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THE EIGENVALUES OF HYPOELLIPTIC OPERATORS

A. MENIKOFF and J. SJÖSTRAND



Let $P = P(x, D)$ be a self-adjoint pseudo-differential operator of order $m > 0$, with principal symbol $p_m(x, \xi) \geq 0$ on a smooth n -dimensional compact riemannian manifold M without boundary. If P is elliptic then P has a discrete set of eigenvalues bounded from below. Denoting by $N(\lambda)$ the number of eigenvalues $\leq \lambda$ (counting multiplicities) the distribution of eigenvalues of P may be described by the formula

$$(1) \quad N(\lambda) \sim \frac{\lambda^{n/m}}{(2\pi)^n} \int_{p_m(x, \xi) \leq 1} dx \wedge d\xi \quad \text{as } \lambda \rightarrow \infty.$$

This result has a long history. It may be obtained by studying the singularities of one of the functions

$$\text{tr}(P - \lambda I)^{-1}, \quad \text{tr}(e^{-tP}), \quad \text{tr}(P^Z), \quad \text{tr}(e^{itP})$$

(see [1], [4] or [8]). Here we would like to consider the same problem for hypoelliptic operators.

A result in this direction has been obtained by Metivier [7], who studied the spectral function of hypoelliptic operators which are the sums of squares of real vector fields. He described the spectral function for operators which have a uniform behavior in the base space, but, for example for the Grušin operator, $D_{x''}^2 + |x''|^2 |D_{x'}|^2$, his results do not give the asymptotic behavior of the eigenvalues. Other results which overlap with ours have been presented at this meeting by Bolley, Camus and Pham [2].

We will discuss the eigenvalues of self-adjoint operators P which are hypoelliptic with the loss of one derivative. Let $\Sigma = \{p_m(x, \xi) = 0\}$ be the characteristic variety of P . We will suppose that Σ is a smooth symplectic submanifold of $T^*(M)$ and that p_m vanishes to exactly second order on Σ . Let $2n' = \dim \Sigma$, $2n'' = \text{codim } \Sigma$ and $\pm i\mu_j$, $j = 1, \dots, n''$, with $\mu_j > 0$ be the eigenvalues of the Hamilton matrix of p_m

(cf. [9]) restricted to the orthogonal space of Σ . Then, P will be hypoelliptic with the loss of one derivative if and only if

$$(2) \quad p_{m-1}^1(x, \xi) + \sum_{j=1}^{n''} \mu_j(x, \xi) (1 + 2\alpha_j) \neq 0$$

for any set of non-negative integers α_j , at every point $(x, \xi) \in \Sigma$. Here p_{m-1}^1 is the subprincipal symbol of P , (cf. [3] or [9]). In fact, P will have a parametrix $Q \in L_{\frac{1}{2}, \frac{1}{2}}^{1-m}$, i.e.

$$(3) \quad QP = I + K$$

where K is a compact operator on $L^2(M)$.

If $m > 1$ and P is hypoelliptic, then P will have only eigenvalues of finite multiplicity whose only limit points can be $\pm \infty$.

We will further suppose that on Σ

$$(4) \quad p_{m-1}^1 + \sum_{j=1}^{n''} \mu_j > 0.$$

It will then follow from a theorem of Melin [5] that there is a constant C such that

$$(5) \quad (Pu, u) \geq -C \|u\|^2$$

and consequently that the spectrum of P is bounded below. Then e^{-tP} is well defined for $t \geq 0$ and our goal will be to show

THEOREM 1. Under the above assumptions

$$(6) \quad \text{tr}(e^{-tP}) \sim \begin{cases} C_1 t^{-n'/(m-1)} & \text{if } n' > n''(m-1) \\ C_2 t^{-n/m} \log t & \text{if } n' = n''(m-1) \\ C_3 t^{-n/m} & \text{if } n' < n''(m-1) \end{cases}$$

as $t \downarrow 0$.

Since $\text{tr}(e^{-tP}) = \sum e^{-\lambda_j t}$ where λ_j are the eigenvalues of P , we may apply Karamata's Tauberian Theorem to conclude.

