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ON THE CONSTRUCTION OF FUNDAMENTAL SOLUTIONS FOR DIFFERENTIAL OPERATORS ON NILPOTENT GROUPS

by

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I shall here outline some steps towards the construction of parametrices (and local fundamental solutions) for some classes of differential operators on nilpotent groups. Some of the results are as yet only established for 2-step groups, but it seems plausable that the technique developed is applicable to any group which is graded and nilpotent. (For other treatments of the rank 2 case we refer to Geller [1] and Miller [3,4]).

1. General considerations

Let $\mathfrak{g}$ be a graded nilpotent Lie algebra of step r. This means that $\mathfrak{g} = \mathfrak{g}_1 \oplus \dots \oplus \mathfrak{g}_r$, a direct sum of vector spaces, where $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$ (=0 if $i+j > r$). We consider $\mathfrak{g}$ as a Lie group at the same time with multiplication $x.y = x + y + \frac{1}{2}[x,y] + \frac{1}{12}(\mathrm{ad}x)^2 y + \frac{1}{12}(\mathrm{ad}y)^2 x + \dots$ given by the Baker-Campbell Hausdorff formula. Let $\mathcal{U}$ be the subspace of $\mathcal{S}'(\mathfrak{g})$ consisting of distributions which are rapidly decreasing at ∞ and smooth outside the origin. Then $\mathcal{U}$ is closed under group convolution and if T is a unitary irreductible representation of $\mathfrak{g}$ then $T(u)$ is defined on the Schwartz space $\mathcal{S}(T)$ of T when $u \in \mathcal{U}$. We have $T(u * v) = Tu \circ Tv$.

The algebra $D(\mathfrak{g})$ of right invariant differential operators on $\mathfrak{g}$ is via the map $P \to P\delta$ identified with the sub algebra of $(\mathcal{U}, *)$ consisting of elements supported by the origin : If $P \in D(\mathfrak{g})$ and $u \in C_o^\infty(\mathfrak{g})$ we have $Pu = P\delta * u$. Thus to find a fundamental solution (or parametrix) for P amounts to finding an inverse for $P\delta$ in $\mathcal{U}$ (modulo the ideal $\mathcal{S}(\mathfrak{g})$). We denote the Euclidean Fourier transform of u by $Fu = \hat{u}$ and $\mathfrak{g}^*$ is the vector space dual of $\mathfrak{g}$. The symbol of P is by definition the polynomial $p(\xi) = (\widehat{P\delta})(\xi)$ on $\mathfrak{g}^*$.

Since $\mathfrak{g}$ is graded there are natural definitions of quasi-homogeneity for operators and functions on $\mathfrak{g}, \mathfrak{g}^*$ etc.. We set $\mathrm{Pol}^m(\mathfrak{g}^*) = \bigoplus_{0 \leqslant j \leqslant m} \mathrm{Pol}_j(\mathfrak{g}^*)$, where $\mathrm{Pol}_j(\mathfrak{g}^*)$ is the set of polynomials on $\mathfrak{g}^*$ which are quasi-homogeneous of degree j. If $\alpha = (\alpha_1, \dots, \alpha_r)$ is a sequence of multi indices corresponding to a choice of bases for the $\mathfrak{g}_j$ then we set $\|\alpha\| = \sum_1^r j\,|\alpha_j|$. For $\xi = \xi_1 + \dots + \xi_r \in \mathfrak{g}^*$

we let $\|\xi\| = \sum_1^r |\xi_j|^{1/j}$ be the homogeneous norm. Then $S^m(\mathfrak{g}^*)$ is the set of all $a \in C^\infty(\mathfrak{g}^*)$ for which

$$\sup_\xi (1 + \|\xi\|)^{\|\alpha\| - m} |D^\alpha a(\xi)| < \infty$$

for all α .

