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A pseudo-differential treatment of general inhomogeneous initial-boundary value problems for the Navier-Stokes equation.

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1. Introduction. Let Ω be a bounded connected open set in $\mathbf{R}^n$ with smooth boundary $\partial\Omega = \Gamma$, and denote by $\vec{n}(x) = (n_1(x), ..., n_n(x))$ the interior unit normal vector field defined near Γ. For a vector field $v = (v_1, \dots, v_n)$ defined near Γ, we denote

$$(1) \qquad v_\nu = \vec{n} \cdot v \ , \qquad v_\tau = v - v_\nu \vec{n};$$

the normal, resp. tangential component. For functions f on $\overline{\Omega}$, we write $f|_\Gamma = \gamma_0 f$ and $\vec{n} \cdot \operatorname{grad} f|_\Gamma = \gamma_1 f$. We denote $\Omega \times]0,T[= Q_T$ and $\Gamma \times]0,T[= S_T$, for some $T > 0$.

The Stokes problem (where $u = (u_1, \dots, u_n)$ stands for the velocity vector and p the pressure):

$$(2) \qquad \begin{array}{lrl} \text{(i)} & \partial_t u - \Delta u + \operatorname{grad} p = f & \text{in } Q_T, \\ \text{(ii)} & \operatorname{div} u = 0 & \text{in } Q_T, \\ \text{(iii)} & u|_{t=0} = u_0 & \text{in } \Omega, \end{array}$$

and the corresponding Navier-Stokes problem (2′), where (2 i) is replaced by

$$(2\text{ i}') \qquad \partial_t u - \Delta u + (u \cdot \operatorname{grad})u + \operatorname{grad} p = f \qquad \text{in } Q_T,$$

have most often been considered with the Dirichlet boundary condition

$$(3_0) \qquad \gamma_0 u = \varphi_0 \quad \text{on } S_T,$$

cf. e.g. Ladyzenskaja [La], Temam [Te], Solonnikov [S 1,2], Giga–Miyakawa [Gi-M] and numerous other works. But also boundary conditions of Neumann type :

$$(3_1) \qquad \chi_1 u - \gamma_0 p \vec{n} = \varphi_1 \quad \text{on } S_T,$$

and of intermediate type:

$$(3_2) \qquad (\chi_1 u)_\tau = \varphi_{2,\tau}, \quad \gamma_0 u_\nu = \varphi_{2,\nu} \quad \text{on } S_T;$$

are of importance (for instance in the study of free boundary problems, cf. e.g. Solonnikov [S 3,4]); here χ_1 is the (normal) first order trace operator defined by

$$(4) \qquad \chi_1 u = \gamma_0 S(u) \vec{n};$$

where the matrix $S(u) = (\partial_i u_j + \partial_j u_i)_{i,j=1,\dots,n}$ is the so-called strain tensor. We observe that $\chi_1 u = \gamma_1 u + \gamma_0 \operatorname{grad} u_\nu - S_0 \gamma_0 u$, with $S_0 = (\gamma_0 \partial_i n_j)_{i,j=1,\dots,n}$; in particular, $(\chi_1 u)_\nu =$

$2\gamma_1 u_\nu$. (Versions of (3_1) and (3_2) with χ_1 replaced by γ_1 can also be considered.) (2 i), resp. (2 i′), can be generalized to

$$\text{(2 i}'') \qquad \partial_t u - \Delta u + Bu + \operatorname{grad} p = f \quad \text{in } Q_T, \quad \text{resp.,}$$

$$\text{(2 i}''') \qquad \partial_t u - \Delta u + Bu + (u \cdot \operatorname{grad})u + \operatorname{grad} p = f \quad \text{in } Q_T,$$

with a first order differential operator $Bu = \sum B_j(x)\partial_j u$; in particular, when the B_j are constant scalars, one gets Oseen's equation [Os]. We denote these problems $(2'')$ resp. $(2''')$.

