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THE NASH-MOSER INVERSE MAPPING THEOREM

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To prove a local embedding theorem for strongly pseudo-convex CR structures (of dimension $\geq q$) (cf. [2]) we used a variant of Nash-Moser inverse mapping theorem. We try to explain in general terms how it was done, wilhout bothering too much about technical details.

For a $\varepsilon_o > 0$ we define $\varepsilon_\nu > 0$ inductively by

(1) $$\varepsilon_{\nu+1} = \varepsilon_\nu^a \quad , \ a = 3/2.$$

The Nash-Moser inverse mapping theorem (cf. [3]) is based on the following :

LEMMA :

Let $s,t \geq 0$ be given. Pick λ, $\mu > 0$ so large that

(2) $$s + (a-2) \leq 0$$
$$t + a^2\mu + (1-a)\lambda \leq -a.$$

Let $p_\nu > 0$ be a sequence. Assume that for a constant $C^* > 0$

(3) $$p_\nu \leq C^*(\varepsilon_\nu^{-s}(p_{\nu-1})^2 + \varepsilon_\nu^\lambda \varepsilon_{\nu-1}^{-\lambda-t})$$
$$\varepsilon_1 \leq 1/(2C^*)^2 \ , \ p_o \leq \varepsilon_1^\mu/2C^*.$$

Then

(4) $$p_\nu \leq (\varepsilon_{\nu+1})^\mu/2\, C^*$$

Proof goes as follows : we set $g_\nu = \varepsilon_{\nu+1}{}^{-\mu}\, p_\nu$. Then

$$g_\nu \leq C^*(\varepsilon_\nu{}^{-(s+(a-2)\mu)} g_{\nu-1}{}^2 + \varepsilon_{\nu-1}{}^{-(t+a^2\mu+(1-a)\lambda)}).$$

Hence $g_\nu \leqslant C^{\#}((g_{\nu-1})^2 + \varepsilon_\nu)$. We now prove $g_\nu \leqslant 1/2\ C_{\#}$ by induction on ν.

We apply the above lemma in the following setting : we consider open sets F', G' in Frechet spaces F, G and a map

$$\Phi : F' \to G'$$

Each of these Frechet spaces is assumed to be endowed with an increasing sequence of semi-norms $|| \ ||_k$ which defines its topology. In practice, we consider the Frechet spaces of C^∞ sections of vector bundles over a manifold M. $|| \ ||_k$ is defined by measuring the partial derivatives up to degree k of sections. Φ is given by a non-linear partial differential operator involving partial derivatives up to order, say r. This is translated into an assumption

$$||\Phi(f)||_k \leqslant C_k(1 + ||f||_{k+r}) \tag{5}$$

For k sufficiently large any map with the above assumption is called tame (cf. R. Hamilton [1] for more details). We assume that Φ is infinitely differentiable and all partial derivatives are tame. In particular there is for each $f \in F'$ a continuous linear map.

$$d_f\, \Phi : F \to G$$

such that with $R_f(h) = \Phi(f+h) - \Phi(f) - d_f\, \Phi(h)$

$$||R_f(h)||_k \leqslant C_k(||f||_{k+r}\ ||h||^2_{k_o} + ||h||^2_{n+r}) \tag{6}$$

for $k \geqslant k_1$. We also assume that there is a mollifier M_ε ($\varepsilon > 0$) with the standard properties : for $s \geqslant 0$

$$(7) \qquad \begin{aligned} ||M_\varepsilon f||_{k+s} &\leqslant C_{k,s}\ \varepsilon^{-s}\ ||f||_k \\ ||f - M_\varepsilon f||_k &\leqslant C_{k,s}\ \varepsilon^{s}\ ||f||_{k+s} \end{aligned}$$

We now wish to show that an element g G' is in the image of Φ. We may assume that g = 0. We solve the problem by a successive approximation. Namely, for a_ν - th approximation f_ν we define $f_{\nu+1}$ as follows : note that $\Phi(f_\nu+h)$ is very closed to $\Phi(f_\nu) + d_{f_\nu}\Phi(h)$. Hence we solve the equation :

$$(8) \qquad \Phi(f_\nu) + d_{f_\nu}\Phi(h) = 0$$

However, in the process we usually lose derivatives. We compensate this by setting

$$(9) \qquad f_{\nu+1} = f_\nu + M_{\varepsilon_{\nu+1}} h_\nu$$

where h_ν is a solution of (8) and where ε_ν is given in (1). In fact, we assume that we can find h_ν with

$$(10) \qquad ||h_\nu||_{k-r_1} \leqslant C_k||\Phi(f_\nu)||_k$$

This estimate is essential for this method to work. In order to show that f_ν converge to a solution f of our problem, it is enough to show that $p_\nu = ||\Phi(f_\nu)||_k$ satisfy (3) in the lemma. If this is the case, p_ν has estimate (4). In view of (9) and (10) it then follows that f_ν will also converge. Now :

$$\Phi(f_\nu + M_{\varepsilon_{\nu+1}} h_\nu) = \Phi(f_\nu) + d_{f_\nu}\Phi(M_{\varepsilon_{\nu+1}} h_\nu) + R_{f_\nu}(M_{\varepsilon_{\nu+1}} h_\nu)$$

$$= R_{f_\nu}(M_{\varepsilon_{\nu+1}} h_\nu) - d_{f_\nu}\Phi(h_\nu - M_{\varepsilon_{\nu+1}} h_\nu)$$

Note (7) and (6). From the first term (resp. the second term) we obtain terms $C^{\#} \varepsilon_{\nu+1}{}^{-s}(p_\nu)^2$ (resp. $\varepsilon_{\nu+1}^{\lambda}\,\varepsilon_\nu{}^{-\lambda-t}$) for a choice of s and t.