Definition 1.1 : There is a self adjoint differential operator $\Phi = \Phi(\xi,\eta, D_\xi, D_\eta) = \Sigma\, \Phi_j(\xi_j,\eta_j,D_{\xi_1},D_{\eta_1},\dots,D_{\xi_{j-1}},D_{\eta_{j-1}})$ on $\mathfrak{g}^* \times \mathfrak{g}^*$ such that

$$(u * v)^\wedge(\xi) = \hat{u} \# \hat{v}(\xi) \overset{\text{def}}{=} (e^{i\Phi}\, \hat{u} \otimes \hat{v})(\xi,\xi)$$

when $u,v \in \mathcal{S}(\mathfrak{g})$. Moreover, Φ is linear in ξ,η and quasi homogeneous of degree 0.

Proposition 1.2 : The multiplication # extends to $S(\mathfrak{g}^*) = \bigcup_{m \in \mathbb{R}} S^m(\mathfrak{g}^*)$ and $a \# b \in S^{m'+m''}(\mathfrak{g}^*)$ if $a \in S^{m'}(\mathfrak{g}^*)$ and $b \in S^{m''}(\mathfrak{g}^*)$.

Definition 1.3 : We say that $p \in \mathrm{Pol}^m(\mathfrak{g}^*)$ is elliptic w.r.t # if its principal part is invertible in $S(\mathfrak{g}^*)/\mathcal{S}(\mathfrak{g}^*)$.

Recall Rocklands conditions for P and P^* :

(Ro) $T(P_m)$ and $T(P_m^*)$ are injective on $\mathcal{S}_T$ when T is a non trivial unitary irreductible representation of $\mathfrak{g}$.

We want to prove the following (which is still unproved for $r > 2$).

(C) p is elliptic if $F^{-1}p$ satisfies (Ro).

Remark 1.4 : One would instead consider right and left inverses . Under the assumptions above it follows from the calculus for # that it suffices to consider the case when p is quasi-homogeneous. Non elliptic polynomials p might be invertible w.r.t. # after adding lower order terms to p.

2. The induction step

Assume that the result (C) is already proved for all groups of lower dimensions.

Let $p \in \mathrm{Pol}_m(\mathfrak{g}^*)$ satisfy (Ro). Choose a vector $e \neq 0$ in $\mathfrak{g}_r$ and consider the quotient group $\tilde{\mathfrak{g}} = \mathfrak{g}/\mathbb{R}e$. The image $\tilde{P}$ of P under the projection $\mathfrak{g} \to \tilde{\mathfrak{g}}$ will then satisfy (Ro) and the symbol of $\tilde{P}$ is the restriction of p to $W : \langle\xi,e\rangle = 0$. This implies that one can find $q \in S^{-m}(\mathfrak{g}^*)$ so that $p \# q = 1+b$ with b in $S^{0,1}(\mathfrak{g}^*)$ so that $S^{k,\tau}(\mathfrak{g}^*)$ is the set of all $b \in S^k(\mathfrak{g}^*)$ so that

$$\sup_{\xi} \left(\frac{1 + |\langle\xi ,e\rangle|}{1 + \|\xi\|^n}\right)^{-\tau} (1 + \|\xi\|)^{\|\alpha\| -k} | D^{\alpha} b(\xi) | < \infty$$

for all α . The calculus then allows us to replace $S^{0,1}(\mathfrak{g}^*)$ by $S^{0,\infty}(\mathfrak{g}^*)$.

