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Estimates for a number of negative eigenvalues of the Schrödinger operator with intensive magnetic field

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Estimates for a number of negative eigenvalues of the Schrödinger operator with intensive magnetic field.

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1. In this lecture I give estimates from above and from below for a maximal dimension $N \leq \infty$ of a linear subspace $\mathcal{L} \subset C_0^\infty(X)$ on which a quadratic form

$$Q(u) = \int \Big[g^{jk}(D_j - V_j)u \cdot \overline{(D_k - V_k)u} + V|u|^2 \Big] dx \tag{1}$$

is negative definite; here X is a domain in $\mathbb{R}^d$, $d = 3$, $g^{jk} = g^{kj}$, V_j, V are real-valued and g^{jk}, $g^{jk}V_j$, $g^{jk}V_jV_k + V \in L^1_{loc}(X)$; we use Einstein' summation rule. Then Q is correctly defined on $C_0^\infty(X)$. Let us assume that

(H_1) $\quad c^{-1} \leq |\xi|^{-2} g^{jk}(x)\xi_j\xi_k \leq c \quad \forall x \in X,\ \xi \in \mathbb{R}^d \setminus 0$

and there are given functions γ, $\rho = \rho_0$, ρ_1, ... on X such that

(H_2) $\quad \rho_m(x) \geq 0,\ \gamma(x) \geq 0,\ |\gamma(x) - \gamma(y)| \leq |x - y|$

and for every point $y \in X' = \{X,\ \rho\gamma > 1\}$ in $X \cap B(y, \gamma(y))$ the following conditions are fulfilled:

(H_3) $\quad c^{-1} \leq \rho_m(x)/\rho_m(y) \leq c, \quad |D_j\rho_m| \leq c\rho_m/\gamma$,

$\qquad m = 0, 1, \dots, \quad \rho_1 \geq \rho/\gamma, \quad \rho \geq \rho_2 \geq 1/\gamma$,

(H_4) $\quad |D^\beta g^{jk}| \leq c\gamma^{-|\beta|}$,

$\qquad |D^\beta F^j| \leq c\,\rho_1\gamma^{-|\beta|}$,

$\qquad |D^\beta V| \leq c\,\rho^2\gamma^{-|\beta|} \qquad \forall \beta: \ |\beta| \leq K < \infty$,

(H_5) $\quad \partial X \cap B(y, \gamma(y)) = \{x_k = z(x_{\hat{k}})\} \cap B(y, \gamma(y))$ with

$\qquad |D^\beta z| \leq c\gamma^{1-|\beta|} \quad \forall \beta: \ |\beta| \leq K$

where $B(y, \gamma)$ is an open ball with a center y and a radius γ, $x_{\hat{2}} = (x_1, x_3)$ etc, $k = k(y) = 1, 2, 3$ in (H_5), $F^j = i\,\varepsilon^{jkl}D_kV_l$

are components of the vector intensity of the magnetic field, ε^{jkl} is absolutely skew-symmetric pseudo-tensor with $\varepsilon^{123} = 1/\sqrt{g}$, $g = \det(g_{jk})$, $(g_{jk}) = (g^{jk})^{-1}$. Let $F = (g_{jk}F^jF^k)^{1/2}$ be a scalar intensity of the magnetic field. Let us further assume that

$$(H_6) \quad \forall\, y \in \{X',\ \rho_1 > c\,\rho/\gamma\} \quad \text{in } X \cap B(y, \gamma(y)) \quad F \geq c^{-1}\rho_1,$$

$$(H_5)' \quad \forall\, y \in \{X',\ \rho_1 > c\,\rho/\gamma,\ V + F \leq \varepsilon\rho^2\} \quad B(y, \gamma(y)) \subset X$$

with $\varepsilon > 0$.

