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ON THE DECIDABILITY OF THE EQUIVALENCE PROBLEM FOR MONADIC RECURSIVE PROGRAMS *

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Abstract. We present a uniform and easy-to-use technique for deciding the equivalence problem for deterministic monadic linear recursive programs. The key idea is to reduce this problem to the well-known group-theoretic problems by revealing an algebraic nature of program computations. We show that the equivalence problem for monadic linear recursive programs over finite and fixed alphabets of basic functions and logical conditions is decidable in polynomial time for the semantics based on the free monoids and free commutative monoids.

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Informally, the equivalence problem for programs is to find out whether two given programs have the same behavior. Taking various formalizations of the terms "program" and "behavior", we get numerous variants of this problem. Some variants of the equivalence problem are decidable [1, 3, 7, 11–13, 18, 20, 21, 24, 26]. There are also many cases when the equivalence problem was proved to be undecidable [4, 8, 13, 16–19]. The study of the frontier between decidable and undecidable cases of the equivalence problem for models of computations is of fundamental interest in computer science, since it significantly influences both the theory and practice of programming. By tackling the equivalence problem we understand better the relationship between the syntactic and semantic components of computer programs and comprehend to what extent specific changes in the structure of program affect its behavior.

The decidability of the equivalence problem essentially depends on the expressive power of computational model and the exact meaning of the term "the

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same behavior". When programs under consideration are deterministic, it is assumed that two programs have the same behavior if for every valid input they output identical results (if any). The functional equivalence thus defined is undecidable for the universal computational models whose program are capable to compute all recursive functions. At the same time it is decidable for the less powerful models such as Yanov's schemata [20, 26], finite multi-tape and push-down deterministic automata [10,23], monadic functional schemata [1,6], and some others. Sometimes the frontier between the decidable and undecidable are rather subtle: by changing a bit the syntax of computational models, we jump from the decidable cases to the undecidable ones. Thus, the equivalence problem is decidable for deterministic multi-tape automata [10], one counter automata [25], and monadic recursive programs w.r.t. semantics based on free monoids of basic functions [1], whereas it is undecidable for nondeterministic multi-tape automata [18], multi-counter automata [11], and for polyadic recursive programs [4]. In this paper we refine the decidability/undecidability frontier for the equivalence problem for monadic recursive programs w.r.t. some natural semantics based on the monoids of basic functions.

The concept of monadic recursive programs (MRPs) was introduced by de Bakker and Scott in [4]. They raised the decision problems for recursive programs and proved, using the results of Paterson [16], that the equivalence problem for polyadic recursive programs is undecidable. The first decidability results for MRPs were obtained in [1,6]. Garland and Luckham [6] proved the equivalence problem to be decidable for linear MRPs w.r.t. the semantics $\mathcal{U}$ based on the free monoid of basic functions (built-in procedures). Ashcroft *et al.* [1] obtained a somewhat more general result, establishing the decidability of the equivalence problem for free MRPs w.r.t. to the semantics $\mathcal{U}_{\text{const}}$ based on the free monoid of basic functions augmented with constants. Both proofs use techniques from formal language and automata theories; time complexity of the decision algorithms is exponential of program size. Sabelfeld [22] proposed a complete equational calculus for linear MRPs. Lisovik [14,15] proved the decidability of the equivalence problem for meta-linear MRPs w.r.t. $\mathcal{U}_{\text{const}}$. He proposed an original technique, reducing the equivalence problem for MRPs to the solvability problem for linear Diophantine equations. Friedman [5] showed that the equivalence problem for MRPs w.r.t. $\mathcal{U}_{\text{const}}$ is interreducible with the equivalence problem for deterministic pushdown automata. Recently the latter has been proved to be decidable [23].

All these decidability results focus on the unique semantics $\mathcal{U}_{\text{const}}$ based on the free monoid augmented with constants and yield no result for equivalence w.r.t. other classes of interpretations [2]. The obtained decision algorithms have at least the exponential time complexity even when the signature of programs is finite and fixed. To overcome these difficulties and extend the decidability results we propose a uniform and simple technique for deciding the equivalence problem for linear MRP w.r.t. to the interpretations based on the length-preserving monoids of basic functions. The key idea is to reduce the equivalence problem for MRPs to some known algebraic problems (such as the identity problem for semigroups) by revealing the algebraic properties of program computations. When the corresponding algebraic problems are efficiently decidable, this method yields polynomial-time decision procedures as a result. By applying our approach to some classes of program interpretations, we show that the equivalence problem for linear MRPs w.r.t. semantics based on the free monoids and free commutative monoids is decidable in polinomial time.

1. Preliminaries

In this section we define the syntax and the semantics of monadic recursive programs (MRPs). To simplify notation we modify a little the original concept of MRP presented in [1,4,6].

1.1. Syntax of MRPs

Fix two finite alphabets $\mathcal{A} = \{a^1, \dots, a^N\}$, $\mathcal{C} = \{c^1, \dots, c^M\}$ and an infinite alphabet $\mathcal{P} = \{F_1, F_2, \dots\}$.

The elements of $\mathcal{A}$ are called *basic actions*. Intuitively, basic actions stand for elementary built-in procedures, like $\sin(x)$ or $tail(L)$. A finite sequence of basic actions is called a *basic term*. The set of all basic terms is denoted by $\mathcal{A}^*$. We write λ for the empty sequence of actions and call it the *empty term*.

Symbols of $\mathcal{C}$ are called *conditions*. Every condition may be viewed as a finite tuple $\langle \sigma_1, \dots \sigma_k \rangle$ of truth-values of primitive relations on program data, like $x < y$ or $L = nil$. It is assumed that the set of primitive relations used in programs is finite and fixed, therefore the internal structure of conditions is of no importance.

Elements of $\mathcal{P}$ are called *procedures*; they stand for the names of recursive procedures defined in programs. A finite sequence of basic actions and procedures is called a *term*. The set of all terms over $\mathcal{A} \cup \mathcal{P}$ is denoted by $Term$. As usual, we write $|t|$ for the length of a term t, and $t_1 t_2$ for the concatenation of t_1 and t_2. The notation $F \in t$ is used to indicate that a procedure F occurs in t.

