Journées Équations aux dérivées partielles

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Journées Équations aux dérivées partielles (1988), p. 1-12
<http://www.numdam.org/item?id=JEDP_1988____A14_0>

RESONANCE FUNCTIONS OF TWO-BODY SCHRÖDINGER OPERATORS

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We consider the Schrödinger operator $-\Delta + V$ in $L^2(\mathbb{R}^n)$, $n \geq 3$, where V is a short-range, dilation-analytic potential in an angle S_α. A resonance λ_0 appears as a discrete eigenvalue of the complex-dilated Hamiltonian [2], a pole of the S-matrix [3] and as a pole of the analytically continued resolvent, acting from an exponentially weighted space to its dual [4,5]. In [2] resonance functions are obtained as square-integrable eigenfunctions of the complex-dilated Hamiltonian, corresponding to the eigenvalue λ_0, in [5] they are defined as certain exponentially growing solutions f of the Schrödinger equation $(-\Delta + V - \lambda_0)f = 0$. In [6] it is proved that for a dilation-analytic multiplicative potential V with resonance λ_0, the resonance functions of [2] and [5] are simply the restrictions of one analytic, $L^2(S^{n-1})$-valued function f on S_α to rays $e^{i\varphi}\mathbb{R}^+$ with $2\varphi > -\mathrm{Arg}\,\lambda_0$ and to $\mathbb{R}^+$, respectively.

Moreover, the precise asymptotic behaviour $f(z) = e^{ik_0 z} z^{\frac{1-n}{2}}(\tau + 0(|z|^{-\varepsilon}))$ with $\tau \in L^2(S^{n-1})$, where $k_0^2 = \lambda_0$, is established together with asymptotics for $f'(z)$. These imply exponential decay in time of resonance states, defined as suitably cut-off resonance functions, as proved in [8].

In this note we shall give a brief account of results on resonance functions, referring for details to [5] and [6].

1. Analytic continuation of resolvent and S-matrix

We introduce the weighted L^2-spaces $L^2_{\delta,b} = L^2_{\delta,b}(\mathbb{R}^n)$ for $\delta, b \in \mathbb{R}$ by

$$L^2_{\delta,b} = \{f \mid \|f\|^2_{\delta,b} = \int_{\mathbb{R}^n} |f(x)|^2 (1+r^2)^{\delta} e^{2br} dx < \infty\}$$

where $x \in \mathbb{R}^n, r = |x|$. The weighted Sobolev spaces are defined by

$$H^2_{\delta,b} = \{f \mid \|f\|^2_{2,\delta,b} = \sum_{|\alpha| \leqq 2} \|D^{\alpha} f\|^2_{\delta,b} < \infty\}$$

We set $L^2_{\delta,0} = L^2_{\delta}$, $H^2_{\delta,0} = H^2$ and $h = L^2(S^{n-1})$, $S^{n-1} = \{x \in \mathbb{R}^n \mid |x| = 1\}$. We assume that the dimension $n \geqq 3$

$$\mathbb{C}^+ = \{k \in \mathbb{C} \mid \operatorname{Im} k > 0\} , \tilde{\mathbb{C}}^+ = \overline{\mathbb{C}^+} \setminus \{0\} .$$

$\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ and $\mathcal{C}(\mathcal{H}_1, \mathcal{H}_2)$ denote the spaces of bounded and compact operators from $\mathcal{H}_1$ into $\mathcal{H}_2$, respectively.

The free Hamiltonian H_0 in L^2 is defined for $u \in \mathcal{D}_{H_0} = H^2$ by $H_0 u = -\Delta u$ with resolvent $R_0(k) = (H_0 - k^2)^{-1} \in \mathcal{B}(L^2)$ for $k \in \mathbb{C}^+$.