Moreover, let us assume that for every $y \in X''' = \{X',\ \rho_1 > c^{-1}\rho^2,\ \rho_2 < c_1^{-1}\rho,\ V + F \leq \varepsilon\rho^2\}$ in $B(y, \gamma(y))$ the following inequalities are fulfilled:

$$(H_7) \quad |D^\beta(V + (2j+1)F)| \leq c\,\rho_2^2\gamma^{-|\beta|} \quad \forall\, \beta: |\beta| \leq K$$

with $j = j(y) \in \mathbb{Z}^+$, $c_1 > 8c$; then X''' is a union of the disjoint domains X_j'''.

Let

$$\gamma_1 = \gamma^2 |\nabla(V/F)|\,\rho_1/\rho^2 + \gamma \min_{j \in \mathbb{Z}^+} |V + (2j+1)\mu hF|^{1/2}/\rho \qquad (\mu = h = 1 \text{ here})$$

on X' and $\gamma_2 = \gamma_1\,\rho/\rho_2$ on X'''.

Finally, let us assume that

$$(H_8) \quad Q(u) \geq c^{-1}\int (|\nabla u|^2 - W|u|^2)\,dx \qquad \forall\, u \in C_0^\infty(X'')$$

where $X'' = \{X,\ \rho\gamma < 2\}$, $W \in L^1_{loc}(X)$, $W \geq 0$.

Our the first principal result is

Theorem 1. Let conditions (H_1)-(H_8) be fulfilled. Then

$$\mathcal{N} - CR_1 - C'R_2 \leq N \leq \mathcal{N} + C(R_1 + R_3) + C'R_2$$

where

$$\mathcal{N} = (1/2\ \pi^2) \sum_{j=0}^{\infty} \int_{X'} (V + (2j+1)F)_-^{1/2}\ F\sqrt{g}\, dx\ ,$$

$$z_{\pm} = \max\,(\pm z,\ 0), \quad R_1 = \sum_{t=1}^{5} R_{1t}\ , \quad R_2 = \sum_{t=1}^{2} R_{2t}\ ,$$

$$R_{11} = \int_{\{X' \setminus X_0''',\ V+F\ \leq\ \varepsilon\rho^2\}} \rho^2 \gamma^{-1}\, dx\ ,$$

$$R_{12} = \int_{\{X' \setminus X''',\ V+F\ \leq\ \varepsilon\rho^2,\ \gamma_1\ \geq\ \rho^{-1/2+\sigma}\gamma^{1/2+\sigma}\}} \rho_1 \gamma_1^{-1}\, dx\ ,$$

$$R_{13} = \int_{\{X' \setminus X''';\ V+F\ \leq \varepsilon\rho^2,\ \gamma_1\ \leq\ \rho^{-1/2+\sigma}\gamma^{1/2+\sigma}\}} \rho_1 \rho^{1/2+\sigma}\gamma^{-1/2+\sigma}\, dx\ ,$$

$$R_{14} = \sum_{j=0}^{\infty} \int_{\{X_j''',\ V+(2j+1)F\ \leq\ \varepsilon\rho_2^2,\ \gamma_2\ \geq\ \rho_2^{-1/2+\sigma}\gamma^{1/2+\sigma}\}} \rho^2 \gamma_2^{-1}\, dx\ ,$$

$$R_{15} = \sum_{j=0}^{\infty} \int_{\{X_j''',\ V+(2j+1)F\ \leq\ \varepsilon\rho_2^2,\ \gamma_2\ \leq\ \rho_2^{-1/2+\sigma}\gamma^{1/2+\sigma}\}} \rho^2 \rho_2^{1/2+\sigma}\gamma^{-1/2+\sigma}\, dx\ ,$$

$$R_{21} = \int_{X'} \rho^3 \rho_1^{-s}\gamma^{-2s}\, dx\ ,$$

$$R_{22} = \int_{X'''} \rho^2 \rho_2^{-s}\gamma^{-1-s}\, dx\ ,$$

$$R_3 = \int_{X''} W^{3/2}\, dx\ ,$$

here and in what follows $\varepsilon > 0$, $\sigma > 0$, s are arbitrary and $C = C(c)$, $C' = C'(c, c_2, \varepsilon, \sigma, s)$, $K = K(\sigma, s)$ in (H_4), (H_7).