A *definition* of a procedure F is an expression D of the form

$$F : (c^1, t_1), (c^2, t_2), \dots, (c^M, t_M), \tag{1}$$

where $t_i \in Term, 1 \le i \le M$. The first occurrence of F in D is called the *head* of D, and the list of pairs $(c^1, t_1), (c^2, t_2), \dots, (c^M, t_M)$ is the *body* of D. For every pair (c^i, t_i) in the body of D, the term t_i is called a *c^i-variant* of the definition.

A *(deterministic) monadic recursive program (MRP)* over the alphabets $\mathcal{A}$, $\mathcal{C}$, $\mathcal{P}$ is a tuple $\pi = \langle G, D_1, D_2, \dots, D_n \rangle$, where

- $G \in Term$ is the *goal* of the program;
- $D_1, D_2, \dots, D_n$ are definitions of pairwise different procedures $F_1, \dots, F_n$.

The set of procedures $\{F_1, \dots, F_n\}$ defined in π is denoted by $\mathcal{P}_\pi$. Given a procedure F in $\mathcal{P}_\pi$ and condition c in $\mathcal{C}$ we write $D_\pi(F)$ for the definition of F in π, and $D_\pi(F, c)$ for the c-variant of $D_\pi(F)$. When a program is understood the subscript π will be omitted. It is assumed that every procedure F occurred in

π is defined in π. The size of π is the total length $\sum_{F\in\mathcal{P}_\pi}\sum_{c\in\mathcal{C}}|D(F,c)|$ of all terms occurred in π.

Given a MRP π and a pair of procedures F', F'' in $\mathcal{P}_\pi$ we say that F' *refers to* F'' if there exists a sequence of procedures $F_1, F_2, \dots, F_m$, $m > 1$, such that $F' = F_1$, $F'' = F_m$, and for every i, $1 \le i < m$, $F_{i+1} \in D(F_i, c)$ for some $c \in \mathcal{C}$. A procedure F in $\mathcal{P}_\pi$ is called

- a *self-referenced* if F refers to itself;
- a *marginal* if F does not refer to any self-referenced procedure in π;
- a *terminated* if either some variant $D(F,c)$ is a basic term or F refers to F' whose variant $D(F',c)$ is a basic term.

Example 1.1. Let us consider a recursive program written in the conventional style of [1,4,6]. This program is intended to compute the disbalance of parantheses [,] in symbolic strings.

$F_0(x)$;
$F_0(x) \Leftarrow$ **if** $x = nil$ **then** $F_2(x)$ **else** $F_1(x)$
$F_1(x) \Leftarrow$ **if** $head(x) =$ [**then** $plus1(F_0(tail(x)))$
else if $head(x) =$] **then** $minus1(F_0(tail(x)))$ **else** $F_0(tail(x))$;
$F_2(x) \Leftarrow zero(x)$.

This program includes basic actions $head(x)$, $tail(x)$, $plus1(x)$, $minus1(x)$ and $zero(x)$. Denote them by a, b, c, d, f respectively. The primitive relations used in this programs are $x = nil$, $head(x) =$ [and $head(x) =$]. Thus, we have 8 conditions $c^1, \dots, c^8$ standing for the tuples of truth-values $\langle\bot,\bot,\bot\rangle, \dots, \langle\top,\top,\top\rangle$ of primitive relations. By the definitions above we present this program as follows:

F_0;
$F_0 : (c^1, F_1), (c^2, F_1), (c^3, F_1), (c^4, F_1), (c^5, F_2), (c^6, F_2), (c^7, F_2), (c^8, F_2)$;
$F_1 : (c^1, F_0b), (c^2, dF_0b), (c^3, cF_0b), (c^4, cF_0b),$
$(c^5, F_0b), (c^6, dF_0b), (c^7, cF_0b), (c^8, cF_0b)$;
$F_2 : (c^1, f), (c^2, f), (c^3, f), (c^4, f), (c^5, f), (c^6, f), (c^7, f), (c^8, f)$.

In this program F_0, F_1 are self-referenced terminated procedures, whereas F_2 is a marginal procedure.

1.2. Dynamic frames and models

The semantics of MRP is defined by means of dynamic Kripke structures (frames and models) (see [9]).

A *dynamic deterministic frame* (or simply a *frame*) over alphabet $\mathcal{A}$ is a triple $\mathcal{F} = \langle S, s_0, R\rangle$, where

- S is a non-empty set of *data states*;
- s_0 is the *initial state*, $s_0 \in S$;
- $R : \mathcal{A} \times S \to S$ is an *updating function*.

For every $a \in \mathcal{A}$, $s \in S$ the state $R(a,s)$ is interpreted as the result of application of the action a to the data state s. An updating function R can be naturally extended to the set $\mathcal{A}^*$ of basic terms as follows: $R^*(\lambda, s) = s$, $R^*(at, s) = R(a, R^*(t,s))$.

A state s'' is said to be *reachable* from s' if $s'' = R^*(t, s')$ for some $t \in \mathcal{A}^*$. Denote by $[t]_\mathcal{F}$ the state $s = R^*(t, s_0)$ reachable from the initial state by means of t. As usual, the subscript $\mathcal{F}$ will be omitted when the frame is understood. Since we will deal only with data states reachable from the initial state, it is assumed that every state $s \in S$ is reachable from the initial state s_0, *i.e.* $S = \{[t] : t \in \mathcal{A}^*\}$.

A frame $\mathcal{F}_s = \langle S', s, R' \rangle$ is said to be a *subframe* of $\mathcal{F} = \langle S, s_0, R \rangle$ induced by a state $s \in S$ if $S' = \{R^*(s,t) : t \in \mathcal{A}^*\}$ and R' is the restriction of R to S'. A frame $\mathcal{F}$ is

- a *semigroup* if $\mathcal{F}$ can be mapped homomorphically onto every subframe $\mathcal{F}_s$;
- a *length-preserving* if $[t_1] = [t_2]$ implies $|t_1| = |t_2|$ for every pair $t_1,\ t_2$ of basic terms;
- a *universal* if $[t_1] = [t_2]$ implies $t_1 = t_2$ for every pair $t_1,\ t_2$ of basic terms.