The interaction Q is assumed to be a symmetric, short-range, S_α-dilation-analytic operator in L^2 . Thus, $Q \in \mathcal{C}(H^2_{-\delta_0}, L^2_{\delta_0})$ for some $\delta_0 > \frac{1}{2}$, and if $\{U(\rho)\}_{\rho \in \mathbb{R}^+}$ is the dilation group on L^2 defined by

$$(U(\rho)f)(x) = \rho^{\frac{n}{2}} f(\rho x)$$

then the function $Q(\rho) = U(\rho) Q U(\rho^{-1})$ on $\mathbb{R}^+$ has an analytic, $\mathcal{C}(H^2_{-\delta_0}, L^2_{\delta_0})$-valued analytic extension to the angle

$$S_\alpha = \{\rho e^{i\varphi} \mid \rho > 0 , |\varphi| < \alpha\}$$

Moreover, $Q(z) \in \mathcal{C}(H^2_{-\delta_0,b}, L^2_{\delta_0,b})$ for all $b \in \mathbb{R}$. (This follows from $Q \in \mathcal{C}(H^2_{-\delta_0}, L^2_{\delta_0})$ if Q is local).

The Hamiltonian $H = H_0 + Q$ is self-adjoint on $\mathcal{D}_H = H^2$, and associated with H is a self-adjoint, analytic family of type A, $H(z)$, given by

$$H(z) = z^{-2} H_0 + Q(z) ,$$

and $H(\rho e^{i\varphi}) = U(\rho) H(e^{i\varphi}) U(\rho^{-1})$, so $\sigma(H(z)) = \sigma(H(e^{i\varphi}))$ for $\rho > 0$, $z = \rho e^{i\varphi}$.

We define the operators H_z and their resolvents $R_z(k)$ by

$$H_z = H_0 + z^2 Q(z) = z^2 H(z) , \; R_z(k) = (H_z - k^2)^{-1} .$$

We note that $R_z(zk) = z^{-2}(H(z) - k^2)^{-1}$.

We have $\sigma_e(H(z)) = e^{-2i\varphi}\overline{\mathbb{R}^+}$ and $\sigma_d(H(z)) \setminus \mathbb{R} \subset \{\lambda \mid -2\varphi < \operatorname{Arg}\lambda < 0\}$.

We define $\mathcal{R}(\varphi)$ by $\mathcal{R}(\varphi) = \{k \mid 0 > \operatorname{Arg} k > -\varphi , \; k^2 \in \sigma_d(H(z))\}$, $\mathcal{R} = \bigcup_{0<\varphi<\alpha} \mathcal{R}(\varphi)$. The points $\lambda = k^2$, $k \in \mathcal{R}$, are called resonances. For our analysis we need the following result, proved in [5]:

<u>Lemma 1.1</u>. For $\delta > 0$ let $S_\alpha^\delta = \{k \in S_\alpha \mid \operatorname{Im}(e^{i(\alpha-\delta)}k) < \varepsilon\}$. There exists S_α-dilation-analytic interactions V_ε and W_ε with $Q = V_\varepsilon + W_\varepsilon$, such that $H_0 + V_\varepsilon$ has no resonances outside $(S_\alpha^\delta)^2$ and W_ε decays faster than any exponential. This holds with $W_\varepsilon = g_\varepsilon Q g_\varepsilon$, where $g_\varepsilon(r) = \exp(-\varepsilon r^\beta)$, $\beta = \frac{\pi}{2\alpha}$, for ε small.

Using Lemma 1.1 one can prove all results for fixed $\delta > 0$ with S_α replaced by $S_\alpha - S_\alpha^\delta$, using the splitting $Q = V_\varepsilon + W_\varepsilon$, and then let $\delta \downarrow 0$. To simplify the presentation, we assume from the outset (although this can strictly speaking not be obtained) that $H_1 = H_0 + V$ has no resonances and fix ε, setting $g = g_\varepsilon$, $W = Qg$, $V = Q - gW$. We denote by H_{1z}, $R_{1z}(k)$ etc. the operators obtained by replacing Q by V.

Basic to our approach is an extended limiting absorption principle proved in [7] and generalized in [5] to non-symmetric, short-range potentials like Q_z. The idea is to consider $-\Delta$ and $-\Delta + Q_z$ as operators H_0^{-b} and H_z^{-b} acting in the space $L^2_{0,-b}$, $b \geq 0$. The spectrum of H_0^{-b} and the essential spectrum of H_z^{-b} coincide with the parabolic region $P_b = \{k^2 \mid |\mathrm{Im}\,k| \leq b\}$, and it is then proved that the resolvents $(H_0^{-b} - (a+ib+i\varepsilon)^2)^{-1}$ and $(H_z^{-b} - (a+ib+i\varepsilon)^2)^{-1}$ have boundary values as $\varepsilon \downarrow 0$ in $B(L^2_{\delta,-b}, H^2_{-\delta-b})$ for $\frac{1}{2} < \delta \leq \delta_0$, except at the so-called singular points.

