JOURNÉES ÉQUATIONS AUX DÉRIVÉES PARTIELLES

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Journées Équations aux dérivées partielles (1987), p. 1-17

<http://www.numdam.org/item?id=JEDP_1987____A19_0>

ILL POSED CAUCHY PROBLEMS FOR IDEAL GAS EQUATIONS AND THEIR REGULARIZATION

par V.P. MASLOV

Exposé n° XIX

The classical Cauchy problems for linear equations of mathematical physics are well posed. But for nonlinear equations it is not always so. As it turns out, the Cauchy problem for equations of ideal barotropic gas is posed improperly (I shall call such problems incorrect). And in some sense, this problem is more incorrect, than the inverse problem for the heat equation.

Before we formulate the results about the system of equations of ideal barotropic gas, we recollect the main properties of this problem. Evidently, the solution of inverse problem for the heat equation is equivalent to the solution of the Cauchy problem in inverse time.

We consider the following example:

$$\frac{\partial u}{\partial t} = -\frac{\partial^2 u}{\partial x^2},$$
$$u\big|_{t=0} = \sin x = u_0(x), \tag{1}$$
$$u\big|_{x=0} = u\big|_{x=\pi} = 0, \quad u = u(x,t).$$

Evidently, the solution of this problem exists.

We consider problem (1) with a perturbed initial condition:

$$u_0(x) \Longrightarrow u_0(x) + \frac{1}{n^{s+1}}\sin nx = u_0^{(n)}(x)$$

Evidently,

$$u_c^{(n)}(x) \xrightarrow[W_2^{s-1}]{} u_c(x)$$

The solution $u(x, t)$ of problem (1), which corresponds to the initial condition $u_c^{(n)}$ has the form:

$$u(x, t) = e^{t} \sin x + \frac{1}{n^{s+1}} e^{n^2 t} \sin nx$$

Evidently, for $t = t_n \sim \frac{(1+s) \ln n}{n^2}$ the value $u^{(n)}(x, t) - u(x, t)$ which is the solution perturbation due to the perturbation of the initial conditions, has the order $O(1)$ for $n \to \infty$. Just this property means, that the Cauchy problem for (1) is not well-posed (incorrect). Moreover, by setting any number s, we obtain, that a perturbation of initial conditions, which is however smooth (in the sense of the Sobolev spaces W_2^s scale) and small as $n \to \infty$, leads to the perturbation of order "one" of the solution of problem (1).

Thus, the operator inverse to the operator $E(t)$, which is the resolving operator of problem (1), is unbounded with respect to the scale of spaces W_2^s, namely, for any s, s' the operator $E(t)^{-1} : W_2^s \to W_2^{s}$ is unbounded.
Such ill-posed (incorrect) Cauchy problem is called strong incorrect.

The values of time for which the small perturbations of the solution achieve the values of order $\sim O(1)$ are an important characteristic of incorrectness. In our example, the definition of a sequence t_n evidently depends on the way of numbering of the terms of the sequence, which perturbs the initial conditions. If in formula for t_n we change the numbering n by any other numbering (for example, $n \to [\ln n]$ where $[\ ]$ the square brackets denote the integer part of a number), we obtain another sequence of "times of swinging" $\{t_n\}$.
We can introduce an invariant characteristic of incorrectness as follows.

DEFINITION. The limit:

$$\alpha = \lim_{n\to\infty} \frac{\ln t_n}{-\ln \|u_n(t_n)\|_{W_2^1} + \ln \|u_n(t_n)\|_{L_2}}$$

will be called the degree of incorrectness α.

In the example of inverse heat conduction we have $\alpha = 2$.

The property of incorrectness of the Cauchy problem is equivalent to the fact, that the solution is not continuous with respect to some topology. In particular, if the solution is not continuous with respect to a weak topology, we have weak incorrectness.

For example, if for the nonlinear Cauchy problem:

$$\frac{\partial u}{\partial t}(t,u_0) + L\left(u(t,u_0)\right) = 0$$
$$u|_{t=0} = u_0$$

a weak convergence of a sequence of initial conditions $u_0^{(n)}$ to a function v_0, $u_0^{(n)} \xrightarrow[n\to\infty]{} v_0$, yields a weak convergence of the solution:

$$u(t,u_0^{(n)}) \xrightarrow[n\to\infty]{} u(t,v_0)$$

then the problem is called weakly continuous.
If the latter relation does not hold, the problem is called weakly incorrect.