Taking the initial state $s_0 = [\lambda]$ for the unit, one may regard a semigroup frame $\mathcal{F}$ as a finitely generated monoid $\langle S, * \rangle$ such that $[t_1] * [t_2] = [t_1 t_2]$. Clearly, the universal frame $\mathcal{U}$ corresponds to the free monoid on $\mathcal{A}$. In this paper we deal with the length-preserving semigroup frames.

A *dynamic deterministic model* (or simply a *model*) over alphabets $\mathcal{A}$, $\mathcal{C}$ is a pair $M = \langle \mathcal{F}, \xi \rangle$ such that

- $\mathcal{F} = \langle S, s_0, R \rangle$ is a frame over $\mathcal{A}$;
- $\xi : S \to \mathcal{C}$ is a *valuation function*, indicating for every data state $s \in S$ a condition $c \in \mathcal{C}$ satisfiable at s.

Let $\pi = \langle G, D_1, D_2, \dots, D_n \rangle$ be some MRP and $M = \langle \mathcal{F}, \xi \rangle$ be a model based on a frame $\mathcal{F} = \langle S, s_0, R \rangle$. A finite or infinite sequence of triples

$$r = (t_1, s_1, c_1), (t_2, s_2, c_2) \dots, (t_i, s_i, c_i), \dots, \tag{2}$$

where $t_i \in Term,\ s_i \in S,\ c_i \in \mathcal{C},\ i \geq 0$, is called a *run of* π *on* M if $t_1 = G$ and for every $i,\ i \geq 1$, one of the following requirements is satisfied:

1. if t_i is a basic term then $s_i = R^*(t_i, s_{i-1}),\ c_i = \xi(s_i)$, and the triple (t_i, s_i, c_i) is the last element of (2);
2. if t_i is a non-basic term of the form $t_i = TFt$, where $F \in \mathcal{P}_\pi,\ t \in \mathcal{A}^*$, then $s_i = R^*(t, s_{i-1}),\ c_i = \xi(s_i),\ t_{i+1} = TD(F, c_i)$.

When r is finite and a triple (s_m, c_m, t_m) is its last element we say that r *terminates* having s_m as the *result*. When run r is an infinite sequence we say that r *loops*. Since MRPs and frames under consideration are deterministic, every program π has the unique run $r(\pi, M)$ on a given model M. We denote by $[r(\pi, M)]$ the result of $r(\pi, M)$, assuming that $[r(\pi, M)]$ is undefined, when $r(\pi, M)$ loops.

Let π' and π'' be some MRPs, M a model, and $\mathcal{F}$ be a frame. Then π' and π'' are called

- *equivalent on* M ($\pi' \sim_M \pi''$ in symbols) if $[r(\pi', M)] = [r(\pi'', M)]$, *i.e.* either both runs $r(\pi', M)$ and $r(\pi'', M)$ loop (and hence have no results) or they both terminate with the same data state s as their results;
- *equivalent on* $\mathcal{F}$ ($\pi' \sim_\mathcal{F} \pi''$ in symbols) if $\pi' \sim_M \pi''$ for every model $M = \langle \mathcal{F}, \xi \rangle$ based on $\mathcal{F}$.

For a given frame $\mathcal{F}$ the *equivalence problem* w.r.t. $\mathcal{F}$ is to check for an arbitrary pair π_1, π_2 of MRPs whether $\pi' \sim_{\mathcal{F}} \pi''$ holds. When the decidability and complexity aspects of the equivalence problem are concerned the frame $\mathcal{F}$ under consideration is assumed to be effectively characterized in logic or algebraic terms. It is worth noticing that if a frame $\mathcal{F}_1$ can be homomorphically mapped onto a frame $\mathcal{F}_2$ then for every pair of MRPs π_1, π_2 the equivalence $\pi_1 \sim_{\mathcal{F}_1} \pi_2$ implies $\pi_1 \sim_{\mathcal{F}_2} \pi_2$. So, we have

Proposition 1.2. *If $\mathcal{U}$ is a universal frame then $\pi_1 \sim_{\mathcal{U}} \pi_2 \Rightarrow \pi_1 \sim_{\mathcal{F}} \pi_2$ holds for an arbitrary frame $\mathcal{F}$ and every pair π_1, π_2.*

1.3. Linear MRPs

A term t is said to be *linear* if it contains at most one occurrence of some procedure $F \in \mathcal{P}$. A definition D of F is called *linear* if each variant of D is a linear term. A MRP $\pi = \langle G, D_1, D_2, \dots, D_n \rangle$ is called *meta-linear* (*linear*) if every definition D_i, $1 \le i \le n$, (and the goal G as well) is linear. In this paper we study the equivalence problem for the linear MRPs only. A linear MRP π is called *normal* if its goal G is in $\mathcal{P}_\pi$ and for every procedures F in $\mathcal{P}_\pi$, each variant $D(F, c)$, $c \in \mathcal{C}$, is either a basic term or one of the form $tF'a$, where $t \in \mathcal{A}^*$, $F' \in \mathcal{P}$, $a \in \mathcal{A}$.

Proposition 1.3. *For every MRP π there exists a normal MRP π' such that $|\pi'| \le |\pi|^2$ and $\pi \sim_{\mathcal{U}} \pi'$.*

Every linear MRP π can be normalized efficiently in linear time by adding some auxiliary procedures and definitions to π. Taking into account Propositions 1.2, 1.3, we assume in what follows that every linear MRP under consideration is in the normal form.

Some useful properties of MRP's computations on the length-preserving models are revealed by the following propositions:

Proposition 1.4. *Let $\mathcal{F}$ be a length-preserving frame and π be some normal linear MRP having G as its goal. Suppose $c_0, c_1, \dots, c_n$ is a sequence of conditions and $F_0, F_1, \dots, F_n$ is a sequence of procedures such that $F_0 = G$ and $D(F_{i-1}, c_{i-1}) = t_i F_i a_i$, $1 \le i \le n$. Then the sequence of triples*

$$(G_0, s_0, c_0), (T_1, s_1, c_1), \dots, (T_i, s_i, c_i), \dots, (T_n, s_n, c_n),$$

where $s_0 = [\lambda]$ is the initial state of $\mathcal{F}$, $T_i = t_1 \dots t_i F_i a_i$, and $s_i = [a_i \dots a_1]$, $1 \le i \le n$, is a prefix of a run $r(\pi, M)$ of π on some model M based on $\mathcal{F}$.