The singular sets Σ_z^c, Σ_z^r and Σ_z are defined for $z = \rho e^{i\varphi}$, $\varphi > 0$, by

$$\Sigma_z^c = \{k \in \mathbb{C}^+ \mid k^2 = z^2\lambda \ , \ \lambda \in \sigma_d(H(z))\} \ ,$$

$$\Sigma_z^r = zR \cap \mathbb{R}^+ \ , \quad \Sigma_z = \Sigma_z^c \cup \Sigma_z^r \ ,$$

and for $\varphi < 0$ by $\Sigma_z^c = -\overline{\Sigma_{\bar{z}}^c}$ and similar for Σ_z^r and Σ_z.
For $\varphi = 0$, $\Sigma = \Sigma^c \cup \Sigma^r = \{k \in \tilde{\mathbb{C}}^+ \mid k^2 \in \sigma_p(H)\}$.

The extended limiting absorption principle for H_z can then be formulated as follows:

Theorem 1.2. For fixed $z \in S_\alpha$, $0 < \delta \leq \delta_0$, there exists a meromorphic $B(L^2_\delta, H^2_{-\delta})$-valued function $R_z^-(k)$ in $\mathbb{C}^+$, continuous in $\mathbb{C}^+ \setminus \Sigma_z$, such that for $k \in \mathbb{R} \setminus \Sigma_z^r \cup \{0\}$

$$R_z^-(k) = e^{ikr} R_z(k+i0) e^{-ikr}$$

where

$$R_z(k+i0) = \lim_{\varepsilon \downarrow 0} R_z(k+i\varepsilon)$$

in the operator-norm topology of $B(L^2_\delta, H^2_{-\delta})$, locally uniformly in k.

For $f \in L^2_{\delta,-b}$, $u = e^{-ikr} R_z^-(k) e^{ikr} f$ is the unique solution in $L^2_{\delta,-b}$ of the equation $(H_z^{-b} - k^2) u = f$, such that $\mathcal{D}u \in L^2_{\delta-1,-b}$, where $b = \mathrm{Im}\, k$ and

$$\mathcal{D}u = (\mathcal{D}_1 u, \ldots, \mathcal{D}_n u) \ , \ \mathcal{D}_j = \frac{\partial}{\partial x_j} + \frac{n-1}{2r^2} x_j - ik \frac{x_j}{r}$$

(the radiation condition).

<u>Proof</u>. We refer to [5] for the proof of the Theorem. It utilizes the result of [7] for H_0 , analytic Fredholm theory and control of the singular points using analyticity in k and z .

The trace operators $T_0(k)$, $T_z(k) \in \mathcal{B}(L^2_\delta, h)$ are defined for $z \in S_\alpha$, by

$$(T_0(\pm k) f)(k,\cdot) = (F_\pm f)(k,\cdot) \ , \quad k \in \mathbb{R}^+$$

where

$$(F_\pm f)(k,\omega) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{\mp ik\omega\cdot x} f(x)\, dx \ ,$$

$$T_z(k) = T_0(k)(1 - Q_z R_z(k+i0)) \ , \quad k \in \mathbb{R} \setminus \Sigma_z^r \ .$$

We set

$$T_0(k) = T_0(k) e^{ikr} \ , \ T_z^+(k) = T_z(k) e^{ikr} \ .$$

The following result is proved in [5].

<u>Theorem 1.3</u>. For $\frac{1}{2} < \delta \leqq \delta_0$, $z \in S_\alpha$, the $\mathcal{B}(L^2_\delta, h)$-valued function $T_z^+(k)$ has a continuous extension to $\tilde{\mathbb{C}}^+ \setminus \Sigma_z$ meromorphic in $\mathbb{C}^+$ with poles at Σ_z^c . The function $T_{\bar{z}}^{+*}(\bar{k})$ defined for $k \in \tilde{\mathbb{C}}^- \setminus (-\Sigma_z)$ is analytic in $\mathbb{C}^- \setminus (-\Sigma_z^c)$ and continuous in $\tilde{\mathbb{C}}^- \setminus (-\Sigma_z)$ as a $\mathcal{B}(h, H^2_{-\delta})$-valued function.