Remark 2. If conditions (H_1)-(H_8) are fulfilled and if $\mathcal{N} + R_1 + R_2 + R_3 < \infty$ then Q is semi-bounded from below on $L^2(X)$ and hence it generates a self-adjoint Schrödinger operator $A = (D_j - V_j)g^{jk}(D_k - V_k) + V$ on X with the Dirichlet boundary condition; then N is a dimension of its invariant negative subspace.

Theorem 1 is a more refined and general version of the principal theorems announced in [1,2]. Moreover, under a certain condition of a global nature concerning integral curves of the vector field (F^1, F^2, F^3) one can derive a more precise estimates. If A depends on parameters then theorem 1 implies asymptotics of N with respect to these parameters (see e.g. [1,2]).

2. The following assertion is the crucial step in the proof of theorem 1:

Theorem 3. Let

$$A_{\mu,h} = (hD_j - \mu V_j)g^{jk}(hD_k - \mu V_k) + V \tag{2}$$

with the Dirichlet boundary condition be a self-adjoint semi-bounded Schrödinger operator with the discrete spectrum an with the polynomial growth of the eigenvalue counting function $N(\lambda)$ as $\lambda \to \infty$; here $h \in (0,1]$, $\mu \geq 1$; let $e(x,y,\lambda,\mu,h)$ be a Schwartz kernel of its spectral projector. Let $y \in X$ and in $B(y,1) \subset X$ conditions (H_1)-(H_6) be fulfilled with $\gamma = \rho = \rho_1 = 1$; moreover, let $\psi \in C_0^K(B(y,1/2))$, $0 \leq \psi \leq 1$, $|D^\beta \psi| \leq c_2 \gamma^{-|\beta|}$ $\forall \beta : |\beta| \leq K$.
Then

(i) The following estimate holds:

$$\mathcal{R} = \left| \int (e(x,x,0,\mu,h) - S(x,\mu h)h^{-d}) \ \psi^2(x)\, dx \right| \leq$$

$$Ch^{-2}(1 + \mu h \int_{\{B(y,1),\ \gamma_1 \geq h^{1/2-\sigma}\}} \gamma_1^{-1}\, dx \quad +$$

$$\mu h^{1/2-\sigma} \operatorname{mes} \{B(y,1),\ \gamma_1 \leq h^{1/2-\sigma}\}) + C'h^{-1}$$

where

$$S(x,\mu h) = (1/2\pi^{d-1}) \sum_{j=0}^{\infty} (V + (2j+1)\mu hF)_{-}^{(d-2)/2} \mu hF \sqrt{g}.$$

(ii) If $V + \mu hF \geq \varepsilon$ in $X \cap B(y,1)$ (and not necessarily $B(y,1) \subset X$ here) then

$$e(x,x,0,\mu,h) \leq C'h^{s}\mu^{-s} \qquad \forall\, x \in X \cap B(y,1/2).$$

(iii) If $V = -(2j+1)F + \zeta^2 V'$ with $\zeta \in (h,1]$ and $j \in \mathbb{Z}^+$ and if V' satisfies $(H_4)_3$ with $\gamma = \rho = 1$ then for $\mu = h^{-1}$

$$\mathcal{R} \leq Ch^{-2}(1 + \int_{\{B(y,1),\ \gamma_2 \geq (h/\zeta)^{1/2-\sigma}\}} \gamma_2^{-1}\, dx \quad +$$

$$(h/\zeta)^{-1/2-\sigma} \operatorname{mes} \{ B(y,1),\ \gamma_2 \leq (h/\zeta)^{1/2-\sigma}\}) + C'h^{-1}\zeta^{-1} ;$$

here $\rho_2 = \zeta$ in the definition of γ_2.