We can say, that the problems are strong incorrect, if the solution perturbations caused by the initial conditions perturbations, which are smooth enough, do not converge to zero with respect to a strong topology.
For example,

$$u\left(t, u_0^{(n)} + \frac{1}{n^{1-\delta}} f\right) - u\left(t, u_0^{(n)}\right) \underset{n\to\infty}{\not\to} 0, \quad f \in C_0^\infty.$$

We define the strong incorrectness for a nonlinear evolutional Cauchy problem:

$$\frac{\partial \vec{u}}{\partial t} + L(\vec{u}) = 0, \quad \vec{u}\,\big|_{t=0} = \vec{u}_0 . \tag{2}$$

Let for any $s, M \in R$ a set $\mathfrak{M}$ of such sequences $\{u_n\}$, $\mathfrak{M} = \{\{u_n\}\}$ exist, that

$$\frac{\partial \vec{u}_n}{\partial t} + L(\vec{u}_n) = \sigma_n \xrightarrow[W_2^s]{} 0$$

$$\vec{u}_n\,\big|_{t=0} = \vec{u}_0^{(n)}$$

and $\vec{u}_0^{(n)} \xrightarrow[n\to\infty]{} \vec{u}_0$. Denote $\vec{u}_n \overset{def}{=} \vec{u}_n\left(t, \vec{u}_0^{(n)}\right)$

DEFINITION. Problem (2) is called strong incorrect, if a sequence $\{t_n\} \subset R_+$ exists, and for any sequence $\{u_n\} \in \mathfrak{M}$ such a sequence $\{\varepsilon_n\}$, $\varepsilon_n \xrightarrow[W_2^s]{} 0$, $n \to \infty$, and such a number $\delta > 0$ exist, that:

$$\|\vec{u}_n(t_n, \vec{u}_0^{(n)} + \mathcal{E}_n) - \vec{u}_n(t_n, \vec{u}_0^{(n)})\|_{L_2} \geqslant \delta,$$
$$t_n \leqslant \|\mathcal{E}_n\|_{W_2^s}^{1/M} \tag{3}$$

Relations (3) in the Definition mean, that for strong incorrect problems some sufficiently smooth small perturbations ($\mathcal{E}_n \to 0$ in W_2^s for $n \to \infty$) of the initial conditions cause the perturbations of order $\sim O(1)$ in the solution. And the solution perturbations achieve the values of order $\sim O(1)$ even at small times:

$$t_n \leqslant \|\mathcal{E}_n\|_{W_2^s}^{1/M} \longrightarrow 0, \quad n \to \infty.$$

The following theorem holds.

THEOREM. The system of equations of ideal barotropic gas:

$$\frac{\partial u}{\partial t} + \langle u, \nabla \rangle u + \frac{1}{\rho} \nabla P = 0,$$
$$\frac{\partial \rho}{\partial t} + \langle \nabla, \rho u \rangle = 0, \quad P = P(\rho) \tag{4}$$

is strong incorrect. The degree of incorrectness is equal to $\alpha = \frac{1}{M+1}$.

An important characteristic of incorrectness of a nonlinear problem is the sufficient completeness of a set of initial conditions, the "perturbations" of which "swing" the solution, and the sufficient completeness of a "set of errors", which arise, when the solution "swings".

For the system of equations of ideal barotropic gas the set of initial conditions satisfies the following conditions (here we give the properties of initial data for the velocity only):

a) the set of weak limits $\vec{V}_0$ of the sequences $\{\vec{u}_0^{(n)}\}$ is dense in L_2 ,

b) we denote by σ_j , j = 1,2,3, a weak limit of the sequence $\{(\vec{u}_{0\,j}^{(n)})^2\}$ of the squares of the components of the velocity vector, then the set of vectors

$$\left\{\begin{pmatrix}\vec{V}_0\\ \vec{\sigma}\end{pmatrix}\right\}$$

is dense in a strip from L_2 .

We remind, that a set of vectors $\{\vec{w}\}\subset L_2$ is called a strip in L_2 , if such vectors $\vec{A},\vec{B}\not\equiv 0$ exist, that for the components of these vectors the following inequalities hold almost everywhere:

$$\vec{A}_j \le \vec{W}_j \le \vec{B}_j .$$

c). We denote by $\{(\vec{\chi})_p\}$ the set of weak limits:

$$\left\{ w - \lim_{n\to\infty} (\vec{u}_0^{(n)})^{2p} \right\} = \left\{ (\vec{\chi})_p \right\}, \quad p=\overline{1,2}$$

Then for any t the set of vectors

$$\{\vec{\chi}\}$$

is dense in a strip from L_2 .