Proof. Since $[a_i \dots a_1] \ne [a_j \dots a_1]$ for every pair i, j, $0 \le i < j$, we arrive at the conclusion of proposition by taking $M = \langle \mathcal{F}, \xi \rangle$ such that $\xi(s_0) = c_0$ and $\xi([a_i \dots a_1]) = c_i$, $1 \le i \le n$. □

Proposition 1.5. *Let $\mathcal{F}$ be a length-preserving frame and π_1, π_2 be some normal linear MRPs having G^1, G^2 as their goals. Given two sequences of conditions $c_0^1, c_1^1, \dots, c_{n_2}^1$ and $c_0^2, c_1^2, \dots, c_{n_2}^2$ suppose that $F_0^1, F_1^1, \dots, F_{n_1}^1$ and $F_0^2, F_1^2, \dots, F_{n_2}^2$*

are two sequences of procedures, such that $F_0^k = G^k$ *and* $D(F_{i-1}^k, c_{i-1}^k) = t_i^k F_i^k a_i^k$, $k = 1, 2,\ 1 \le i \le n_k$.

Then the sequences of triples

$$\begin{array}{ll} (G_0^1, s_0, c_0^1), (T_1^1, s_1^1, c_1^1), \dots, (T_i^1, s_i^1, c_i^1), \dots, (T_{n_1}^1, s_{n_1}^1, c_{n_1}^1), & \text{and} \\ (G_0^2, s_0, c_0^2), (T_1^2, s_1^2, c_1^2), \dots, (T_i^2, s_i^2, c_i^2), \dots, (T_{n_2}^2, s_{n_2}^2, c_{n_2}^2), & \end{array}$$

where $s_0 = [\lambda]$ *is the initial state of* $\mathcal{F}$, $T_i^k = t_1^k \dots t_i^k F_i^k a_i^k$, $s_i^k = [a_1^k \dots a_i^k]$, $k = 1, 2,\ 1 \le i \le n_k$, *are the prefixes of the runs* $r(\pi_1, M)$ *and* $r(\pi_2, M)$ *on the same model* M *based on* $\mathcal{F}$ *iff* $c_0^1 = c_0^2$ *and for every* i, $1 \le i < \min(n_1, n_2)$, *either* $c_i^1 = c_i^2$ *or* $[a_i^1 \dots a_1^1] \ne [a_i^2 \dots a_1^2]$.

Proof. Follows from Proposition 1.4 and the inherent property of the length-preserving frames. □

2. The equivalence problem w.r.t. the length-preserving frames

In this section we present a novel straightforward approach to the equivalence problem for linear MRPs w.r.t. some length-preserving semigroup frames. Its key idea is as follows. Given a frame $\mathcal{F}$ and a pair of MRPs π_1, π_2 we choose at first some specific semigroup W to encode all pairs of states $\langle s', s'' \rangle$. Then, using this coding, we construct a graph structure Γ to represent all pairs of runs $r(\pi_1, M)$, $r(\pi_2, M)$ of π_1, π_2 on the models based on $\mathcal{F}$. It will be proved that only a boundary fragment of $\Gamma(\pi_1, \pi_2)$ should be analyzed to check the equivalence of π_1 and π_2. Thus, the equivalence problem w.r.t. $\mathcal{F}$ can be reduced to the identity problem "$w' = w''$?" on W. When the latter is decidable in polynomial time and W is a group then the equivalence problem for MRPs w.r.t. $\mathcal{F}$ is decidable in polynomial time as well. Applying this technique, we demonstrate that the equivalence problem is decidable for the frames associated with the commutative monoids and it is decidable in polynomial time for the frames associated with the free monoids (the universal frames) and the free commutative monoids.

Given a semigroup frame $\mathcal{F} = \langle S, s_0, R \rangle$ we consider $\mathcal{F}$ as a monoid and write $\mathcal{F} \times \mathcal{F}$ for the direct product of the monoids.

Suppose W is a finitely generated monoid, U is a submonoid of W, and w^+, w^* are the distinguished elements in W. Denote by $\circ$ and e a binary operation on W and the unit of W respectively. The quadruple $K = \langle W, U, w^+, w^* \rangle$ is said to be a *criterial system* for $\mathcal{F}$ if K and $\mathcal{F}$ meet the following requirements:

$(R1)$ there exists a homomorphism φ of $\mathcal{F} \times \mathcal{F}$ in U such that

$$[t_1] = [t_2] \Leftrightarrow w^+ \circ \varphi(\langle [t_1], [t_2] \rangle) \circ w^* = e$$

holds for every pair t_1, t_2 in $\mathcal{A}^*$;

$(R2)$ for every element w in the coset $U \circ w^*$ (in the coset $w^+ \circ U$) the equation $X \circ w = e$ ($w \circ X = e$) has at most one solution X in the coset $w^+ \circ U$ ($U \circ w^*$ respectively).

It is worth noting that if W is a group then $(R2)$ is always satisfied.

Let $K = \langle W, U, w^+, w^* \rangle$ be a criterial system for a semigroup frame $\mathcal{F}$. Given a pair of normal linear MRPs π_i, $i = 1, 2$, such that $\mathcal{P}_{\pi_1} \cap \mathcal{P}_{\pi_2} = \emptyset$, define a labelled directed graph Γ. The nodes of Γ are the quadruples of the form (H^1, H^2, w', w''), where $H^i \in \mathcal{P}_{\pi_i} \cup \{\lambda\}$, $i = 1, 2$, and w', w'' are the elements of cosets $w^+ \circ U$ and $U \circ w^*$ respectively. The node (G^1, G^2, w^+, w^*), where G^1, G^2 are the goals of the MRPs, is called the *root* of Γ. The set of nodes is divided into three subsets X_1, X_2, and X_3 such that

$X_1 = \{(H^1, H^2, w', w'') : \ w^+ \circ w'' \neq e, \ H^i \in \mathcal{P}_{\pi_i}, \ i = 1, 2\}$,

$X_2 = \{(H^1, H^2, w', w'') : \ w^+ \circ w'' = e, \ \text{or} \ \ H^i \in \mathcal{P}_{\pi_i}, H^{3-i} = \lambda, \ i \in \{1, 2\}\}$,

and all other nodes are in X_3.