The above shows that we can solve the equation $\Phi(f) = g$ for a given g provided we find a very good approximation f_o so that the last inequality in (30) is satisfied. In particular, we find that a small neighborhood of f_o is covered by Φ.

For a local embedding theorem mentioned in the beginning we have a following more general setting. Namely, we have a manifold M and for each open $U \subset M$ we have :

$$F'(U) \overset{\Phi}{\to} G'(U) \overset{\Psi}{\to} H'(U)$$

with $\Psi.\Phi = 0$. They are related by compatible restriction maps. We are given $g \in G'(M)$ with $\Psi(g) = 0$ and a reference point p_o in M. We wish to show that the restriction of g to a suitable open neighborhood $\mathcal{U}$ of p_o is in the image of Φ. We may assume that $g = 0$. The existence of Ψ means that we may not be able to solve the equation (8). We have to replace $\Phi(f_\nu)$ by its projection to the image of $d_{f_\nu}\Phi$. Moreover, we could find such projection only for $\mathcal{U}$ satisfying certain conditions which also depend on f_ν. Namely, for each $f \in F(U_1)$, where $p_o \in \mathcal{U}_1$, we have a way to define $r_f > 0$ and a distance function t_f to p_o with the following properties : for $0 < r < r_f$ set

(11) $$\mathcal{U}(f,r) = \{p \in \mathcal{U}_1 \ ; \ t_f(p) < r\}.$$

Let f' be the restriction of f to $F(U(f,r))$, $f' = f|U(f,r)$. Then there is

(12) $$V_{f'} : G(U(f,r)) \to F(U(f,r))$$

such that with $h' = V_{f'}(\Phi(f'))$

(13) $$-\Phi(f') = (d_{f'}\Phi)(h') + A(\Phi(f'))$$

where $A(\varphi)$ is given by a composition.

(14) $$A(\varphi) = A_1 \circ A_2(\varphi)$$

A_1 is a linear map. A_2 is a non-linear partial differential operator starting with quadratic terms. Since our error term $A(\Phi(f'))$ is of quadratic nature as R_f in (6) we may try to solve our problem by the same method as in the standard case.

We first find $f_o \in \mathcal{F}(U_o)$ such that :

(15) $$||\Phi(f_o)|\mathcal{U}(f_o,r)||_k \leqslant O(r^N)$$

for all N. This is achieved by solving the differential equation $\Phi(f) = 0$ as a formal power series centered at p_o whose Taylor series agree with the solution formal power series will satisfy our requirement. For $\alpha > \beta$ we set for $0 < r_o < r_{f_o}$

(16) $$\varepsilon_o = r_o^{\alpha} \quad , \quad \delta_o = r_o^{\beta}$$

and define ε_ν and δ_ν as in (1). We then set

(17) $$r_{\nu+1} = r_\nu - 3\,\delta_\nu$$

Starting from $f_o | U(f_o, r_o)$ we construct f_1 as in the standard case replacing h in (8) by h' in (13). We the, show that, if r_{f_o} is properly chosen, $r_1 + \delta_1 \leqslant r_{f_1}$ and $U(f_1, r_1 + \delta_1) \subseteq U(f_o, r_o - \delta_o)$. We then consider $f_1 | U(f_1, r_1)$ and proceed inductively. We do this construction for all r_o in $]0, r_{f_o}[$.

Thus the success of our method depends essentially on the nature of $V_{f'}$ in (12) which solves the equations (13) and how we could handle the new error term $A(\Phi(f'))$. In our case $V_{f'}$ is obtained by using the solution mapping $N_{f'}$ of generalized Neumann boundary value problem on $U(f,r)$ associated with $d_{f'}, \Phi$. N_f also enters in the construction of A_1 in (14). The fact is we could only find N_f for $U(f,r)$ as in (11), where t_f satisfies certain conditions. This is the reason why we had to change U as each step of the successive approximation. On the other hand, since $U(f_{\nu+1}, r_{\nu+1}) \Subset U(f_\nu, r_\nu)$, we could use the interior estimate. In such estimate a factor $(\delta_\nu)^{-\ell}$ (cf. (17)) will come in the constant of the inequality. However, we can admit such factor in view of (3). Using estimates for N_f on $U(f,r)$ as well as interior estimate, we prove the first inequality in (3) for all r in $]0, r_{f_o}[$, provided $p_o, \ldots, p_{\nu-1}$ is sufficiently small. We now need the second and the third inequality in (3). In view fo (16) the second is satisfied for sufficiently small r_o. Similarly the third is satisfied in view of (15).

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