In the various problems with $k = 0, 1, 2$, the data are assumed divergence free; more precisely:

$$(5_k) \qquad \begin{array}{ll} \operatorname{div} f = 0,\ \operatorname{div} u_0 = 0 & \text{for } k = 1; \\ \operatorname{div} f = 0,\ \gamma_0 f_\nu = 0,\ \operatorname{div} u_0 = 0,\ \gamma_0 u_{0,\nu} = 0,\ \varphi_{k,\nu} = 0 & \text{for } k = 0 \text{ or } 2. \end{array}$$

One of the main difficulties is that the system L

$$(6) \qquad L\begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} \partial_t - \Delta & \operatorname{grad} \\ -\operatorname{div} & 0 \end{pmatrix}\begin{pmatrix} u \\ p \end{pmatrix}$$

is only *degenerate* parabolic, in the sense that its symbol

$$l(\xi,\tau) = \begin{pmatrix} i\tau + |\xi|^2 & \dots & 0 & i\xi_1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & i\tau + |\xi|^2 & i\xi_n \\ -i\xi_1 & \dots & -i\xi_n & 0 \end{pmatrix}$$

(considered for $(\xi,\tau) \in \mathbf{R}^{n+1}$) has the determinant $-(i\tau + |\xi|^2)^{n-1}|\xi|^2$, that is zero not only when $(\xi,\tau) = (0,0)$ but also when $\xi = 0, \tau \neq 0$. Then the standard parabolic theory is not directly applicable; and various techniques have been invented to circumvent the difficulty. Most of the known methods involve working in a space of "solenoidal vectors" (vectors satisfying the condition $\operatorname{div} u = 0$), and one then has to make sure that the reductions, one makes, preserve this space, which can be troublesome e.g. if one wants to change variables, localize by cut-off functions, etc. Moreover, when the boundary problems are considered in terms of symbolic calculus, the fact that the roots in the characteristic polynomial do not stay neatly apart gives difficulties, increasing with the dimension (cf. e.g. Mogilevski [Mo]).

We shall here describe a new method that: 1) eliminates the condition $\operatorname{div} u = 0$, 2) eliminates the need to estimate the position of the roots of the characteristic polynomial. The method consists in some sense of "dividing out" the degeneracy; and the price one pays is that this gives rise to pseudo-differential terms in the equation (more precisely a singular Green operator term); however, the resulting problem is nondegenerate parabolic. Then we can use the general solvability theory for pseudo-differential parabolic boundary problems set up in Grubb [Gr], which gives existence and regularity results for the linearized equation. Solvability of the nonlinear problem for small times or data is obtained from this by suitable iteration procedures.

One type of pseudo-differential method was introduced by Giga [Gi 1] for the Dirichlet problem (2)–(3_0), where the construction of the resolvent of the Stokes operator (acting in the solenoidal space) was reduced to a study of parameter-dependent ps.d.o.s *on the boundary* Γ. Then the resolvent was used to construct fractional powers [Gi 3], and a solution of the Navier-Stokes problem with homogeneous boundary condition was obtained in an L^p setting via semigroup theory [Gi-M], extending methods of Fujita–Kato [Fu-K]. See also Miyakawa [Mi] and [Gi 2] for treatments of variants of the intermediate problem (3_2), and see Fabes–Lewis–Riviere [F-L-R] for a singular integral operator approach. In those problems, only u enters in the boundary condition. General L^p estimates for (3_0) were obtained earlier in Solonnikov [S 2] (for $n = 3$).

Our approach was aimed originally towards the Neumann condition (3_1), but was modified afterwards to include also (3_0) and (3_2). It uses not only ps.d.o.s but also ps.d. boundary operators, "Green operators", as introduced by Boutet de Monvel [BM] and generalized to a parameter-dependent setting in Grubb [Gr]; it transforms the problem to a parabolic initial-boundary value problem in full Sobolev spaces. This leads to solvability of the inhomogeneous problem in a scale of $H^{l,l/2}(Q_T)$ spaces.

So far, the method of pseudo-differential boundary problems has only been developed in the L^2 setting, but as soon as an extension to L^p has been worked out, the reductions we present here would also generalize.

We have here only mentioned works that are directly related to ours, and refer to surveys such as [La], [Te] and Giga [Gi 4] for more complete information.

