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The Boltzmann–Enskog equation with large data;
wellposedness and regularity.

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In 1921 the Swedish mathematician David Enskog suggested an ad hoc equation to describe moderately dense gases. That model has ever since been known as the Enskog equation, and is quite successful in physical applications. It is an equation for the one-particle distribution function of the following type; $(\partial_t + v\partial_x) f = Qf$, where the right hand side describes the evolution due to collisions, and the density f is a function of space x, velocity v, and time t.

The molecules are colliding as billiard balls. In a collision the velocities v, v_* of two colliding molecules are transformed into the velocities v', v'_*, which besides v, v_* also depend on a collision parameter u, conveniently taken on the unit sphere $\mathcal{S}$. The difference from the Boltzmann equation lies in the collision term Qf, equal to the difference between a gain term and a loss term, in the Enskog case

$$Qf = \sigma^2 \int_{R^3 \times \mathcal{S}} (f'f'_- \kappa_- - ff_+ \kappa_+) B dv_* du.$$

Here σ is a radius, $\kappa_\pm$ are high density factors, and the weight B is

$$B(v, v_*, u) = (v - v_*, u)_+ = \max((v - v_*, u), 0).$$

The coordinates in the various densities are as follows;

$$f = f(x, v), \quad f_+ = f(x + \sigma u, v_*), \quad f' = f(x, v'), \quad f'_- = f(x - \sigma u, v'_*).$$

As a comparison the Boltzmann equation has the factor $\sigma^2 \kappa =$ constant and the radius $\sigma = 0$ in the density variables.

The talk is based on two papers "On the convergence of solutions of the Enskog equation to solutions of the Boltzmann equation" by C. Cercignani and L. Arkeryd, to appear in Comm. in PDE, and "On the Enskog equation with large initial data" by L. Arkeryd, which is to appear in SIAM Journ. Math. Anal. For clarity the discussion is limited to the special case of a constant high density factor, the so called Boltzmann-Enskog equation.

We start by constructing approximate solutions when the collision operator Q is substituted by a truncated one Q^j, where the high velocities are removed by inserting an extra cut-off factor $\chi = 1$ for $v^2 + v_*^2 \leq 2^{2j}$, $\chi = 0$ otherwise. Set $f^\#(x,v,t) = f(x+vt,v,t)$ and $F(x,v) = \sup_{0 \leq t \leq T} f^\#(x,v,t)$. The density product in the loss term can be estimated by

$$(ff_+)^\#(x,v,t) \leq F(x,v)F(x+t(v-v_*) + \sigma u, v_*).$$

Also

$$\sigma^2 (v-v_*, u)_+ dudt = dy$$

where $\quad y = x + t(v-v_*) + \sigma u.$ This gives the loss term estimate

$$\int_0^t \int \int \chi ff_+ \sigma^2 (v-v_*,u)_+ dudsdvdv_* dx \leq \sup_{x,v} \int_{M_{xv}} F(y,v_*)dydv_* \int F(x,v)dxdv.$$

Here $\text{vol}(M_{xv})$ is proportional to T, and that makes the sup-factor small for small T. A similar estimates holds for the gain term. Together they imply a contraction mapping argument for local existence, when the initial value is in L_+^1. Assuming initially finite entropy this can be carried into a global result.

The solution of the truncated equation also satisfies the equation in iterated integral form, i.e. $\int_{\Lambda\times R^3} f^{j\#}(t)\phi(t)dxdv = \int_{\Lambda\times R^3} f^j(0)\phi(0)dxdv +$

$+ \int_{\Lambda\times R^3} dxdv \int_0^t (\partial_t\phi f^{j\#})ds + \int_{\Lambda\times R^3} dxdv \int_0^t ds\,\phi(s)(Q^jf^j)^{\#}(s)$ for all $\phi \in C^1(R_+, L^\infty(\Lambda\times R^3))$

with bounded support, and with ϕ in $C^1(R_+)$ for a.e. (x,v). This form is under slight restrictions equivalent to the mild, renormalized, and exponential forms of the diPerna, Lions proof of existence for the Boltzmann eqation via the averaging technique. The talk next discusses a quick way to such averaging results via the iterated integral form, here illustrated through the convergence of the above approximate solutions f^j to a solution of the Boltzmann-Enskog equation in iterated integral form when $j \to \infty$, in periodic physical space $\Lambda = R^3/Z^3$, and with symmetrized weight function $B = |(v-v_*,u)|$. That case is simplified by its having the same entropy properties as the Boltzmann equation.

Theorem. _Assume that the function_ f_0 _satisfies_

$$\sup_{t,j} \int_{\Lambda\times R^3} f^j(x,v,t)(1 + v^2 + |logf(x,v,t)|)dxdv < \infty.$$

Then there exists a function f _in_ $C(R_+, L^1(\Lambda\times R^3))$ _satisfying the Boltzmann-Enskog equation in iterated integral (mild, renormalized, exponential) form and with initial value_ f_0.

This result can be substantially generalized without a great change in the framework. Such generalizations are not yet known for the final questions addressed in the talk, namely those of wellposedness and regularity. Here the setting so far understood is for the original unsymmetrized Boltzmann-Enskog equation with constant high density factor in full physical space. The following result holds.

Theorem. _Suppose that_ $f_0|v|^r$ _belongs to_ L^1_+ _for all_ $r \geq 0$, _and that_ x^2f_0, f_0logf_0 _belong to_ L^1. _Then the Enskog equation with_ $\kappa = constant$ _has a unique solution_ f _on_ R_+ _with_ F _integrable, mass and v-moments conserved, and energy,_ x^2_-moment and_

entropy locally bounded in time. If, moreover, $D^{\alpha}f_0$ *has all v-moments finite in* L^1*-norm for all* $|\alpha| \leq k$, *then the same holds for* $f(t)$.

The proof starts from fairly involved estimates in the L^1-norm for $|v|^r F$, first proving that for large r's and small T's the above sequence f^j is Cauchy in such a norm, and then proceeding to the global result using the control of mass, energy and entropy. Since the equation for a derivative formally is linear in that derivative, the same procedure can then be employed to prove differentiability using the already obtained estimates of the solution.