We recall the following formulas from the stationary scattering theory:

$$R(k+i0) = R(-k+i0) + \pi i k^{n-2}\, T^*(k) T(k) \ , \ k \in \mathbb{R}^+ \setminus \Sigma_r \qquad (1.2)$$

$$T(k) = S(k)RT(-k) \tag{1.3}$$

where $(R\tau)(\omega) = \tau(\omega)$ for $\tau \in h$.

Inserting (1.3) in (1.2), we obtain

$$R(k+i0) = R(-k+i0) + \pi i k^{n-2} T^*(k)S(k)RT(-k) \tag{1.4}$$

The S-matrix $S(k)$ of (H_0,H) is given for $k \in \mathbb{R}^+ \setminus \Sigma_r$ by

$$S(k) = 1 - \pi i k^{n-2} T_0(k)(Q - QR(k+i0)Q)T_0^*(k) \tag{1.5}$$

and the S-matrix $S_1(k)$ of (H_0,H_1) by (1.5) with Q and R replaced by V and R_1 .

The following result is proved in [3].

<u>Theorem 1.4</u>. The S-matrix $S(k)$ has a meromorphic extension $\tilde{S}(k)$ from $\mathbb{R}^+$ to S_α with poles at $\mathcal{R}$. The S-matrix $S_1(k)$ has an analytic extension $\tilde{S}_1(k)$ from $\mathbb{R}^+$ to S_α . Moreover, for $k>0$, $0<\varphi<\alpha$, $\tilde{S}_1(ke^{-i\varphi}) = S_{1,e^{i\varphi}}(k)$, where $S_{1,e^{i\varphi}}(k)$ is the S-matrix of $(H_0,H_{1,e^{i\varphi}})$ at the point k .

From (1.4) and Theorem 1.2 we obtain

<u>Theorem 1.5</u>. For any $b>0$ the $\mathcal{B}(L^2_{0,b}, H^2_{0,-b})$-valued function $R(k)$ has a meromorphic continuation $\tilde{R}(k)$ from $\mathbb{C}^+$ across $\mathbb{R}^+$ to $S_{\alpha,b} = \{k \in S_\alpha \mid -b < \operatorname{Im} k < 0\}$, given by

$$\tilde{R}(k) = R(-k) + \pi i k^{n-2} T^*(\bar{k})\tilde{S}(k)T(-k) \tag{1.6}$$

The $\mathcal{B}(L^2_{0,b}, H^2_{0,-b})$-valued function $R_1(k)$ has an analytic continuation $\tilde{R}_1(k)$ from $\mathbb{C}^+ \setminus \Sigma_{1c}$ across $\mathbb{R}^+$ to $S_{\alpha,b}$, given by (1.6) with R,T and S replaced by R_1, T_1 and S_1 .

The functions $\tilde{R}(k)$ and $\tilde{R}_1(k)$ are connected by the analytically continued symmetrized resolvent equation

$$\tilde{R}(k) = \tilde{R}_1(k) - \tilde{R}_1(k)g(1 + W\tilde{R}_1(k)g)^{-1} W\tilde{R}_1(k) \tag{1.7}$$

The following result is proved in [5]:

Theorem 1.6. $\widetilde{R}(k)$ and $\widetilde{S}(k)$ have the same poles and of the same order in $S_{\alpha,b}$.

2. Resonance functions

Let k_0^2 be a resonance, and fix $b > -\mathrm{Im}\,k_0$. Then k_0 is a pole of $\tilde{R}(k) \in \mathcal{B}(L^2_{0,b}, H^2_{0,-b})$, defined in Theorem 1.5. Let C be a circle separating k_0 from other poles and let

$$P = -\frac{1}{2\pi i}\int_C \tilde{R}_2(k)\,dk^2$$

be the residue of $\tilde{R}_2(k)$ at k_0, $P \in \mathcal{B}(L^2_{0,b}, H^2_{0,-b})$ is of finite rank.