The arcs of Γ are marked with pairs (c^1, c^2) in $\mathcal{C} \times \mathcal{C}$. For every node x in Γ we define the set Δ_x as follows:

$$\Delta_x = \begin{cases} \{(c^1, c^2) : \ c^i \in \mathcal{C}, \ i = 1, 2\}, \text{ if } x \in X_1, \\ \{(c, c) : c \in \mathcal{C}\}, \text{ if } x \in X_2, \\ \emptyset, \text{ if } x \in X_3. \end{cases}$$

Each node x in X_1 has $|\mathcal{C}|^2 = M^2$ outgoing arcs, and each node x in X_2 has $|\mathcal{C}| = M$ outgoing arcs marked with pairs in Δ_x. The nodes in X_3 have no outgoing arcs. The arcs connect the nodes in Γ as follows.

Suppose $x = (F^1, F^2, w, u)$ is a node in $X_1 \cup X_2$, and $(c^1, c^2) \in \Delta_x$. Then the arc marked with (c^1, c^2) leads from x to $x' = (H^1, H^2, w', u')$, where $w' = w \circ \varphi(\langle [T^1], [T^2] \rangle)$, $u' = \varphi(\langle t^1, t^2 \rangle) \circ u$, and the terms H^k, t^k, T^k, $k = 1, 2$, are defined as follows. If $F^k \in \mathcal{P}_{\pi_k}$ and the term $D(F^k, c^k)$ is one of the form TFb, where $T \in \mathcal{A}^*$, $F \in \mathcal{P}_{\pi_k}$, $b \in \mathcal{A}$, then $H^k = F$, $t^k = b$, $T^k = T$. If $F^k \in \mathcal{P}_{\pi_k}$ and $D(F^k, c^k)$ is a basic term T, then $H^k = t^k = \lambda$, $T^k = T$. If $F^k = \lambda$ then $H^k = f^k = T^k = \lambda$.

To the end of the section we assume that $\mathcal{F}$ is a length-preserving frame, $K = \langle W, U, w^+, w^* \rangle$ is a criterial system for $\mathcal{F}$, programs π_1 and π_2 are normal linear MRPs having G^1, G^2 as their goals, and Γ is graph structure for π_1 and π_2 defined above.

Lemma 2.1. *Suppose $x_0, x_1, \ldots, x_m, x_{m+1}$, $m \geq 0$, is a finite sequence of nodes in Γ such that x_0 is the root of Γ and $x_i = (F_i^1, F_i^2, w_i, u_i)$, $1 \leq i \leq m+1$. Then*

$$x_0 \xrightarrow{(c_0^1, c_0^2)} x_1 \xrightarrow{(c_1^1, c_1^2)} \cdots \xrightarrow{(c_m^1, c_m^2)} x_{m+1}$$

is a directed path in Γ iff there exists a model M based on $\mathcal{F}$ such that the runs $r(\pi_k, M)$, $k = 1, 2$, have the prefixes

$$(G^k, s_0, c_0^k), (T_1^k, s_1^k, c_1^k), \ldots, (T_{m_k}^k, s_{m_k}^k, c_{m_k}^j),$$

and the following requirements are satisfied

1. *if $m_k \leq m$ then $T_{m_k}^k$ is a basic term;*

2. *for every i, $0 \le i \le m_k$, either T_i^k is a basic term t_i^k and $i = m_k$ holds, or T_i^k is one of the form $t_i^k F_i^k b_i^k$, such that*

$$\begin{array}{rcl} u_i & = & \varphi(\langle s_i^1, s_i^2 \rangle) \circ w^* \;=\; \varphi(\langle [b_i^1 \dots b_1^1], [b_i^2 \dots b_1^2] \rangle) \circ w^*, \\ w_i & = & w^+ \circ \varphi(\langle [t_1^1 \dots t_i^1], [t_1^2 \dots t_i^2] \rangle) \end{array}$$

hold.

Proof. By induction on m, using Proposition 1.4 and the definition of Γ. □

Lemma 2.2. *Suppose $c_1, \dots, c_m$, $m \ge 0$, is a finite sequence of conditions and $F_0^k, F_1^k, \dots, F_m^k$ are two sequences of elements in $\mathcal{P}_{\pi_j} \cup \{\lambda\}$, $k = 1, 2$, such that $F_{i+1}^k \in D(F_i^k, c_{i+1})$ if $D(F_i^k, c_{i+1})$ is not a basic term, and $F_{i+1}^k = \lambda$ otherwise, hold for every i, $0 \le i < m$. Then for every node $x_0 = (F_0^1, F_0^2, w, u)$ in Γ there exists a directed path*

$$x_0 \xrightarrow{(c_1,c_1)} x_1 \xrightarrow{(c_2,c_2)} \cdots \xrightarrow{(c_m,c_m)} x_m,$$

leading to the node $x_m = (H^1, H^2, w', u')$, such that $w' = w \circ \varphi(\langle [T^1], [T^2] \rangle)$, $u' = \varphi(\langle [t^1], [t^2] \rangle) \circ u$ for some basic terms t^1, t^2, T^1, T^2. The terms H^k, t^k, T^k, $k = 1, 2$, depend on F_0^1, F_0^2 and the sequence of pairs $(c_1, c_1), \dots, (c_m, c_m)$ only.

Proof. Follows from the definition of Γ. □

A node x in Γ is said to be *rejected* if either $x = (\lambda, \lambda, w', w'')$ and $w' \circ w'' \ne e$, or $x = (F^1, F^2, w', w'')$ is such that one of the terms F^1, F^2 is λ, whereas the other is a non-marginal procedure. A node $x = (F^1, F^2, w, u)$ is called a *marginal* if each of the terms F^1, F^2 is either a basic term λ or a marginal procedure.