Property a) is the property of completeness of weak limits of sequences $\{\vec{u}_0^{(n)}\}$. It does not inform about the "oscillating" parts of initial data, which tend to zero as we pass to a weak limit.

Generally speaking, the even powers of these oscillations do not vanish as we pass to a weak limit, and they contribute to the components of the vectors $\vec{\sigma}$ and $\vec{\chi}$. Thus, the conditions b) and c) are the conditions of sufficient completeness of the set of oscillations in the initial conditions, as well they show, that these oscillations do not depend in a way on the non-oscillating part of initial data.

The condition of sufficient completeness of the set of errors has the form:
the set of weak limits of the scalar products:

$$\Big\{ w-\lim_{n\to\infty} \Big(\Big\langle \vec{u}_n(t_n, u_0^{(n)}+\varepsilon_n) - \vec{u}_n(t_n, \vec{u}_0^{(n)}),\ \vec{u}_n(t_n, \vec{u}_0^{(n)}+\varepsilon_n') - \vec{u}_n(t_n, \vec{u}_0^{(n)}) \Big\rangle \Big) \Big\} = \{\gamma\}$$

is dense in L_2.

The weak limits γ are analogues of correlations of the solutions errors caused by the perturbations ε_n and ε_n' of the initial conditions.

The initial conditions, which generate the set $\mathfrak{M}$ for the system of equations of ideal barotropic gas, can be written explicitly:

$$\vec{u}_n\Big|_{t=0} = \underbrace{\vec{U}_0(x) + \vec{a}(x, n g_0(x))}_{\vec{u}_0^{(n)}} + \underbrace{h^{-1+\frac{1}{M+1}}\vec{\varphi}(x)}_{\varepsilon_n},$$

$$\rho_n\Big|_{t=0} = \rho^0(x),$$

and

$$\vec{a}(x,\eta) = \vec{a}(x,\eta+2\pi), \quad \bar{a} = \frac{1}{2\pi}\int_0^{2\pi} a(x,\eta)\,d\eta = 0,$$

$$\langle \vec{a}, \nabla g_0 \rangle = 0$$

If we solve the system of equations of ideal barotropic gas in the domain Ω, then the boundary condition (impenetrability condition) has the form:

$$\langle \vec{u}, \vec{\nu} \rangle\Big|_{\Gamma} = 0$$

where $\vec{\nu}$ is a normal to the boundary Γ of the domain Ω. This boundary condition induces the following conditions on the functions $\vec{U}_0$, $\vec{a}$, $\vec{\varphi}$, g_0:

$$\langle \vec{U}_0, \vec{\nu} \rangle\Big|_{\Gamma} = 0, \quad \langle \vec{a}(x,0), \vec{\nu} \rangle\Big|_{\Gamma} = 0,$$

$$\langle \vec{\varphi}, \vec{\nu} \rangle\Big|_{\Gamma} = 0, \quad g_0\Big|_{\Gamma} = 0$$

In order to solve the incorrect problems we apply the regularization. To regularize a sequence of solutions means to calculate a limit of this sequence in the sense of generalized functions (distributions) over some space of basic functions.

Now we return to inverse heat conduction:

$$\frac{\partial u}{\partial t} = -\frac{\partial^2 u}{\partial x^2}, \quad u\big|_{x=0} = u\big|_{x=\pi} = 0,$$

$$u\big|_{t=0} = \sin x + \frac{\alpha}{n^{s+1}} \sin nx, \quad \alpha = const$$

The perturbation of the initial condition ($\sin x$) tends to zero as $n \to \infty$ in the space W_2^s .

The solution $u = u_n(t,x)$ has the form:

$$u_n(t,x) = e^{t} \sin x + \frac{\alpha\, e^{n^2 t}}{n^{s+1}} \sin nx$$

We consider "small" times. We set $t = t_n = \frac{s+1}{n^2} \ln n$.

Then

$$u_n(t_n, x) = e^{t_n} \sin x + \alpha \sin nx$$

The last summand in the right-hand side converges weakly to zero as $n \to \infty$. The first summand $e^{t_n} \sin x \stackrel{def}{=} V(t_n, x)$ for $t \leq t_n$ satisfies the initial equation of inverse heat conduction:

$$\frac{\partial V}{\partial t} = - \frac{\partial^2 V}{\partial x^2}$$

Thus, In our example the regularization means, that we pass to a weak limit (in $\mathcal{D}'$) as $n \to \infty$.