2. The reduction. The main step in our method is the following reduction:

THEOREM 1. *Each of the problems* (2)–(3_k) *(for* $k = 0, 1$ *or* 2*) with data satisfying* (5_k) *can be reduced to a problem of the form*

$$(7_k)\qquad \begin{array}{lrl} \text{(i)} & \partial_t u - \Delta u + G_k u = f_k & \text{in } Q_T, \\ \text{(ii)} & u|_{t=0} = u_0 & \text{in } \Omega, \\ \text{(iii)} & T_k u = \psi_k & \text{on } S_T; \end{array}$$

where T_k *is a normal differential trace operator,* G_k *is a singular Green operator of the form* $K_{k0}\gamma_0 + K_{k1}\gamma_1$ *with Poisson operators* K_{kj} *(no higher order normal derivatives occur), and the problem* (7_k) *is parabolic in the sense of* [Gr]. *The relation between* (2)–(3_k) *and* (7_k) *is as explained below; in particular, one has that when* u *solves* (7_k) *with* f_k, u_0 *and* ψ_k *satisfying* (5_k), *then* $\operatorname{div} u = 0$.

The normality of T_k and the normal order ≤ 1 for G_k are essential for the applicability of [Gr]; they assure that a certain limit condition on the boundary symbolic problem for $\xi' \to 0$ is satisfied. Besides this, the parabolicity requirements of [Gr] for $\xi' \neq 0$ are easily verified.

Because of the novelty, we explain the reduction for the Neumann type problem (the case $k = 1$) in most detail: Let $\{u, p\}$ be a solution of (2)–(3_1). Application of div to (2 i), and of $\vec{n}\cdot$ to (3_1), shows that p must solve the Dirichlet problem (for each t)

$$(8)\qquad \begin{aligned} &-\Delta p = 0 \quad \text{in } \Omega, \\ &\gamma_0 p = (\chi_1 u)_\nu - \varphi_{1,\nu}\, [= 2\gamma_1 u_\nu - \varphi_{1,\nu}] \quad \text{on } \Gamma. \end{aligned}$$

Thus $p = K_0(2\gamma_1 u_\nu - \varphi_{1,\nu})$, where K_0 is the Poisson operator for the Dirichlet problem for Δ; and insertion of this in (2) leads to (7_1) with

$$(9)\qquad \begin{aligned} G_1 u &= 2 \operatorname{grad} K_0 \gamma_1 u_\nu, & f_1 &= f + \operatorname{grad} K_0 \varphi_{1,\nu}, \\ T_1 u &= \begin{pmatrix} (\chi_1 u)_\tau \\ \gamma_0 \operatorname{div} u \end{pmatrix}, & \psi_1 &= \begin{pmatrix} \varphi_{1,\tau} \\ 0 \end{pmatrix}. \end{aligned}$$

In (7_1), one has dropped the full condition $\operatorname{div} u = 0$ (only $\gamma_0 \operatorname{div} u = 0$ remains), and therefore it is crucial that one can show that divergence free data give a divergence free solution; this is seen by showing that $\operatorname{div} u$ satisfies the homogeneous Dirichlet heat problem when $\operatorname{div} f_1$ and $\operatorname{div} u_0$ are 0.

For the cases (3_0) and (3_2) one finds a Neumann problem for p with data defined from u by applying both div and $\gamma_0 \vec{n}\cdot$ to (2 i). Expressing p by use of the Neumann solution operator K_1 and inserting in (2), one obtains pseudo-differential problems, where the singular Green term has the form $Gu = \operatorname{grad} K_1 \gamma_0(\vec{n} \cdot \Delta u)$. This has a too high normal order, but is further reduced by use of the other equations in (2)–(3_k). The trace operators in (7_k) are here simply taken over from (3_k),

$$T_0 u = \gamma_0 u, \qquad \text{resp.} \qquad T_2 u = \begin{pmatrix} (\chi_1 u)_\tau \\ \gamma_0 u_\nu \end{pmatrix}. \tag{10}$$

To prove that divergence free data give divergence free solutions, one shows that $\operatorname{div} u$ then satisfies a homogeneous Neumann heat problem.

For the more general problems $(2'')$–(3_k) ($k = 0, 1, 2$), the abovementioned reductions work with small modifications (the equation $-\Delta p = 0$ being replaced by $-\Delta p = \operatorname{div} Bu$), and the principal parts of the operators are the same as before.

