JOURNÉES ÉQUATIONS AUX DÉRIVÉES PARTIELLES

LOUIS NIRENBERG

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Journées Équations aux dérivées partielles (1981), p. 1-4
<http://www.numdam.org/item?id=JEDP_1981____A13_0>

REMARKS ON THE NAVIER-STOKES EQUATIONS

by L. NIRENBERG

This talk is a report of work in progress of L. Caffarelli, R. Kohn and L. Nirenberg [1] extending the results of V. Scheffer [2-4] . It concerns weak solutions of the incompressible Navier-Stokes equations in 3 space dimensions of fluid flow. The velocity vector

$$u = (u^1, u^2, u^3)$$

satisfies (using summation convention and subscripts t, i to denote differentiation with respect to time or x_i)

$$u^j_t - \Delta u^j + u^i u^j_i + p_j = 0 \qquad j = 1,2,3 \tag{1}$$

$$\nabla . u = u^j_j = 0 . \tag{2}$$

Here Δ is the Laplace operator in the space variables, and p represents the pressure; viscosity has been normalized to be one . For simplicity we assume here that the forcing term on the right of (1) is zero.

The initial value problem consists in prescribing

$$u(x,0) = u_0(x) .$$

We suppose u_0 has finite energy $E_0 = \int |u_0|^2 dx$.

If we consider a flow in a fixed domain G rather than all of R^3, we also prescribes some boundary conditions, for example the values of u(x,t) for $x \in \partial G$, $t \geq 0$, say zero. Since the classical work of J. Leray and subsequently, E. Hopf, one knows the existence of weak solutions of (1), (2) for $t > 0$ with finite energy for any time :

$$\int |u(x,t)|^2 dx \leq C(T) \qquad \text{for } 0 \leq t \leq T. \tag{3}$$

and

$$\int_0^T \int |Du|^2 dx\, dt \leq C(T) \tag{4}$$

where $|Du|^2 = \sum_{i,j} (u_i^j)^2$. Furthermore one has the energy inequality which we express in the following form; $\forall\, T > 0$, for $\varphi \in C_0^\infty$ $(t \leqslant T)$, $\varphi(x,t) \geqslant 0$, we have

$$(5) \qquad \int_{t=T} \varphi|u|^2 dx + \iint \varphi|Du|^2 dx\, dt \leqslant \iint \tfrac{1}{2}|u|^2(\varphi_t + \Delta\varphi) + (|u|^2 + p)u^i\varphi_i\, dxdt .$$

Formally, one obtains this, with equality, if one multiplies (1) by φu^j, integrates in (x,t) and sums over j. However up to now one has only proved the existence of a weak solution of (1), (2) satisfying the inequality (5).

Since the early 50's (see the books by O. Ladyzhenskaya [5], J. L. Lions [6] and R. Temam [7]) there has been much work devoted to the following questions :

1. Is the weak solution, described above, of the initial (or initial boundary value) problem unique ? If one can prove some further regularity of the solution then it is (see [5-7]).

2. Can the solution(s) develop singularities? If so can they eat energy in the sense that one may have strict inequality in (5) ? For the initial value problem one knows that after some time $t = T_0(E_0)$ the velocity $|u|$ is finite. u is then C^∞ with respect to the space variables .

A weaker form of 2 is :

3. Can the solution develop singularities in case $u_0(x)$ is a nice function ?

Let S be the complement of the largest open set (in space-time) in which $u \in L^\infty_{loc}$, i.e. S is the set where $|u|$ becomes infinite. Treating the initial value problem in all of R^3 in [1], and the initial boundary value problem in [3], Scheffer proved the following result.

<u>Theorem 1</u> (Scheffer) : <u>The 5/3-Hausdorff measure of S is finite</u> : $H^{5/3}(S) < \infty$.

In [1] we localize his arguments and extend them to give the following improvement.

<u>Theorem 2</u> : <u>If</u> u <u>is a weak solution in an open set in</u> $R^3 \times R$ <u>satisfying</u> (3) - (5) <u>then</u> $H^1(S) = 0$.

The proof is based on two propositions. The first is a local form of the key proposition of Scheffer in [2]. In the following, if $P = (x_0, t_0)$ we denote by $Q_r = Q_r(P)$ a circular cylinder in (x,t) space given by

$$Q_r = \{(x,t) \mid |x - x_o| \leq r,\ t_o - r^2 \leq t \leq t_o\} \quad .$$

Proposition 1 : There is an absolute constant $\delta > 0$ such that if u is a solution of (1), (2) in $Q_1(P)$, $P = (0,0,0,1)$ with

$$\int_{Q_1}\int (|u|^3 + |u||p|)dx\,dt + \int_0^1 (\int_{|x|<1} |p|dx)^{3/2} dt \leq \delta$$

then $|u|$ is finite in a neighborhood of P in $Q_1(P)$.

A reformulation of this result is the following obtained by scaling - if u is a solution , so is $\lambda u(\lambda x, \lambda^2 t)\ \forall \lambda > 0$.

Proposition 1' : If $P \in S$ then

$$r^{-2}\iint_{Q_r(P)} (|u|^3 + |u||p|)dxdt + r^{-7/2}\int_{t_o - r^2}^{t_o} (\int_{|x-x_o|<r} |p|dx)^{3/2} dt > \delta \ .$$

Making use of this, and various interpolation estimates, as well as the relationship

$$\Delta p = - u^i_j\, u^j_i \ ,$$

we prove

Proposition 2 : There is an absolute constant $\delta' > 0$ such that if $P \in S$ then

$$\overline{\lim_{r \to 0}}\ r^{-1}\iint_{Q_r(P)} |Du|^2 dxdt > \delta' .$$

Proposition 1 is proved with the aid of special test functions φ in (5) approximating the fundamental solution of the backward heat equation, with singularity at P.

REFERENCES

[1] L. Caffarelli, R. Kohn, L. Nirenberg, to appear.

[2] V. Scheffer, Comm. Math. Phys. 55 (1977), p.97-112.

[3] V. Scheffer, Comm. Math. Phys. 61 (1978) p.41-68.

[4] V. Scheffer, Comm. Math. Phys. 73 (1980) p.1-42.

[5] O. A. Ladyzhenskaya (her book on fluid dynamics).

[6] J. L. Lions, livre paru chez Dunod.

[7] R. Temam (his recent book on Navier-Stokes equation).

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