COROLLARY 2. Denoting the number of eigenvalues $\leq \lambda$ by $N(\lambda)$ we have

$$(7) \quad N(\lambda) \sim \begin{cases} a_1 \lambda^{n/(m-1)} & \underline{\text{if}} \quad n' > n''(m-1) \\ a_2 \lambda^{n/m} \log \lambda & \underline{\text{if}} \quad n' = n''(m-1) \\ a_3 \lambda^{n/m} & \underline{\text{if}} \quad n' < n''(m-1) \end{cases}$$

as $\lambda \rightarrow \infty$ (a_3 , incidently is the same constant as in formula (1)).

1. THE ELLIPTIC CASE.

We will begin our discussion of Theorem 1 by rederiving formula (1) for the elliptic case in a way amenable to generalization. To approximate $\exp(-tP)$ we will seek a solution of

$$(1.1) \quad \begin{aligned} D_t w &= i P(x, D_x) w & \text{or } \mathbb{R}^+ \times M \\ w(x, 0) &= u(x), \end{aligned}$$

micro-locally of the form

$$(1.2) \quad w(x, t) = A_t u(x) = (2\pi)^{-n} \int e^{i\varphi(t, x, \eta)} a(t, x, \eta) \hat{u}(\eta) d\eta.$$

Applying $D_t - i P(x, D_x)$ to (1.2) and grouping terms as if φ were homogenous of degree 1 in η we will get an eikonal equation of the form

$$(1.3) \quad \varphi_t - i p_m(x, \varphi'_x) = 0 \quad ; \quad \varphi(0, x, \xi) = x \cdot \eta$$

and various transport equations. Making the change of variables $t = |\eta|^{m-1} s$, (1.3) will become

$$(1.4) \quad \varphi_s - i p'(x, \varphi'_x) = 0 \quad \text{where } p' = p_m(x, \varphi'_x) / |\eta|^{m-1}$$

for which we will try to find a solution which is homogenous of degree 1 in η . Expanding φ as a power series in s we can find

$$(1.5) \quad \varphi(s, x, \eta) = \langle x, \eta \rangle + i P^1(x, \eta) s + \psi_2(x, \eta) s^2 + \dots$$

which satisfies (1.4) modulo an arbitrarily high power of s . From the first transport equation we find that $a = 1 + O(s)$. Since φ leaves the real axis rapidly we may modify φ and a for large s so as to get a solution of (1.1) modulo an operator with C^∞

kernel in x and t .

As a result

$$e^{-tP}u(x) \approx A(t)u(x) = (2\pi)^{-n} \int e^{i\langle x-y, \eta \rangle - tP_m(x, \eta) + \dots} a(t, x, \eta) u(y) dy d\eta$$

and

$$\begin{aligned} \text{tr}(e^{-tP}) &\approx (2\pi)^{-n} \iint e^{-tP_m(x, \xi)} dx \wedge d\xi + \dots \\ &= (2\pi)^{-n} t^{-n/m} \frac{n}{m} \Gamma\left(\frac{n}{m}\right) \iint_{P_m(x, \xi) \leq 1} dx \wedge d\xi + \dots \end{aligned}$$

modulo a function less singular in t . Applying Karamata's Tauberian Theorem gives (1).

2. THE HYPOELLIPTIC CASE.

We will now attempt to find a solution of (1.1) micro-locally of the form (1.2) when P satisfies the assumption of Theorem 1. The eikonal equation will be of the form

$$(2.1) \quad \varphi'_t = iP_m(x, \varphi'_x)$$

again. We make the same change of variables as before to make (2.1) homogenous. But this time it will be necessary to solve (2.1) as $s \rightarrow \infty$. This is because the solutions of (2.1) will not leave the real axis everywhere. In fact, bicharacteristics starting in Σ stay in Σ giving a point where $\text{Im } \varphi$ stays 0.