3. Solution of the linear problems. The general theory of [Gr] shows that (7_k) is solvable in anisotropic Sobolev spaces of the form

$$\begin{aligned} H^{r,s}(Q_T, \underline{E}) &= L^2(\,]0,T[\,; H^r(\Omega, E)) \cap H^s(\,]0,T[\,; L^2(\Omega, E)) \\ &= H^{r,0}(Q_T, \underline{E}) \cap H^{0,s}(Q_T, \underline{E}); \end{aligned} \tag{11}$$

the boundary data given in similar spaces with Ω replaced by Γ, Q_T replaced by S_T. Here E stands for a vector bundle over Ω and $\underline{E}$ is its lifting to $\Omega \times\,]0,T[= Q_T$, etc. The bundles over Ω and Q_T will usually be trivial, e.g. $E = \Omega \times \mathbf{C}^n$, where we shall write $H^r(\Omega, E)$ as $H^r(\Omega)^n$, etc. The bundles over Γ are of the form F_{kj}, $j = 0, 1$, where

$$\begin{aligned} &F_{00} = \Gamma \times \mathbf{C}^n, \quad F_{01} = \Gamma \times \{0\}; \\ &F_{10} = \Gamma \times \{0\}, \quad F_{11} = \Gamma \times \mathbf{C}^n; \\ &F_{20} = \{\,(x,z) \in \Gamma \times \mathbf{C}^n \mid \vec{n}(x) \cdot z = 0\,\}, \quad F_{21} = \Gamma \times \mathbf{C}; \end{aligned} \tag{12}$$

introduced in order to have a unified treatment of the cases $k = 0, 1, 2$.

For $l \geq 0$, one finds that for a set of data $\{f_k, u_0, \psi_k\}$ in the space

$$\mathcal{H}_{l,k} = H^{l,l/2}(Q_T)^n \times H^{l+1}(\Omega)^n \times (H^{l+3/2,\,l/2+3/4}(S_T, \underline{F}_{k0}) \times H^{l+1/2,\,l/2+1/4}(S_T, \underline{F}_{k1})), \tag{13}$$

there will exist a unique solution $u \in H^{l+2,\,l/2+1}(Q_T)^n$ of (7_k), *provided that* the data satisfy a set of compatibility conditions, linking together those traces of the data that have a sense at the "corner" $\Gamma \times \{t = 0\}$:

Let $u^{(j)}$ be defined successively by

$$u^{(0)} = u_0, \qquad u^{(j+1)} = \Delta u^{(j)} - G_k u^{(j)} + \partial_t^j f_k|_{t=0}, \tag{14}$$

then the condition is e.g. for $k = 1$:

$$T_1 u^{(j)} = \partial_t^j \psi_1|_{t=0}, \qquad j = 0, \ldots, [(l - 1/2)/2]. \tag{15}$$

These traces are well defined for $j < [(l-1/2)/2]$, and when $(l-1/2)/2$ is integer, the condition (15) for $j = (l-1/2)/2$ is interpreted to mean that

$$
(16) \qquad \begin{aligned} &I[\partial_t^j \psi_{1,\tau}, (S(u^{(j)})\vec{n})_\tau] < \infty, \quad I[\partial_t^j \psi_{1,\nu}, \operatorname{div} u^{(j)}] < \infty, \quad \text{where} \\ &I[\varrho,\omega] = \int_0^T dt \int_\Gamma dS_x \int_\Omega \frac{|\varrho(x,t)-\omega(y)|^2}{(|x-y|^2+t)^{(n+2)/2}}\,dy. \end{aligned}
$$

When $l < 1/2$, the set of compatibility conditions for (7_1) is empty. For $k = 0$, the compatibility conditions are that $\gamma_0 u^{(j)} = \partial_t^j \psi_0|_{t=0}$ for $j \le [(l+1/2)/2]$. For $k = 2$, the conditions are that $(\chi_1 u^{(j)})_\tau = \partial_t^j \psi_{2,\tau}|_{t=0}$ for $j \le [(l-1/2)/2]$ and $\gamma_0 u_\nu^{(j)} = \partial_t^j \psi_{2,\nu}|_{t=0}$ for $j \le [(l+1/2)/2]$. (The integer cases are interpreted by use of $I(\varrho,\omega)$.)