The space $\mathcal{F}$ of resonance functions associated with k_0 is defined by

$$\mathcal{F} = \{f \in \mathcal{R}_P \mid (-\Delta + Q - k_0^2)f = 0\} \quad .$$

The following result is proved in [5]:

<u>Theorem 2.1</u>. $\mathcal{F}$ is the isomorphic image of $N(\tilde{S}^{-1}(k_0))$ and of $N(1 + W\tilde{R}_1(k_0)g)$ via the following maps:

$$N(\tilde{S}^{-1}(k_0)) \ni \tau \to T^*(\bar{k}_0)\tau = f \in \mathcal{F}$$

$$N(1 + W\tilde{R}_1(k_0)g) \ni \phi \to \tilde{R}_1(k_0)g\phi = f \in \mathcal{F}$$

<u>Remark</u>. From the representation $f = T^*(\bar{k}_0)\tau$ we conclude by Theorem 1.3 and the uniqueness part of Theorem 1.2 that $f \in H^2_{-\delta,-b_0} \setminus L^2_{\delta-1,-b_0}$ for every $\delta > \frac{1}{2}$ and $b_0 = -\mathrm{Im}\,k_0$. A further analysis yields precise asymptotic estimates. We first establish the analyticity properties, using the second isomorphism.

Applying (1.4) to the operator H_{1z} at a point zk_0 with $\mathrm{Arg}\,zk_0 = 0$ and noting that by Theorem 1.4, $S_{1z}(zk_0) = \tilde{S}_1(k_0)$ we obtain

$$R_{1z}(zk_0 + i0) = R_{1z}(-zk_0 + i0) + \pi i (zk_0)^{n-2}$$
$$T^*_{1\bar{z}}(\overline{zk}_0)\ \tilde{S}_1(k_0)\ RT_{1z}(-zk_0) \tag{1.7}$$

By Theorems 1.2 and 1.3 we obtain from (1.7)

<u>Theorem 2.2</u>. The $\mathcal{B}(L^2_\delta, H^2_{-\delta})$-valued function $e^{-izk_0 r}$ $R_{1z}(zk_0 + i0)e^{-izk_0 r}$ has an analytic extension from $\{z \in zk_0 \mid \mathbb{R}^+\}$ to $\{z \in S_\alpha \mid \operatorname{Arg} zk_0 < 0\}$, given by

$$e^{-izk_0 r}\tilde{R}_{1z}(zk_0)e^{-izk_0 r} = e^{-izk_0 r} R_{1z}(-zk_0)e^{-izk_0 r} +$$
$$\pi i (zk_0)^{n-2}\ T^*_{1z}(\overline{zk}_0)\tilde{S}_1(k_0)RT_{1z}(-zk_0) \tag{1.8}$$

Recalling that $W_z = Q_z\, g(rz)$, where $g(rz) = \exp\{-\varepsilon(rz)^\beta\}$ with $\beta > 1$, we obtain from Theorem 2.2

<u>Theorem 2.3</u>. The $\mathcal{C}(L^2)$-valued function $W_z R_{1z}(zk_0)g(rz)$ has an analytic continuation from $\{z \in S_\alpha \mid \operatorname{Arg} zk_0 > 0\}$ to $\{z \in S_\alpha \mid \operatorname{Arg} zk_0 \leqq 0\}$, given by $W_z\tilde{R}_{1z}(zk_0)g(rz)$.

By standard dilation-analytic arguments $\sigma(W_z\tilde{R}_1(zk_0)g(rz)$ is constant. Let C be a circle separating -1 from the rest of $\sigma(W_z\tilde{R}_1(zk_0)g(rz))$ and set

$$P(z) = -\frac{1}{2\pi i}\int_C (-\lambda + W_z\tilde{R}_{1z}(zk_0)g(rz))^{-1}\, d\lambda\ .$$

Then $P(z)$ is a dilation-analytic $\mathcal{B}(L^2)$-valued function of z, and $P(z)$ is a projection on the finite-dimensional algebraic null space of $1 + W_z\tilde{R}_{1z}(zk_0)g(rz)$. Let $\phi \in N(1 + W\tilde{R}_1(k_0)g(rz))$ and pick an S_α-dilation-analytic vector η in L^2 such that

$\phi = P(1)\eta$. Then $\phi(z) = P(z)\eta(z) \in N(1 + W_z\tilde{R}_{1z}(zk_0)g(rz)$, and $\phi(z)$ is dilation-analytic.