We'll solve (2.1) using Hamilton-Jacobi Theory. We'll make a series of canonical transformations to simplify our problem. To begin with let us choose new canonical coordinates so that $\Sigma = \{x'' = \xi'' = 0\}$ where $(x, \xi) = (x', x'', \xi', \xi'')$, $x' \in \mathbb{R}^{n'}$, $x'' \in \mathbb{R}^{n''}$ etc. Setting $t = s |\eta'|^{m-1}$, (2.1)

becomes

$$(2.2) \quad \varphi'_s = iP_m(x, \varphi'_x) / |\eta'|^{m-1} = i p'(x, \varphi'_x).$$

Expanding p' as a Taylor's series in (x'', ξ'') we find

$$(2.3) \quad p'(x, \xi) = \sum_{|\alpha+\beta|=2} a_{\alpha\beta}(x', \xi') x''^\alpha \xi''^\beta + O(|\xi|^m (|x''| + |\xi''| |\xi|)^3).$$

The quadratic terms in (2.3) may be expressed as

$$\sigma((x'', \xi''), H(x'', \xi''))$$

where H , the (transversal) Hamilton matrix of p is skew-symmetric with respect to the standard symplectic form σ in $\mathbb{R}^{2n''}$.

Recalling the results of [9], H has eigenvalues of the form $\pm i\mu_j(x', \xi')$ with $\mu_j > 0$ for $j = 1, \dots, n''$, and if V_+ (V_-) denotes the span of the positive (negative) eigenvectors of H in $\mathbb{C}^{2n''}$, then V_+ (V_-) is a positive (negative) definite Lagrangean plane in $\mathbb{C}^{2n''}$, and

$$\mathbb{C}^{n''} \oplus \mathbb{C}^{n''} = V_+ \oplus V_-.$$

Since V_{\pm} depend smoothly on (x', ξ') we may make a complex canonical change of variables so that $V_- = \{x'' = 0\}$ and $V_+ = \{\xi'' = 0\}$. In terms of these new coordinates

$$(2.4) \quad H = \frac{i}{2} \begin{pmatrix} A & 0 \\ 0 & -A^t \end{pmatrix}$$

where A is a matrix with only positive eigenvalues.

Since we have made a complex change of variable the following considerations will be only formal and will required justification.

Equation (2.2) now takes the form

$$(2.5) \quad \varphi'_S = - \langle A(x', \varphi'_{x'}) \rangle x'', \varphi'_{x''} \rangle + \sum_{3 < |\alpha+\beta|} b_{\alpha\beta} x''^\alpha \varphi'_{x''}{}^\beta.$$

It is possible to find one more canonical transformation so as to make the higher order term in (2.5) takes the form $O(|x''| |\varphi'_{x''}| (|x''| + |\varphi''|))$. Solving (2.5) by using formal power series in (x'', η'') we will get a solution

$$\varphi = \langle x', \eta' \rangle + \langle e^{-SA} x'', \eta'' \rangle + \text{cubic term in } (x'', \eta'').$$

The phase function of A_t is

$$\psi = \langle e^{-SA} x'' - y'', \eta'' \rangle + \langle x' - y', \eta' \rangle + \dots$$

where the other higher order terms converge to 0 exponentially fast.

Denoting by $C_S = \{(x, \varphi'_{x'} - \varphi'_\eta, \eta)\}$ the canonical relation generated by ψ we may note the C_0 is the graph of the identity and $C_\infty = \{(x', x'', \xi', 0), (x', 0, \xi', \xi'')\}$.

The first transport equation is

$$(2.7) \quad \frac{da}{ds} + \left(\frac{1}{2} \operatorname{tr} A + p'_{m-1}\right)a = 0(|x''| + |\xi''|)$$

whose solution is

$$a(s, x, \xi) = e^{-s\left(\frac{1}{2} \operatorname{tr} A + p'_{m-1}\right)} + 0(|x''| + |\xi''|).$$

The leading term of the solution $A_t u$ is

$$(2\pi)^{-n} \int e^{i\langle e^{-sA} x'' - y'', \xi'' \rangle + \langle x' - y', \xi' \rangle - s(\operatorname{tr}^+ H + p'_{m-1})} u(y) dy d\xi.$$

The leading term of $\operatorname{tr}(e^{-tP})$ is then

$$(2.8) \quad (2\pi)^{-n} \int e^{i\langle (e^{-sA} - I)x'', \xi'' \rangle} e^{-s(\operatorname{tr}^+ H + p'_{m-1})} dx d\xi.$$

When $n' > n''(m-1)$ we will compute the singular part of (2.8).