The results for (7_k) carry over to existence and uniqueness theorems for the original problems (2)–(3_k), when we define p from the other entries by use of K_0 resp. K_1. For the formulation of (5_k), we introduce the solenoidal spaces:

$$
(17) \qquad \begin{aligned} J(\Omega) &= \{\, u \in L^2(\Omega)^n \mid \operatorname{div} u = 0 \,\}, \\ J_0(\Omega) &= \{\, u \in L^2(\Omega)^n \mid \operatorname{div} u = 0,\ \gamma_0 u_\nu = 0 \,\}; \end{aligned}
$$

here $J(\Omega)$ is the L^2-closure of $J(\Omega) \cap C^\infty(\overline{\Omega})^n$, and $J_0(\Omega)$ is the L^2-closure of $J_0(\Omega) \cap C^\infty(\overline{\Omega})^n$ and of $J(\Omega) \cap C_0^\infty(\Omega)^n$. The spaces $J(Q_T)$ and $J_0(Q_T)$ are defined as in (17) with Ω replaced by Q_T (the divergence and γ_0 applied with respect to x). One can show that the orthogonal projections pr_J and pr_{J_0} of $L^2(\Omega)^n$ onto $J(\Omega)$ resp. $J_0(\Omega)$ belong to the calculus of pseudo-differential boundary operators of [BM]; in particular they are continuous in $H^l(\Omega)^n$ for $l \ge 0$ (and the corresponding projections in $L^2(Q_T)^n$ are continuous in $H^{l,l/2}(Q_T)^n$).

Altogether, we find:

THEOREM 2. *Let $k = 0, 1$ or 2, and let $l \ge 0$.*

1° For each set of data $\{f_k, u_0, \psi_k\} \in \mathcal{H}_{l,k}$ satisfying the compatibility conditions described above, (7_k) has a unique solution $u \in H^{l+2,l/2+1}(Q_T)^n$.

2° Let there be given $\{f, u_0, \varphi_k\} \in \mathcal{H}_{l,k}$ such that (5_k) holds, and define f_k and ψ_k from these data as indicated in Section 2. If the data satisfy the compatibility conditions described above, then there is a unique solution $\{u,p\}$ of (2)–(3_k) with

$$
(18) \qquad u \in H^{l+2,l/2+1}(Q_T)^n, \quad p \in H^{l+1,l/2+1/4}(Q_T),
$$

(subject to the side condition $\int_0^T |\int_\Omega p\,dx|^2\,dt = 0$ for $k = 0$ or 2). For $k = 1$, one has moreover that $p \in H^{l+1,l/2+1/2}(Q_T)$ if $\varphi_{1,\nu} = 0$, and for $k = 2$ that $p \in H^{l+2,l/2+3/4}(Q_T)$ if $\varphi_2 = 0$.

One can now also study the evolution operator $S_k(t)$ for (7_k) sending the initial value $u_0(x)$ into the solution $u(x,t)$ when f_k and ψ_k are zero, and draw the consequences for the evolution operator $\widetilde{S}_k(t) : u_0(x) \mapsto u(x,t)$ for (2)–(3_k) with $f = 0$ and $\varphi_k = 0$. Here it is found that $S_k(t)$ satisfies estimates

$$
(19) \qquad \|S_k(t)v\|_{\mathcal{M}_{l,k}} \le C\,t^{-(l-r)/2}\|v\|_{\mathcal{M}_{r,k}} \quad \text{for } 0 \le r \le l,
$$

and $\widetilde{S}_k(t)$ similarly satisfies

$$\|\widetilde{S}_k(t)v\|_{\widetilde{\mathcal{M}}_{l,k}} \leq C\,t^{-(l-r)/2}\|v\|_{\widetilde{\mathcal{M}}_{r,k}} \quad \text{for } 0 \leq r \leq l; \tag{20}$$

where $\mathcal{M}_{l,k}$ is the subspace of $H^l(\Omega)^n$ consisting of the functions satisfying the relevant null compatibility conditions, and $\widetilde{\mathcal{M}}_{l,k}$ equals $\mathcal{M}_{l,k} \cap J(\Omega)$ if $k=1$, resp. $\mathcal{M}_{l,k} \cap J_0(\Omega)$ if $k=0$ or 2. $\mathcal{M}_{l,k}$ and $\widetilde{\mathcal{M}}_{l,k}$ are closed subspaces of $H^l(\Omega)^n$ when $l-1/2$ is not integer, and are a little more complicated when $l-1/2$ is integer (details are given in [Gr-S]).