(iv) Moreover, if $V' \geq \varepsilon$ in $B(y,1)$ then

$$e(x,x,0,\mu = h^{-1},h) \leq C'h^{-2}(h/\zeta)^{s} \qquad \forall\, x \in B(y,1/2).$$

The proof of theorem 3 is complicated and it is based on the quasiclassical microlocal analysis of the non-stationary Schrödinger equation with parameters μ, h. Without conditions $(H_5)'$,(H_6) the similar assertion holds for $\mu \leq c$, $d \geq 2$ and it is the basis of the proof of the principal theorems of [3] . When all these asymptotics are established we generalize them first to arbitrary balls and then coplete the proof of theorem 1 by means of an appropriate partition of unity and Rosenblyum variational estimate for an eigenvalue counting function for operator generated by Q in $L^2(X'', Jdx)$ with an admissible weight function J. One can find the similar procedure in [4].

3. Let us consider now the case $d = 2$. Certainly, now $F = F^3$, $F^3 = i(D_1V_2 - D_2V_1)/\sqrt{g}$. This case is not completely investigated yet. However I have proved the following

Theorem 4. Let $d = 2$ and all the conditions of theorem 3 be fulfilled. Then

(i) If $\gamma_1 \geq c^{-1}$ then

$$\mathcal{R} \leq C\mu^{-1}h^{-1} + C'\mu^{-2}.$$

(ii) Assertion (ii) of theorem 3 holds.

(iii) If $V = -(2j+1)F + \zeta^2 V'$ with $\zeta \in (h,1]$ and $j \in \mathbb{Z}^+$ and if V' satisfies $(H_4)_3$ with $\gamma = \rho = 1$ and if $\gamma_2 \geq c^{-1}$ (here $\rho_2 = \zeta$ in the definition of γ_2) then

$$\mathcal{R} \leq C\zeta^{-2} + C'h\zeta^{-1}.$$

(iv) On the other hand, if

$$|V + (2j+1)\mu hF| \geq \zeta^2 \geq C\mu^{-2} \quad \forall j \in \mathbb{Z}^+ \quad \forall x \in B(y,1)$$

then

$$|e(x,x,\lambda_2,\mu,h) - e(x,x,\lambda_1,\mu,h)| \leq C'(h/\zeta)^s$$
$$\forall x \in B(y,1/2) \quad \forall \lambda_1, \lambda_2 \in (-\zeta^2/C, \zeta^2/C) \qquad (3).$$

(v) Moreover, if

$$|V + (2j+1)\mu hF + \mu^{-2}RV^2/8F| \geq \zeta^2 \geq \varepsilon\mu^{-2}$$
$$\forall j \in \mathbb{Z}^+ \quad \forall x \in B(y,1)$$

where R is a scalar curvature associated with the metrics (g_{jk}/F) and if $\mu^{-1} + \mu h \leq \delta = \delta(c,\varepsilon)$ then inequality (3) holds.

Let us note that in the two-dimensional case the presence of the intensive magnetic field can improve the remainder estimate; on the other hand, in this case the gaps in the quasiclassical limit of the spectrum can appear. The role of Landau levels $E_j = V + (2j+1)\mu hF$ is more important in the two-dimensional case than in three-dimensional one; there is a correction $\Delta E_j = \mu^{-2}R(E_j - V)^2/8F$ to these levels.

4. Finally, it is well-known that if $X = \mathbb{R}^d$, $d = 2,3$, g^{jk}, F^j are constant and $V = 0$ then

$$e(x,x,\lambda,\mu,h) =$$

$$(1/2\pi^{d-1}) \sum_{j=0}^{\infty} (\lambda - (2j+1)\mu hF)_+^{(d-1)/2} \mu h^{1-d} F\sqrt{g};$$

in particular,

$$\sigma(A) = \sigma_{ac}(A) = [\mu hF, \infty) \qquad \text{for} \quad d = 3,$$

$$\sigma(A) = \sigma_{ess}(A) = \sigma_{pp}(A) = \{(2j+1)\mu hF, \ j \in \mathbb{Z}^+\}$$

for $d = 2$.

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