Lemma 2.3. *$\pi_1 \sim_{\mathcal{F}} \pi_2$ iff no rejected node is accessible from the root Γ.*

Proof. Follows from Proposition 1.5, Lemmas 2.1, 2.2, and requirement $(R1)$ of criterial system. □

Lemma 2.4. *Suppose one of the procedures F^1, F^2 is terminated, a node $x = (F^1, F^2, w, u)$ is accessible from the root of Γ, and $w \circ v \circ u \ne e$ holds for every element v in U. Then some rejected node is accessible from the root of Γ.*

Proof. Suppose F^1 is terminated. Then, by Proposition 1.2 and Lemma 2.2, a node $y = (\lambda, H, w', u')$ is accessible from x by some path marked with pairs $(c_1, c_1), \dots, (c_m, c_m)$. If H is a non-marginal procedure then y is a rejected node which has to be found. Otherwise, a node $z = (\lambda, \lambda, w'', u'')$ is accessible from y by some path marked with pairs $(c_{m+1}, c_{m+1}), \dots, (c_n, c_n)$, $m < n$. Note, that, by Lemma 2.2, we have $w'' = w \circ \varphi(\langle [T^1], [T^2] \rangle)$, $u'' = \varphi(\langle [t^1], [t^2] \rangle) \circ u$ holds for some terms t^1, t^2, T^1, T^2. Since $w \circ v \circ u \ne e$ holds for every element v in U, we arrive at the conclusion $w'' \circ u'' \ne e$. Hence, z is a rejected node. □

Lemma 2.5. *Suppose one of the procedures F^1, F^2 is terminated and the nodes $x_1 = (F^1, F^2, w_1, u_1)$, $x_2 = (F^1, F^2, w_2, u_2)$ are accessible from the root of Γ.*

Suppose also that only one of the identities $w_1 = w_2$, $u_1 = u_2$ holds. Then some rejected node is accessible from the root of Γ as well.

Proof. Consider the case $w_1 = w_2$, $u_1 \neq u_2$. Suppose F^1 is a terminated procedure in π_1. Following the proof of Lemma 2.4, we come to the conclusion that either some node $y_1 = (\lambda, H^2, w_1', u_1')$ such that H^2 is a non-marginal procedure, or some node $z_1 = (\lambda, \lambda, w_1'', u_1'')$ is accessible from x_1 by some path marked with pairs $(c_1, c_1), \dots, (c_m, c_m)$. In the former case x_1 is a rejected node accessible from the root of Γ which we have to found. In the latter case two alternatives are possible. If $w_1'' \circ u_1'' \neq e$ then z_1 is a rejected node accessible from the root of Γ. If $w_1'' \circ u_1'' = e$ then, by Lemma 2.2, a node $z_2 = (\lambda, \lambda, w_2'', u_2'')$ is accessible from x_2 by the path marked with pairs $(c_1, c_1), \dots, (c_n, c_n)$. Following this way, we have

$$\begin{array}{rclcrcl} w_1'' & = & w_1 \circ \varphi(\langle [T^1], [T^2] \rangle), & \quad & u_1'' & = & \varphi(\langle [t^1], [t^2] \rangle) \circ u_1, \\ w_2'' & = & w_2 \circ \varphi(\langle [T^1], [T^2] \rangle), & \quad & u_2'' & = & \varphi(\langle [t^1], [t^2] \rangle) \circ u_2 \end{array}$$

for some basic terms t^1, t^2, T^1, T^2. Since $w_1 = w_2$, $u_1 \neq u_2$, and $w_1'' \circ u_1'' = e$, we obtain, as a consequence of $(R2)$ in the definition of criterial system, $w_2'' \circ u_2'' \neq e$. This means that z_2 is a rejected node.

In the case of $u_1 = u_2$, $w_1 \neq w_2$ the proof is analogous. □

Lemma 2.6. *Let $L = \max(|\pi_1|, |\pi_2|) + 1$, and F^1, F^2 be a pair of procedures such that one of them is non-marginal, whereas the other is terminated. Suppose that at least L pairwise different nodes $x_1 = (F^1, F^2, w_1, u_1), \dots, x_L = (F^1, F^2, w_L, u_L)$ are accessible from the root of Γ. Then some rejected node is accessible from the root of Γ as well.*

Proof. By Lemma 2.5, it suffices to consider the case, when the elements $u_1, \dots, u_L$ are pairwise different. Suppose F^1 is terminated and F^2 is non-marginal. Then there exists a sequence of conditions $c_1^1, \dots, c_m^1$ and a sequence of procedures $F_1^1, \dots F_m^1$, such that $m < L$, $F_1^1 = F^1$, $D(F_i^1, c_i^1) = T_i^1 F_{i+1}^1 b_i^1$, $1 \leq i < m$, and $D(F_m^1, c_m^1) = T_m^1 \in \mathcal{A}^*$. On the other hand, since F^2 is a non-marginal procedure, there exists a sequence of conditions $c_1^2, \dots, c_n^2$ and a sequence of procedures $F_1^2, \dots F_n^2$, such that $n < L$, $F_1^2 = F^2$, $D(F_j^2, c_j^2) = T_j^2 F_{j+1}^2 b_j^2$, $1 \leq j \leq n$ and F_n^2 is a self-referenced procedure. To simplify notation we assume $n = m$. Consider a matrix $\{v_{ij}\}_{j=1,L}^{i=1,m}$ whose elements are $v_{ij} = w^+ \circ \varphi(\langle [b_1^1 \dots b_i^1], [b_1^2 \dots b_i^2] \rangle) \circ u_j$. By requirement $(R2)$ of criterial system, for every i, $1 \leq i \leq m$, the list of elements $v_{i1}, v_{i2}, \dots, v_{iL}$ (the i-th row of the matrix) contains at most one element equal to the unit e of W. Since $m < L$, there exists k, $1 \leq k \leq L$, such that all elements of the column $v_{1k}, v_{2k}, \dots, v_{mk}$ are non-unit. Therefore, by definition of Γ, there exists a path from x_k to a node $y = (\lambda, F_k^2, w', u')$. Clearly, y is a rejected node since F_k^2 is non-marginal. □

Theorem 2.7. *Suppose $\mathcal{F}$ is a length-preserving semigroup frame over the alphabet $\mathcal{A}$, and $K = \langle W, U, w^+, w^* \rangle$ is a criterial system for $\mathcal{F}$ such that the identity problem "$w_1 = w_2$?" in W is decidable. Then the equivalence problem "$\pi_1 \sim_{\mathcal{F}} \pi_2$?" w.r.t. $\mathcal{F}$ is decidable for linear MRPs.*

Proof. Suppose π_1 and π_2 are normal MRPs, and $L = \max(|\pi_1|, |\pi_2|) + 1$. Since the identity problem "$w_1 = w_2$?" is decidable, any finite fragment of Γ can be constructed effectively. By Lemma 2.3, to verify the equivalence of π_1 and π_2 one needs only to check the accessibility of the rejected nodes from the root of Γ.