We now obtain, using the second isomorphism of Theorem 2.1,

<u>Theorem 2.4</u>. Let $f \in F$. Then there exists an S_α-dilation-analytic, $H^2_{-\delta}$-valued function $\chi(z)$, such that $f = e^{ik_0r}\chi(1)$ and for $\operatorname{Arg} z\, k_0 > 0$

$$f(z) = e^{ik_0zr}\chi(z) \in N(H(z) - k_0^2) .$$

Moreover, $\chi(z) \notin L^2_{\delta-1}$ for all $z \in S_\alpha$, $\delta > \frac{1}{2}$.

<u>Proof</u>. Define $f(z)$ by

$$f(z) = \begin{cases} e^{izk_0r} R^+_{1z}(zk_0)\, e^{-izk_0r} g(rz)\phi(z) , & \operatorname{Im} zk_0 > 0 \\ e^{izk_0r}\left(e^{-izk_0r}\tilde{R}_{1z}(zk_0)e^{-izk_0r}\right)e^{izk_0r} g(rz)\phi(z) , & \operatorname{Im} zk_0 \leqq 0 \end{cases}$$

where $R^+_{1z}(zk_0)$ is defined similarly to $R^-_{1z}(zk_0)$, replacing $-b$ by b and $e^{\pm iar}$ with $e^{\mp iar}$ in Theorem 1.2. Clearly, $f(z)$ is continuous for $zk_0 \in \mathbb{R}^+$. By Theorem 1.2 and 2.2, $\chi(z) = e^{-izk_0r}f(z)$ is an analytic $H^2_{-\delta}$-valued function in S_α .

It follows from the uniqueness part of Theorem 1.2 that $\chi(z) \notin L^2_{\delta-1}$ for $\operatorname{Im} zk_0 < 0$. The fact that $\chi(z) \notin L^2_{\delta-1}$ for $\operatorname{Im} zk_0 \geqq 0$ then follows by the next Lemma, proved in [6]:

<u>Lemma 2.5</u>. Let $\chi(z)$ be an S_α-dilation-analytic vector, and define $h(\varphi)$ for $\varphi \in (-\alpha,\alpha)$ by

$$h(\varphi) = \inf\{s \mid \chi(e^{i\varphi}) \in L^2_{-s}\} .$$

Then either $h(\varphi) \equiv -\infty$ or $h(\varphi) > -\infty$ and h is convex in $(-\alpha,\alpha)$.

Using this Lemma together with a recent result of Agmon [1], giving the precise asymptotic behaviour of $f(z)$ for $\operatorname{Arg} zk_0 > 0$, we finally obtain the desired asymptotic estimates of f and f'. We refer to [6] for the proof.

<u>Theorem 2.6</u>. Assume that Q is an S_α-dilation-analytic multiplicative potential such that $|Q(z)(x)| \leqq C|x|^{-1-\varepsilon}$ for $z \in S_\alpha$ and $|x| \geqq R$. Let $f \in \mathcal{F}$. Then f is an analytic, h-valued function $f(z,\cdot)$ on S_α of the form

$$f(z,\cdot) = e^{ik_0 z} z^{\frac{1-n}{2}} g(z,\cdot)$$

where

$$g(z,\cdot) = \tau + 0(|z|^{-\varepsilon})$$

$$g'(z,\cdot) = 0(|z|^{-1-\varepsilon})$$

uniformly in any smaller angle S'_α for some $\varepsilon > 0$. Moreover, $\tau \in N(\tilde{S}^{-1}(k_0))$ and $f = CT^*(\bar{k}_0)\tau$,

$$C = k_0^{\frac{n-1}{2}} (-i)^{\frac{1-n}{2}} (2\pi)^{\frac{1}{2}} .$$

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