + o(\varepsilon^2). \qquad (17)$$

5. If $h_1 > 0$ and $\lambda$ equals one of the resonant points $\lambda_1 + K$, $K \in Z$, there arises a non-trivial interaction of the incident wave with the quasi-bound state of the time-dependent well. This interaction does not vanish in the limit $\varepsilon \to 0$. From the mathematical viewpoint the problem is due to the appearence of zero eigenvalues of the operator $\Lambda$. The operator $\Lambda - \varepsilon K$ is invertible for all $\varepsilon > 0$ but some of the matrix elements of $(\Lambda - \varepsilon K)^{-1}$ tend to infinity as $\varepsilon \to 0$. For definiteness we suppose that $0 < h_1 < 1$ and $\lambda$ approaches the point $\lambda_0 = 1 - h_1^2$. In this case the resonant interaction is the most significant. In fact, we shall obtain asymptotic formulas for $S_n(\lambda,\varepsilon)$ which hold uniformly in $\lambda \in I_\delta = [\delta, 1-\delta]$, $\delta > 0$, as $\varepsilon \to 0$.

To bypass the problem of small denominators which appears now we distinguish equation (10) with $n = -1$

$$(h_1 - (1-\lambda)^{1/2}) S_{-1} - \varepsilon(S_0 + S_{-2}) = -\varepsilon \qquad (18)$$

where all coefficients vanish as $\lambda \to \lambda_0$ and $\varepsilon \to 0$. First we consider only equations in (10) which correspond to $n \geq 0$. We shall solve this system with respect to amplitudes $S_n$, $n \geq 0$, with $S_{-1}$ playing the role of a parameter. Since all diagonal elements $i(\lambda + n)^{1/2} + h_1$, $n \geq 0$, are separated from zero, this system can be solved by iteration which gives the relation

$$S_0 = (h_1 + i\lambda^{1/2})^{-1} (\varepsilon S_{-1} + h_{-1} - i\lambda^{1/2})(1 + 0(\varepsilon^2)). \tag{19}$$

We emphasize that quantities as $0(\varepsilon^2)$ are uniform in $\lambda \in I_\delta$. Similarly, solving equations in (10) corresponding to $n \leqslant -2$ with respect to $S_n$, $n \leqslant -2$, we find that

$$S_{-2} = \varepsilon (h_1 - (2-\lambda)^{1/2})^{-1} S_{-1} (1 + 0(\varepsilon^2)). \tag{20}$$

Substituting expressions (19), (20) into (18) we obtain finally the equation for $S_{-1}$. It follows that

$$S_{-1}(\lambda,\varepsilon) = 2i\varepsilon \lambda^{1/2} \Omega^{-1}(\lambda,\varepsilon)(1 + 0(\varepsilon)). \tag{21}$$

where

$$\Omega(\lambda,\varepsilon) = [-h_1 + (1-\lambda)^{1/2} + \varepsilon^2 (h_1 - (2-\lambda)^{1/2})^{-1}](h_1 + i\lambda^{1/2}) + \varepsilon^2$$

Here we have taken into account that

$$|\varepsilon^2 \Omega^{-1}(\lambda,\varepsilon)| \leqslant C.$$

Combining (19) with (21), we find also the asymptotics of $S_0$ :

$$S_0(\lambda,\varepsilon) = (h_1 - i\lambda^{1/2})(h_1 + i\lambda^{1/2})^{-1} + 2i\varepsilon^2 \lambda^{1/2} (h_1 + i\lambda^{1/2})^{-1} \Omega^{-1}(\lambda,\varepsilon) + 0(\varepsilon). \tag{22}$$

Clearly, $|S_0(\lambda,\varepsilon)| = 1$ up to an error of order $\varepsilon$.

If $\lambda$ is separated from the point $\lambda_0$, we can replace $\Omega(\lambda,\varepsilon)$ by $\Omega(\lambda,0)$ which is not zero. In this case we recover the relations (16), (17) (for $m = 0$). In the particular case $\lambda = \lambda_0$ we have that

$$(\lambda_0,\varepsilon) = \varepsilon^2 (h_1 - (1+h_1^2)^{1/2})^{-1} b_1$$

where

$$b_1 = 2h_1 - (1+h_1^2)^{1/2} + i(1-h_1^2)^{1/2}$$

There fore according to (21), (22)

$$S_{-1}(\lambda_0,\varepsilon) = 2i(1-h_1^2)^{1/2}(h_1 - (1+h_1^2)^{1/2}) b_1^{-1} \varepsilon^{-1} + 0(1),$$

$$S_0(\lambda_0,\lambda) = \overline{b_1}\, b_1^{-1} + 0(\varepsilon).$$

As could be expected, the amplitude $S_{-1}(\lambda_0,\varepsilon)$ grows infinitely as $\varepsilon \to 0$. By virtue of (5) it follows that for the corresponding function (4) and any $r > 0$ the integral

tends to infinity as $\varepsilon \to 0$. This is consistent with the decoupling of bound states and scattering states in the stationary case $\varepsilon = 0$ when, by (14), the integral (23) has arbitrary value.

The amplitude $S_0(\lambda_0,\varepsilon)$ has a finite limit $S_0(\lambda_0,0)$ which is, however, different from the scattering matrix (1) at energy $\lambda_0$ for the time-independent boundary condition $u'(0) = -h_1 u(0)$. Therefore, at energy $\lambda_0$ we find an additional resonant phase shift which does not vanish in the limit $\varepsilon \to 0$.

6. In stationary problems resonances are usually defined as complex "eigenvalues" for which the Schrödinger equation has solutions satisfying the outgoing radiation condition at infinity. Similarly, a compex point $\lambda$ can be called [3] resonant point for the problem (2), (3) if there exists its solution of the form

$$u(x,t) = \sum_{n=-\infty}^{\infty} A_n \exp\,[i(\lambda+n)^{1/2}x - i(n+\lambda)t]$$

It is easy to see that at such $\lambda$ the homogeneous system of equations

$$(i(\lambda+n)^{1/2} + h_1)\, A_n - \varepsilon\,(A_{n+1} + A_{n-1}) = 0$$

should have a non-trivial solution. This system can be studied by the method of section 5. In the case $0 < h_1 < 1$ there exist for sufficiently small $\varepsilon$ resonant points obeying the relation

$$\lambda = n - h_1^2 - 2\,\varepsilon^2 h_1((1+h_1^2)^{1/2} + i(1-h_1^2)^{1/2}) + 0(\varepsilon^4)$$

where n is an arbitrary integer. In the limit $\varepsilon \to 0$ these complex points approach real points differing from $\lambda_1 = -h_1^2$ by some integer.

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