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# Existence of solutions for transversally elliptic left invariant differential operators on nilpotent Lie groups.

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1. Introduction and notation. We describe here some recent results, obtained jointly with Lawrence Corwin [3] on solvability of left invariant differential operators on nilpotent Lie groups. For related results see [2], [8], [9], [10], [11], [14], [15], [16], [17].

We consider first operators on a 2-step nilpotent Lie group G, i.e. we assume that the Lie algebra $\mathfrak{G}$ is a vector space direct sum $\mathfrak{G} = \mathfrak{G}_1 + \mathfrak{G}_2$ with $[\mathfrak{G}_1, \mathfrak{G}_1] = \mathfrak{G}_2$ and $[\mathfrak{G}_2, \mathfrak{G}] = (0)$. For $\eta \in \mathfrak{G}_2^*$ let $B_\eta$ be the bilinear form $B_\eta(X_1, X_2) = \eta([X_1, X_2])$ for $X_1, X_2 \in \mathfrak{G}_1$. $B_\eta$ assumes its maximal rank on a Zariski open subset in $\mathfrak{G}_2^*$. Recall that to every $\ell \in \mathfrak{G}^*$ we may associate, by the Kirillov theory, an irreducible unitary representation $\pi_\ell$ of G, realized on a Hilbert space of the form $L^2(\mathbb{R}^k)$ for some $k$.

By a transversally elliptic operator on G we shall mean a left invariant differential operator L on G which is an elliptic polynomial on $\mathfrak{G}_1$, i.e.

$$L = L_m + L_{m-1} + \cdots + L_0, \tag{1.1}$$

with $L_j$ homogeneous of degree $j$, and $L_m$ an elliptic polynomial on $\mathfrak{G}_1$.

2. Necessary conditions for local solvability. We give the following criterion, which generalizes known results [2] for homogeneous operators i.e. those for which $L_j \equiv 0$, $j < m$.

_Theorem 1._ _Let_ $L$ _be a left invariant operator on_ $G$ _which is transversally elliptic._ _Assume that there is a non-empty open set_ $V \subset \mathfrak{G}^*$ _such that_

$$\ker \pi_\ell(L^\tau) \neq 0 \quad \text{for all} \quad \ell \in V, \tag{2.1}$$

or, _equivalently_,

$$\ker L^\tau \cap L^2(G) \neq 0 . \tag{2.2}$$

_Then_ $L$ _is not locally solvable_.

The idea of the proof is as follows. First, if $B_\eta$ is degenerate for all $\eta$, then [1] may be applied to show that the hypothesis is vacuous. So assume $B_\eta$ nondegenerate for $\eta$ in a Zariski open set. We show, using microlocal constructions as in [6] that there is a pseudo-differential operator $\Pi$ not of order $-\infty$ such that $L^\tau\Pi$ is of order $-\infty$. Now for any distribution $\sigma$ for which $Lv - \sigma = 0$ in an open set $U$ for some distribution $v$, $\Pi^\tau(Lv - \sigma)$ is smooth, and hence $\Pi^\tau\sigma$ is smooth. Hence $\sigma$ cannot be arbitrary.

3. _Sufficient conditions for solvability on_ $H$-_groups_. $G$ is called an $H$-_group_ if $B_\eta$ is nondegenerate for $\eta \neq 0$. We prove the following converse to Theorem 2 for $H$-groups. A globally defined differential operator $P$ is _uniformly semi-globally solvable_ if there is an integer $r$ such that for every bounded open neighborhood $U$ of $0$ there exists a distribution $\sigma_U$ of order at most $r$ such that $L\sigma_U = \delta$ in $U$.

_Theorem 2._ _If_ $G$ _is an_ $H$-_group and_ $L$ _a left invariant transversally elliptic operator on_ $G$ _then_ $L$ _is uniformly semi-globally solvable if_

(2.1) *and* (2.2) *do not hold.*

The proof of Theorem 2 is somewhat similar to that of the corresponding result [17] in the case where L is homogeneous. Both rely on the theorem of Lojasiewicz which says that one can divide a distribution by a non-zero analytic function.

*Corollary. If* $L_m$ *is locally solvable, then* L *is locally solvable.*

4. *Existence of global fundamental solutions.* Here we allow G to be any connected Lie group, not necessarily nilpotent.

*Theorem 3. Let* L *be a left invariant differential operator on* G *which is uniformly semi-globally solvable. Suppose that* G *is* L-*convex. Then* L *has a global fundamental solution;* i.e. *there is a distribution* $\sigma$ *on* G *for which* $L\sigma = \delta$.

The proof of Theorem 3 involves a construction similar to that used in proving that L-convexity implies global solvability. The theorem gives a new result even for homogeneous operators.

*Corollary 1. Let* L *be a homogeneous left invariant differential operator on a nilpotent Lie group* G *with dilations. If* L *is locally solvable at* 0 *then* L *has a global fundamental solution.*

*Corollary 2. If* L *is a transversally elliptic operator on an* H-*group which satisfies the hypothesis of* Theorem 2 *then* L *has a global fundamental solution.*

5. Global criteria for hypoellipticity. The various global criteria for local solvability for homogeneous differential operators on nilpotent groups, e.g. $\ker L^{\tau} \cap L^2(G) = (0)$, suggest that the representation-theoretic criterion of Helffer-Nourrigat [5] may be reformulated. Indeed, using a recent Liouville-type theorem of Geller [4] one may obtain the following.

Theorem 4. (Geller, Helffer-Nourrigat). *Let* $G$ *be a stratified nilpotent Lie group and* $L$ *a homogeneous left invariant differential operator on* $G$. *Then* $L$ *is hypoelliptic if and only if there is no non-constant bounded function* $f$ *on* $G$ *such that* $Lf = 0$.

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