Set $S_0^{k,\infty}(\mathfrak{g}^*) = \{b \in S^{k,\infty}(\mathfrak{g}^*) ;\ b = 0$ for $\langle\xi,e\rangle$ small or negative$\}$ and $S_0^k(\mathbb{R}_+ ; \mathcal{S}(W))$ the space of $\mathcal{S}(W)$-valued order k symbols on $\mathbb{R}_+$ vanishing for small t. Considering $t = \langle\xi,e\rangle$ as a parameter we obtain a natural isomorphism

$$S_0^{k,\infty}(\mathfrak{g}^*) \cong S_0^k(\mathbb{R}_+, \mathcal{S}(W)) .$$

In the right hand side we may view $\mathcal{S}(W)$ as the restriction of $\mathcal{S}(\mathfrak{g}^*)$ to $\langle\xi,e\rangle = 1$ and the restriction of # to this space is well defined. The multiplication # will then be respected by the isomorphism above in view of the homogeneity of Φ . By a single partition of unity w.r.t. the variables ξ_2 we may also assume that a has its support in a small conic neighborhood of a vector e^* . This will imply that $|\xi_2| \leq$ Cst. in the support of the image $\tilde{a}$ of a under the isomorphism considered above.

3. The rank 2 case

Let (V,σ,g) be a symplectic vector space with a positively quadratic form g on V with dual form g^* on V^*. We shall assume that we have a bound

$$\sigma(x,y)^2 \leq C_0 g(x) g(y) \tag{3.1}$$

with C_0 fixed all the time. If $u \in \mathcal{S}(V^*)$ we set

$$|u|_k(\xi) = \sum_{j \leq k} \max_{\eta} |u^{(j)}(\xi_j,\eta_1,\ldots,\eta_j)| / g^*(\eta_1)^{1/2} \ldots g^*(\eta_j)^{1/2} .$$

Note that σ defines a differential operator D_σ on $(V^* \times V^*)$ and an associative multiplication $\#_\sigma$ on $\mathcal{S}(V^*)$ is defined by

$$(u \underset{\sigma}{\#} v)(\xi) = e^{iD_\sigma/2}\, u(\xi)\, v(\eta)/\, \xi = \eta \,.$$

After a quantization every $u \in \mathcal{S}(V^*)$ can be viewed as the symbol of a pseudo differential operator and its operator L^2-norm is independent of choice of quantization as well as its Hilbert-Schmidt norm. We denote these by $\|u\|_{\sigma,L^2}$ and $\|u\|_{\sigma,HS}$. Set $\|u\|_{g,k} = \max_{\xi} |u|_k(\xi)$.

Lemma 3.1 : There is a positive $C = C(n)$, $n = \dim V$ so that the following holds : If $u \in \mathcal{S}(V^*)$ and $\|u\|_{\sigma,L^2} < C_o$ then there is a unique v in $\mathcal{S}(V^*)$ so that

$$(1-u) \underset{\sigma}{\#} (1-v) = (1-v) \underset{\sigma}{\#} (1-u) = 1 \,.$$

There are also maps $k \to k'$, C'_k depending on n and C_o so that

$$\|v\|_{g,k} \leq C'_k (1 + \|u\|_{g,k'})^{k'} \|u\|_{g,k'} \,.$$

By using (Ro) and Lemma 3.1 with $\sigma = B_{\xi_2}(x,y)$, $x,y \in \mathcal{G}_1/\mathrm{Rad}\, B_{\xi_2}$ (see also the case $\mathcal{G}$ = the Heisenberg group treated in Melin [2]) one can always modify a so that $\tilde{a}$(= a considered as an element in $S^o(\mathbb{R}_+, \mathcal{S}(W))$ vanishes along some orbit $\mathcal{O}(\xi)$ for the co-adjoint representation with $\xi_2 = \eta_2$ = a fixed element in $\mathcal{G}^*_2$. The norm $\|\tilde{a}\|_{\xi,L^2}$ is not changed much when one pass to nearby orbits. This allows one (by a partition of unity argument) to find a with $\sup\|\tilde{a}\|_{\xi,L^2}$ small when $\xi_2 = \eta_2$ and an application of Lemma 3.1 gives then an a vanishing identically for $\xi_2 = \eta_2$. Finally one has to consider derivation w.r.t. ξ_2 for ξ_2 in a compact set. These derivatives must be smooth where B_{ξ_2} has maximal rank and the estimates we obtain for then are uniform.

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