The theory of [Gr] shows that $S_k(t)$ is a holomorphic semigroup in $L^2(\Omega)^n$ (for $\operatorname{Re} t > 0$), and hence so is $\widetilde{S}_k(t)$ in $J(\Omega)$ resp. $J_0(\Omega)$ for $k=1$ resp. $k=0, 2$.

The various equations and boundary conditions in low regularity spaces of course require a careful interpretation; they are valid in the strong sense (limits of C^∞ functions).

The results extend to the problems (2″)–(3_k); and they also extend to the cases where χ_1 in (3_1) and (3_2) is replaced by γ_1.

4. Solution of the nonlinear problems.

We here describe one of the possible ways to analyze the associated Navier–Stokes problems. In the following, when $l \geq 0$ is fixed, we denote

$$|||u|||_{Q_T} = \|u\|_{H^{l+2,\,l/2+1}(Q_T)^n} + \sup_{t\leq T}\|u\|_{H^{l+1}(\Omega)^n}; \tag{21}$$

it is equivalent to $\|u\|_{H^{l+2,\,l/2+1}(Q_T)^n}$, since $u|_{t=\text{const}}$ is a continuous function of $t \in [0,T]$ valued in $H^{l+1}(\Omega)^n$. For the nonlinear term in (2 i′), (2 i‴), we use the notation

$$\mathbf{K}(u,v) = \sum_{j=1}^{n} u_j \partial_j v, \quad \text{in particular we write } \mathbf{K}(u,u) = \mathbf{K}(u). \tag{22}$$

$\mathbf{K}(u,v)$ can be estimated as follows, by suitable applications of Sobolev inequalities:

THEOREM 3. *Let l be a real nonnegative number with $l+2 > n/2$. There is a constant C such that for any $\varepsilon > 0$ there exists C_ε for which*

$$\begin{aligned}
&\|\mathbf{K}(u,v)\|_{H^{l,\,l/2}(Q_T)} \leq C\,|||u|||_{Q_T}|||v|||_{Q_T},\\
&\left.\begin{aligned}\|\mathbf{K}(u,v)\|_{H^{l,\,l/2}(Q_T)}\\ \|\mathbf{K}(v,u)\|_{H^{l,\,l/2}(Q_T)}\end{aligned}\right\} \leq C\,|||u|||_{Q_T}\big(\varepsilon|||v|||_{Q_T} + C_\varepsilon\int_0^T |||v|||_{Q_t}\,dt\big), \quad \textit{and hence}\\
&\|\mathbf{K}(u,v)\|_{H^{l,\,l/2}(Q_T)} \leq C\,\min(1,\varepsilon + C_\varepsilon T)\,|||u|||_{Q_T}|||v|||_{Q_T}.
\end{aligned} \tag{23}$$

Note here that

$$\mu(\varepsilon,T) = \min(1,\varepsilon + C_\varepsilon T) \tag{24}$$

can be made as small as we want by taking first ε and then $T = T(\varepsilon)$ small enough.