The following observations should be taken into account when searching for the rejected nodes. Let F^1, F^2 be a pair of procedures in π_1, π_2.

1. If both F_1 and F^2 are non-terminated then, clearly, no rejected node is accessible from any node x of the form (F^1, F^2, w, u).
2. If one of F^1, F^2 is a terminated procedure, whereas the other is non-marginal then, by Lemmas 2.5, 2.6, at most L nodes x of the form (F^1, F^2, w, u) should be analyzed to check the accessibility of the rejected nodes from the root of Γ.
3. If some finite fragment of Γ contains l non-marginal nodes then it contains at most $l|\mathcal{C}|^L$ marginal nodes.

Thus, to verify the equivalence of π_1 and π_2 it suffices to analyze only a finite fragment of Γ containing $O(L^3 + L^3|\mathcal{C}|^L)$ nodes. □

Theorem 2.7 provides us with an exponential-time algorithm deciding the equivalence problem for MRPs. To get a polynomial-time algorithm one needs to reduce the searching space of Γ down to the fragment of polynomial size. This is possible when the criterial monoid W is a group. So, we continue the list of lemmas, assuming now that the criterial system under consideration is based on a group W.

Lemma 2.8. *Let $x = (F^1,\ F^2, w_1, u_1)$ and $y = (F^1,\ F^2, w_1, u_1)$ be a pair of marginal nodes in Γ. If no rejected node is accessible from both x and y then $(u_1 \circ w_1)^{-1} = (u_2 \circ w_2)^{-1}$.*

Proof. By the definition of marginal nodes and Lemma 2.2, there exists a sequence of conditions $c_1, \dots c_m$, such that

$$\begin{array}{c} x = x_0 \xrightarrow{(c_1,c_1)} x_1 \xrightarrow{(c_2,c_2)} \cdots \xrightarrow{(c_m,c_m)} x_m = z', \\ y = y_0 \xrightarrow{(c_1,c_1)} y_1 \xrightarrow{(c_2,c_2)} \cdots \xrightarrow{(c^m,c^m)} y_m = z'' \end{array}$$

are directed paths in Γ, leading to $z' = (\lambda, \lambda, w', u')$, $z'' = (\lambda, \lambda, w'', u'')$, and

$$\begin{array}{rclcrcl} w' & = & w_1 \circ \varphi(\langle [T^1], [T^2] \rangle), & & u' & = & \varphi(\langle [t^1], [t^2] \rangle) \circ u_1, \\ w'' & = & w_2 \circ \varphi(\langle [T^1], [T^2] \rangle), & & u'' & = & \varphi(\langle [t^1], [t^2] \rangle) \circ u_2 \end{array}$$

hold for some terms t^1, t^2, T^1, T^2. Since both z' and z'' are not rejected nodes, we have $w' \circ u' = w'' \circ u'' = e$ and hence

$$\varphi(\langle [T^1], [T^2] \rangle) \circ \varphi(\langle [t^1], [t^2] \rangle) = (u_1 \circ w_1)^{-1} = (u_2 \circ w_2)^{-1}.$$

□

Lemma 2.9. *Let $L = \max(|\pi_1|, |\pi_2|)$, and F_0^1, F_0^2 be a pair of marginal procedures in π_1, π_2. Suppose*

$$y_1 = (F_0^1, F_0^2, w_1, u_1), \dots y_L = (F_0^1, F_0^2, w_L, u_L), y_{L+1} = (F_0^1, F_0^2, w_{L+1}, u_{L+1}),$$

are pairwise different marginal nodes accessible from the root of Γ. Suppose also that no rejected nodes are accessible from $y_1, \dots y_L$. Then a rejected node is accessible from y_{L+1} iff $(u_{L+1} \circ w_{L+1})^{-1} \neq (u_1 \circ w_1)^{-1}$.

Proof. ($\Leftarrow$) Follows from Lemma 2.8.
($\Rightarrow$) Suppose $(u_{L+1} \circ w_{L+1})^{-1} = (u_1 \circ w_1)^{-1}$. Consider an arbitrary path

$$y_{L+1} = x_0 \xrightarrow{(c_0^1, c_0^2)} x_1 \xrightarrow{(c_1^1, c_1^2)} \dots \xrightarrow{(c_m^1, c_m^2)} x_{m+1} = z' \tag{3}$$

leading to a node $z' = (\lambda, \lambda, w', u')$, and assume that for every $i, 1 \leq i \leq m$ we have $x_i = (F_i^1, F_i^2, v_i^1, v_i^2)$ such that $D(F_{i-1}^k, c_i^k) = T_i^k F_i^k b_i^k, 1 \leq i \leq m$, and $D(F_m^k, c_m^k) = T_{m+1}^k$, $k = 1, 2$. Clearly, $m + 1 < L$. Then similar to the proof of Lemma 2.6 consider a matrix $\{v_{ij}\}_{j=1,L}^{i=0,m}$ whose elements are $v_{ij} = w^+ \circ \varphi(\langle [b_i^1 \dots b_1^1], [b_i^2 \dots b_1^2] \rangle) \circ u_j$. By $(R2)$ of the definition of criterial system, at most one element in each row is equal to e. Since $m + 1 < L$, there exists k, $1 \leq k \leq L$, such that every element in the column $v_{0k}, v_{1k}, \dots v_{mk}$ is non-unit. Hence, by definition of Γ, there exists a path from y_k to a node $z'' = (\lambda, F_k^2, w'', u'')$. Put $w_0 = \varphi(\langle [b_m^1 \dots b_1^1], [b_m^2 \dots b_1^2] \rangle)$ and $u_0 = \varphi(\langle [T_1^1 \dots T_{m+1}^1], [T_1^2 \dots T_{m+1}^2] \rangle)$. Since z'' is not a rejected node, we have $e = w'' \circ u'' = w_k \circ w_0 \circ u_0 \circ u_k$. Therefore, $w_0 \circ u_0 = (u_k \circ w_k)^{-1}$. Applying Lemma 2.8, we obtain

$$\begin{aligned} w' \circ u' &= w_{L+1} \circ w_0 \circ u_0 \circ u_{L+1} = \\ &= w_{L+1} \circ (u_k \circ w_k)^{-} u_{L+1} = w_{L+1} \circ (u_1 \circ w_1)^{-} u_{L+1} = e. \end{aligned}$$