We note, that if the solution is considered for any values of t , the method of regularization considered above cannot be applied.

The regularization for such values of t exists only as a limit in the sense of distributions over trigonometric polynomials.

Further, for $t = t_n = \frac{s+1}{n^2} \ln n$ the powers of the solution $u_n(t_n, x)$ cannot be regularized. Really, we consider, for example, the square of the solution:

$$u^2(t_n, x) = (e^{t_n} \sin x)^2 + \alpha^2 + 2\alpha \sin x \sin nx$$

The first summand in the right-hand side is the square of the regularized solution, the last summand tends to zero in the sense of weak convergence as $n \to \infty$. Thus, the regularization of the solution square has the form:

$$\operatorname{Reg} u_n^2 = V^2 + \alpha^2$$

and evidently, depends on a choice of the sequence, which perturbs the initial condition.

Therefore, the square of the solution of the inverse heat conduction problem is non-regularizable.

We assume now, that at the initial time an arbitrary enough initial condition is given for the heat equation in the inverse time, for example,

$$u\big|_{t=0} = u_0(x) \in C_0^{\infty}.$$

Then the solution $u(t,x)$ for $t>0$ does not, generally speaking, exist. However, one can construct a formal series with respect to t, which will formally satisfy the equation:

$$\frac{\partial u}{\partial t} = -\frac{\partial^2 u}{\partial x^2}$$

and the initial condition $u\big|_{t=0} = u_0(x)$

Such a formal series has the form:

$$\sum_{i \geq 0} \psi_i(x) t^i$$

The coefficients of the series $\psi_i(x)$ can easily be calculated. For example,

$$\psi_0 = u_0(x), \quad \psi_1 = -u_0'', \quad \psi_2 = u_0^{\overline{\text{IV}}}, \ldots\ldots$$

We define the regularization in such a way, that the regularized solution converges to a formal series in t, which satisfies formally the equation and the initial condition.

It allows us to avoid the assumption, that the solution of the regularized problem exists.

Now we return to our example.

$$\frac{\partial u}{\partial t} = -\frac{\partial^2 u}{\partial x^2}, \quad u|_{t=0} = \sin x + \frac{1}{n^{s+1}} \sin nx$$

Then for $t = t_n = \frac{s+1}{n^2} \ln n$ for any $\varphi \in W_2^{3M+1}$ the equality holds:

$$\left(\varphi, \left[e^{t_n} \sin x + \frac{e^{n^2 t_n}}{n^{s+1}} \sin nx - \sum_{i=0}^{M-1} \frac{t_n^i}{i!} \sin x\right]\right) = = O(t_n^M).$$

This relation means, that in the sense of distributions (in the space W_2^{-3M-1}) for $t = t_{n_{M-1}}$ the functions $e^{t_n} \times \times \sin x + e^{-n^2 t_n} \cdot \sin nx\,(n)^{-(s+1)}$ and $\sum_{i=0} t_n^i \sin x\,(i!)^{-1}$, are close to each other. We show, that:

Formal series $\sum_{i\geq 0} t^i \sin x\,(i!)^{-1} = \mathrm{Reg}\left(e^{t} \sin x + e^{n^2 t} \sin nx\,(n)^{-(s+1)}\right)$

DEFINITION. The sequence $\{\vec{u}_n(x,t)\} \in \mathfrak{M}$ is regularized to the degree m, if such a formal series $\sum_{i=0}^{\infty} \psi_i(x) t^i$ exists, that for any function $\Phi \in W_2^m$ for $n \to \infty$ the following estimate holds:

$$\left(\Phi, \vec{u}_n(t_n, \vec{u}_0^{(n)}) - \sum_{i=0}^{M-1} \psi_i(x) t_n^i\right) = O(t_n^M)$$

So we have, that the solution of inverse heat equation is regularizable to the degree ∞.

In order to formulate the theorem on regularization of the solution of the system of equations of ideal barotropic gas we introduce the following notations.

Let $\vec{v} = \vec{v}(x, t, \eta)$, $\rho = \rho(x, t, \eta)$ be 2π-periodic functions. We denote $\bar{A} = \frac{1}{2\pi} \int_0^{2\pi} A \, d\eta$ for any 2π-periodic function A. The following theorem holds.