Now consider the Navier–Stokes Problem (2′) with one of the boundary conditions (3_k). Since f is given and u is searched for in $J(Q_T)$, resp. in $J_0(Q_T)$, we may write (2 i′) as

$$\partial_t u - \Delta u + \mathbf{Q}_k(u) + \operatorname{grad} p = f, \tag{2 i$'_k$}$$

where $\mathbf{Q}_k(u) = \mathbf{Q}_k(u,u)$, defined by

$$\mathbf{Q}_k(u,v) = P_k \mathbf{K}(u,v), \qquad \text{with} \tag{25}$$
$$P_k = \mathrm{pr}_J \text{ for } k=1, \qquad P_k = \mathrm{pr}_{J_0} \text{ for } k=0,2.$$

By a reduction similar to that in Section 2, one replaces $(2')$ with a problem like (7_k), now with the first line replaced by

$$\partial_t u - \Delta u + G_k u + \mathbf{Q}_k(u) = f_k; \tag{7 i$'_k$}$$

we call the full problem $(7'_k)$. We also modify the compatibility conditions to the nonlinear case, defining

$$u^{(0)} = u_0, \quad u^{(j+1)} = \Delta u^{(j)} - G_k u^{(j)} - \mathbf{Q}_k(u)^{(j)} + \partial_t^j f_k|_{t=0}, \quad \text{where}$$
$$\mathbf{Q}_k(u)^{(j)} = P_k \sum_{s=0}^{j} \binom{j}{s} \sum_{i=1}^{n} u_i^{(s)} \partial_{x_i} u^{(j-s)}. \tag{26}$$

Now Theorem 2 is applied successively for $m = 0, 1, \ldots$, to the problems

$$\begin{aligned} \partial_t v_{m+1} - \Delta v_{m+1} + G_k v_{m+1} &= f_k - \mathbf{Q}_k(v_m), \\ v_{m+1}|_{t=0} &= u_0, \\ T_k v_{m+1} &= \psi_k; \end{aligned} \tag{28}$$

with v_0 chosen as a vector field satisfying the initial conditions

$$\partial_t^j v_0|_{t=0} = u^{(j)}(x), \quad j = 0, \ldots, [(l+1)/2], \tag{29}$$

and $|||v_0||| \le C' \sum_{j \le [(l+1)/2]} \|u^{(j)}\|_{l+1-2j}$ (suitably interpreted if $(l+1)/2$ is integer). One can then use Theorem 3 to show that $w_m = v_m - v_0$ satisfies a quadratic estimate

$$|||w_{m+1}||| \le C_1[M + \mu(\varepsilon,T)|||v_0|||\,|||w_m||| + \mu(\varepsilon,T)|||w_m|||^2], \tag{30}$$

where $M = \|f_k\|_{H^{l,l/2}(Q_T)} +$ boundary norms, and $\mu(\varepsilon, T)$ is defined in (24). Then, taking T and the norms of the data so small that

$$C_1\mu(\varepsilon,T)|||v_0||| < 1, \qquad (1 - C_1\mu(\varepsilon,T)|||v_0|||)^2 > 4C_1^2 M\mu(\varepsilon,T), \tag{31}$$

one finds that $\sum_m v_m$ converges to a (unique) solution of $(7'_k)$. Thus we obtain for the original problem:

THEOREM 4. *Let l be a nonnegative real number with $l+2 > n/2$, and let $k = 0, 1$ or 2. For each set of data $\{f, u_0, \varphi_k\} \in \mathcal{H}_{l,k}$ satisfying (5_k) and the relevant compatibility conditions, there is a unique solution $\{u, p\}$ in the spaces* (18) *etc., provided that the data and T satisfy the smallness condition* (31).

It should be noted here that for data of arbitrary size, T can always be adapted so that (31) is obtained; and on the other hand, there are cases where T can be $+\infty$.

There are similar results for the problems $(2''')$–(3_k).

Of course, the result is more complicated, the larger l is, because of the nonlinear compatibility conditions stemming from the differential equation. Concerning low values of l, note that since the solution is searched for in $H^{l+2,l/2+1}(Q_T)^n$, the initial data are taken in $H^{l+1}(\Omega)^n$, so they must at least belong to $H^1(\Omega)^n$. This can be relaxed somewhat (depending on the dimension), by different methods based on (19)–(20) and another estimate of $\mathbf{K}(u,v)$, much as in [S 2]; cf. also the results of Fujita and Kato [Fu-K], Sobolevski [So], Giga–Miyakawa [Gi-M], Giga [Gi 2], for the problems with $k = 0$ or 2. In these studies f has a certain t-regularity and the boundary condition is homogeneous, whereas Theorem 4 allows quite general f and nonhomogeneous boundary conditions.

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