Therefore z' is not a rejected node. Since an arbitrary path (3) from y_{L+1} is considered, we arrive at the conclusion that no rejected node is accessible from y_{L+1}. □

By putting Lemmas 2.1–2.9 together we obtain:

Theorem 2.10. *Suppose $\mathcal{F}$ is a length-preserving semigroup frame over the alphabet $\mathcal{A}$, and $K = \langle W, U, w^+, w^* \rangle$ is a criterial system for $\mathcal{F}$. Suppose also that W is a group, and the identity problem "$w_1 = w_2$?" in W is decidable in time $\tau(m)$, where $m = \max(|w_1|, |w_2|)$. Then the equivalence problem "$\pi_1 \sim_{\mathcal{F}} \pi_2$?" is decidable for linear MRPs in time $C_1 n^3(\tau(C_2 n^3) + \log n)$, where $n = \max(|\pi_1|, |\pi_2|)$. Constants C_1, C_2 depend on $|\mathcal{A}|$, $|\mathcal{C}|$, and the homomorphism φ.*

Remark 2.11. We would like to emphasize that Theorem 2.10 guarantees the polynomial-time decidability of the equivalence problem w.r.t. $\mathcal{F}$ ONLY when the recursive procedures have definitions of the form (1). The concept of MRP introduced in [1, 4, 6] has somewhat different syntax. The MRPs considered in [1, 4, 6] can be translated (or, it is better to say, adapted) to the MRPs we deal with in this paper, but the size of the target MRP will be exponential of the size of the source one.

3. Applications

The following examples demonstrate the capability of Theorems 2.7, 2.10.

Example 3.1. Let $\mathcal{F}$ be a universal frame $\mathcal{U}$ over the alphabet $\mathcal{A} = \{a^1, \dots, a^N\}$. Consider a monoid $W = (\mathcal{U} \times \mathcal{U}) \cup \{w^+\}$ characterized by the identities

$$\begin{aligned}&\langle [T_1], [T_2] \rangle \circ \langle [t_1], [t_2] \rangle = \langle [T_1 t_1], [T_2 t_2] \rangle,\\ &w^+ \circ w^+ = e, \quad w^+ \circ \langle [t], [t] \rangle = \langle [t], [t] \rangle \circ w^+ = w^+.\end{aligned}$$

It is not difficult to verify that $K = \langle W, \mathcal{U} \times \mathcal{U}, w^+, w^+ \rangle$ is a criterial system for $\mathcal{U}$. Therefore, we have:

Corollary 3.2. [27]. *The equivalence problem w.r.t. the universal frames is decidable for linear MRPs in time $O(n^3 \log n)$.*

Proof. The decidability of the equivalence problem w.r.t. $\mathcal{U}$ follows from Theorem 2.7. To establish a polynomial upper bound, it is worth noting that for every pair of terms t_1, t_2 of the same length, equation $w^+ \circ X \circ \varphi(\langle [t_1], [t_2] \rangle) \circ w^+ = e$ has a solution X in $\mathcal{U} \times \mathcal{U}$ iff $w^+ \circ \varphi(\langle [t_1], [t_2] \rangle) = w^+$. Then, by Lemmas 2.4, 2.5, for every pair of procedures F^1, F^2, one of whose is terminated, no more then two nodes of the form (F^1, F^2, w, u) should be checked in Γ to verify the accessibility of some rejected node. □

Example 3.3. Let $\mathcal{F}_{fc}$ be a frame associated with a free commutative monoid. Suppose $\mathcal{A} = \{a^1, \dots, a^N\}$ and denote by Z a free Abelian group of the range N generated by some elements $q_1, \dots, q_N$. Then $K = \langle Z, Z, e, e \rangle$ is a criterial system for $\mathcal{F}_{fc}$, assuming $\varphi(\langle [a_i], [\lambda] \rangle) = q_i$ and $\varphi(\langle [\lambda], [a_j] \rangle) = q_j^{-1}$ for every pair of actions a_i, a_j.

Corollary 3.4. *The equivalence problem w.r.t. $\mathcal{F}_{fc}$ is decidable for linear MRPs in time $O(n^3 \log n)$.*

Proof. Follows from Theorem 2.10. □

Example 3.5. Suppose a frame $\mathcal{F}_c^I$ is associated with a partially commutative monoid characterized by the identities $[a^i a^j] = [a^j a^i]$, $(i, j) \in I$, $I \subseteq \{1, \dots, N\} \times \{1, \dots, N\}$. Consider a monoid W whose elements are all pairs in $\mathcal{F}_c^I \times \mathcal{F}_c^I$ together with specific elements w^+, w^*. The binary operation $\circ$ is defined as follows: $w^+ \circ w^* = e$, $\langle [T_1], [T_2] \rangle \circ \langle [t_1], [t_2] \rangle = \langle [T_1 t_1], [T_2 t_2] \rangle$, $w^+ \circ \langle [a], [a] \rangle = w^+$, Then $K = \langle W, \mathcal{F}_c^I \times \mathcal{F}_c^I, w^+, w^* \rangle$ is a criterial system for $\mathcal{F}_c^I$. Thus, we obtain:

Corollary 3.6. *The equivalence problem w.r.t. $\mathcal{F}_c^I$ is decidable for linear MRPs.*

However, we still do not know if this problem can be decided in polynomial time.

Using our algebraic technique we obtain thus some new decidable cases of the equivalence problem for monadic recursive programs. Our approach to the designing of the efficient decision algorithms for the equivalence problem can be extended to the more general classes of interpretations and MPRs.

4. Note

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