THEOREM. The system of equations of ideal barotropic gas is regularizable to the degree 0, and the formal series

$$\begin{pmatrix} \vec{v} \\ \rho \end{pmatrix} = \sum_{i=0}^{\infty} \Psi_i(x) t^i$$

satisfies the following system of equations:

$$(1)' \qquad \vec{U}_t + \langle \vec{U}, \nabla \rangle \vec{U} = -\frac{1}{\rho} \nabla P - \overline{\langle \vec{v}, \nabla \rangle \vec{v}} - \overline{\vec{v} \langle \nabla, \vec{v} \rangle},$$

$$(2)' \qquad \vec{v}_t + \langle \vec{U}, \nabla \rangle \vec{v} + \langle \vec{v}, \nabla \rangle_\eta \vec{v} + \langle \vec{v}, \nabla \rangle \vec{U} =$$

$$= \frac{1}{\rho} \mathcal{P}_\eta \nabla g + \frac{1}{\rho} \vec{v}_\eta \int_0^\eta \langle \nabla, \rho \vec{v} \rangle d\eta' + \langle \vec{v}, \nabla \rangle \vec{v} +$$

$$+ \vec{v} \langle \nabla, \vec{v} \rangle .$$

$$\rho_t + \langle \nabla, \rho \vec{U} \rangle = 0, \quad g_t + \langle \vec{U}, \nabla \rangle g = 0,$$

$$\langle \vec{v}, \nabla g \rangle = 0, \quad P = P(\rho), \quad \overline{\vec{v}} = 0,$$

$$\vec{U} = \vec{U}(x,t), \quad \rho = \rho(x,t), \quad g = g(x,t)$$

The boundary conditions have the form:

$$\langle \vec{U}_j, \vec{\nu} \rangle \big|_\Gamma = 0, \qquad \langle \vec{v}, \vec{\nu} \rangle \big|_{\Gamma,\, \eta = 0} = 0$$

The products of powers of the components of the velocity vector can be regularized:

$$\mathrm{Reg} \prod_{i=1}^{3} (\vec{u}_n)_i^{m_i} = \overline{\prod_{i=1}^{3} (\vec{U} + \vec{v})_i^{m_i}}$$

We consider in the whole space the system of the Navier-Stokes equations:

$$\frac{\partial \vec{u}}{\partial t} + \langle \vec{u}, \nabla \rangle \vec{u} = - \frac{1}{\rho} \nabla P + \frac{1}{Re} \Delta \vec{u},$$

$$\frac{\partial \rho}{\partial t} + \operatorname{div} \rho \vec{u} = 0, \qquad P = P(\rho), \quad Re \gg 1.$$

We assume, that for $t=0$ the solution satisfies the same initial conditions as we had for the system of equations of ideal barotropic gas:

$$\vec{u}\Big|_{t=0} = \vec{U}_0(x) + \vec{a}\,(x, n g_0(x)) + n^{-1+1/M+1} \varphi(x),$$

$$\rho\Big|_{t=0} = \rho^0(x)$$

and the integer-valued parameter n is connected with the Reynolds number Re by the equality: $n = [\varepsilon Re^{1/2}]$, where the square brackets denote the integer part of a number.

By regularizing the Navier-Stokes equations, we obtain, that the formal series $\begin{pmatrix} U \\ \rho \end{pmatrix} = \sum\limits_{i=0}^{\infty} \Psi_i(x) t^i$ satisfies a system of equations similar to system (1), (2).

If we take the viscosity into account, then in the second equation the following additional summand arises:

$$\varepsilon^2 (\nabla g)^2 \vec{v}_{\eta\eta}$$

Then our system of equations has the form:

$$(1) \qquad \vec{U}_t + \langle \vec{U}, \nabla \rangle \vec{U} = - \frac{1}{\rho} \nabla P - \boxed{\overline{\langle \vec{v}, \nabla \rangle \vec{v}} - \overline{\vec{v} \langle \nabla, \vec{v} \rangle}},$$

$$(2) \qquad \vec{v}_t + \langle \vec{U}, \nabla \rangle \vec{v} + \langle \vec{v}, \nabla \rangle \vec{v} + \langle \vec{v}, \nabla \rangle \vec{U} =$$

$$= -\frac{1}{\rho}\mathcal{P}_\eta \nabla g + \varepsilon^2(\nabla g)^2 \vec{v}_{\eta\eta} + \frac{1}{\rho}\vec{v}_\eta \int_0^\eta \langle\nabla,\rho\vec{v}\rangle d\eta' +$$

$$+ \boxed{\overline{\langle\vec{v},\nabla\rangle\vec{v}} + \overline{\vec{v}\langle\nabla,\vec{v}\rangle}}$$

$$\rho_t + \langle\nabla,\rho\vec{U}\rangle = 0, \qquad g_t + \langle\vec{U},\nabla\rangle g = 0,$$

$$\langle\vec{v},\nabla g\rangle = 0, \quad P = P(\rho), \quad \overline{\vec{v}} = 0 .$$