Evaluate the integral with respect to (x'', ξ'') in (2.8) by the "method of stationary phases" (thinking of $s^{-1} = |\xi|^{m-1}/t$ as the large parameter). This gives that the leading term of $\operatorname{tr}(\exp(-tP))$ is

$$(2.9) \quad (2\pi)^{-n'} \int \frac{e^{-s(\operatorname{tr}^+ H + p'_{m-1})}}{\det(I - e^{-sA})} dx' \wedge d\xi'.$$

It is easily seen that

$$\begin{aligned} \det(I - e^{-sA})^{-1} &= \prod (1 - e^{-2s\mu_j})^{-1} \\ &= \sum_{0 \leq \alpha \in \mathbb{Z}^n} e^{-2(\alpha \cdot \mu)s} \end{aligned}$$

where $2\mu_1, \dots, 2\mu_n$ are the eigenvalues of A . When $n' > (m-1)n''$ the integral

(2.9) is convergent and equals

$$(2.10) \quad \frac{t^{-\frac{n'}{m-1}}}{(2\pi)^{n'}} \frac{n'}{m-1} \Gamma\left(\frac{n'}{m-1}\right) \int_{\Sigma \cap \{F(x', \xi') \geq 1\}} dx' \wedge d\xi'$$

where

$$(2.11) \quad F(x', \xi') = \sum_{0 \leq \alpha \in \mathbb{Z}^{n''}} (p'_{m-1}(x', \xi') + (1 + 2\alpha_j)\mu_j(x', \xi'))^{-n'/m-1}$$

(P is hypoelliptic if and only if $F \neq \infty$ for all $(x', \xi') \in \Sigma$).

Applying a Tauberian theorem will yield

$$(2.12) \quad N(\lambda) \sim \frac{\lambda^{n'/(m-1)}}{(2\pi)^{n'}} \int_{\{F \geq 1\} \cap \Sigma} dx' \wedge d\xi'.$$

This completes a sketch of the proof of Theorem 1. A justification of our formal changes and variable and complete details of the proof will appear in a future publication.

After this conference we learned that Trèves has also constructed exponential e^{-tP} for the same class of operators considered here. Trèves' construction is different from ours. As an application he proves the local analytic hypoellipticity of the $\bar{\partial}$ -Neuman-problem.

REFERENCES

- [1] S. AGMON and Y. KANNAI. On the asymptotic behavior of the spectral functions and resolvent kernels of elliptic operators. *Israel J. Math.* 5 (1967) 1-30.
- [2] C. BOLLEY, J. CAMUS and T. PHAM. In this volume.
- [3] L. BOUTET de MONVEL and F. TREVES. On a classe of pseudo-differential operators with double characteristics. *Invent. Math.* 24 (1974) 1-34.
- [4] L. HÖRMANDER. The spectral function of an elliptic operator. *Acta Math.* 121 (1968) 193-218.
- [5] A. MELIN. Lower bounds for pseudo-differential operators. *Ark. Mat.* 9 (1971) 117-140.
- [6] A. MELIN and J. SJÖSTRAND. Fourier integral operators with complex-valued phase functions. *Springer L.N.* 459, 255-282.
- [7] G. METIVIER. Fonction spectrale et valeurs propres d'une classe d'opérateurs non elliptiques. *Comm. PDE* 1 (1976) 467-519.
- [8] R. SEELEY. Complex powers of an elliptic operator. *AMS Proc. Symp. in Pure Math.* 10 (1967) 288-307.
- [9] J. SJÖSTRAND. Parametrices for pseudodifferential operators with multiple characteristics. *Ark. Mat.* 12 (1974) 85-130.