A particular case of the obtained system of equations is the well-known in hydrodynamics system of the Prandtl equations. In order to derive these equations we assume, that the initial conditions are non-periodic with respect to the "rapid" variable $\eta = n g_0(x)$, but they are rapidly decreasing :

$$a(x,\eta) = O(\eta^{-N}), \quad \eta \to \infty, \quad N \gg 1 .$$

In this case the averaging procedure (the integration over the period) in the derivation of these equations is changed by the calculation of a limit as $\eta \to \infty$, and the equations (1), (2) have the form:

$$\vec{U}_t + \langle\vec{U},\nabla\rangle\vec{U} = -\frac{1}{\rho}\nabla P ,$$

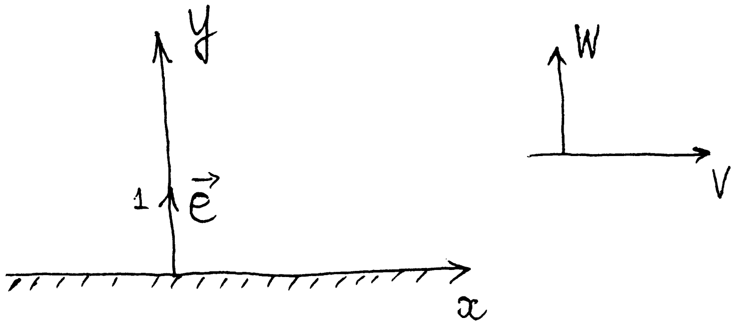
$$\vec{v}_t + \langle\vec{U},\nabla\rangle\vec{v} + \langle\vec{v},\nabla\rangle\vec{v} + \langle\vec{v},\nabla\rangle\vec{U} =$$

$$= -\frac{1}{\rho}\mathcal{P}_\eta \nabla g + \varepsilon^2(\nabla g)^2 \vec{v}_{\eta\eta} + \frac{1}{\rho}\vec{v}_\eta \int_0^\eta \langle\nabla,\rho\vec{v}\rangle d\eta',$$

$$\rho_t + \langle\nabla,\rho\vec{U}\rangle = 0, \qquad g_t + \langle\vec{U},\nabla\rangle g = 0$$

$$\langle\vec{v},\nabla g\rangle = 0, \quad P = P(\rho) , \quad \overline{\vec{v}} = 0$$

Now we show, how we can obtain from this system of equations the Prandtl equations, which describe a flow along a smooth surface, namely along the plane $y = 0$.



We set $g \equiv y$. Then we obtain from the conditions:

$$\langle \vec{v}, \nabla g \rangle = 0, \quad g_t + \langle \vec{V}, \nabla \rangle g = 0$$

that the scalar equation, which corresponds to the projection on the axis O_y in the vector equation (2), is the identity of the form $0 = 0$.

The equation, which corresponds to the projection on the axis O_x , has the form:

$$V_t + V V_x = -\frac{1}{\rho} V_\eta \int_0^\eta (\rho V)_x d\eta' - \frac{1}{\rho} \mathcal{P}_\eta + \varepsilon^2 V_{\eta\eta} .$$

By setting

$$W = -\frac{1}{\rho} \int_0^\eta (\rho V)_x d\eta'$$

we obtain the following system of equations (the Prandtl equations):

$$V_t + V V_x + W V_\eta = -\frac{1}{\rho} P_\eta + \varepsilon V_{\eta\eta}$$

$$(\rho W)_\eta + (\rho V)_x = 0, \quad P_\eta = 0.$$

Equation (1) is the equation, which describes the flow outside the boundary layer:

$$\vec{U}_t + \langle \vec{U}, \nabla \rangle \vec{U} = -\frac{1}{\rho} \nabla P,$$

$$\rho_t + \langle \nabla, \rho \vec{U} \rangle = 0, \qquad P = P(\rho),$$

Here $\vec{U}$ is the velocity outside the boundary layer:

$$\vec{U} = \lim_{\eta \to \infty} \vec{u}.$$