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A CHARACTERIZATION OF POLY-SLENDER CONTEXT-FREE LANGUAGES *

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Abstract. For a non-negative integer k, we say that a language L is *k-poly-slender* if the number of words of length n in L is of order $\mathcal{O}(n^k)$. We give a precise characterization of the k-poly-slender context-free languages. The well-known characterization of the k-poly-slender regular languages is an immediate consequence of ours.

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INTRODUCTION

An infinite sequence $(\#_L(n))_{n\geq 0}$ can be associated in a natural way to a language L: $\#_L(n)$ is the number of words of length n in L. The idea is by no means new; for instance, in the first ICALP, Berstel [3] considered the notion of the *population* function of a language L which associates, to every n, the number of words of length *at most* n in L. The notion of the number of words of the same length is certainly very basic one in language theory and this is why some results have been proved several times. We recall briefly in the following the history of such results.

When $\#_L(n)$ is bounded from above by a fixed constant, such languages are called *semidiscrete* in Kunze *et al.* [11] and *slender* in Andraşiu *et al.* [1]. The slender regular languages have been characterized as finite unions of sets of the

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form uv^*w in [11] but the result was not well known and it was proved again independently by Păun and Salomaa [16] and Shallit [19]. A similar situation is in the context-free case. The characterization of the slender context-free languages as finite unions of sets of the form $\{uv^nwx^ny \mid n \geq 0\}$ was proved by Latteux and Thierrin [9] (they called such finite unions *iterative* languages) but, again, the result was not widely known and the same characterization was conjectured in [16] and shown to be true by Ilie [12]. The proof of [12] is completely different from the one of [9]. The characterization has been strengthened in [10] where some upper bounds on the lengths of the words u, v, w, x, y are given.

The case when $\#_L(n)$ is bounded by a polynomial (we say L is *poly-slender*) has been considered by Latteux and Thierrin [13] who proved that for context-free languages the notion of poly-slenderness coincides with the one of boundedness. Once more, the result was proved again by Raz [17]. In the case of regular languages, Szilard *et al.* [20] gave a fine characterization based on the order of the polynomial which bounds $\#_L(n)$.

Besides the above mentioned results, there has been recently a lot of attention devoted to other aspects of slenderness. Some applications of the slender languages to cryptography are shown in [1], Shallit [19] investigated slender regular languages in connection with numeration systems, the slenderness of L-languages has been considered by Dassow *et al.* [5] and Nishida and Salomaa [15] and Honkala studied in [7] and [8] a generalization of the notion of slenderness, called Parikh slenderness, by considering languages for which the number of words with the same Parikh vector is bounded from above by a constant.

In this note, we consider the situation when the degree of the polynomial bounding $\#_L(n)$ is a fixed non-negative integer; the obtained languages are called *k-poly-slender*. Generalizing the result of [9] and [12], we give a characterization of the k-poly-slender context-free languages. The structure of the Dyck language is the base for the structure of these languages. The corresponding characterization given in [20] for regular languages follows immediately from ours.

Poly-slender languages

We first fix some notations. For a word w and a letter a, $|w|$ is the length of w, $|w|_a$ is the number of occurrences of a in w, and $\rho(w)$ denotes the primitive root of w. The conjugacy relation is denoted $\sim$; for two words u, v, $u \sim v$ iff $u = pq, v = qp$, for some p, q. The empty word is denoted by ε. For basic notions and results of combinatorics on words and formal languages we refer to [4, 14] and [18], respectively.

For a language L, denote $\#_L(n) = \text{card}(\{w \in L \mid |w| = n\})$; this is referred to as the **complexity** (function) of L. For an integer $k \geq 0$, L is called **k-poly-slender** if $\#_L(n) = \mathcal{O}(n^k)$. L is **poly-slender** iff it is k-poly-slender, for some $k \geq 0$.

A language $L \subseteq \Sigma^*$ is called **bounded** if there are some words $w_1, w_2, \ldots, w_n \in \Sigma^*$ such that $L \subseteq w_1^* w_2^* \ldots w_n^*$. It is clear that the class of poly-slender languages is the same with the class of languages with the population function polynomially limited. Therefore, the characterization theorem of [13] can be written as below (it appears in this form in [17]).

Theorem 1. *[13, 17] A context-free language is poly-slender iff it is bounded.*

Dyck loops

Consider the **Dyck** language of order $k, k \geq 1$, $D_k \subseteq \{[_i,]_i \mid 1 \leq i \leq k\}^*$. D_k is generated by $S \longrightarrow SS \mid [_i S]_i \mid \varepsilon$. Consider also a word $z \in D_k$ with $|z|_{[_i} = |z|_{]_i} = 1, 1 \leq i \leq k$, and some words $u_i, v_i, w_i \in \Sigma^*$, where $\Sigma \cap \{[_i,]_i \mid 1 \leq i \leq k\} = \emptyset$. For any integers $n_i \geq 0, 1 \leq i \leq k$, define the morphism $h_{n_1, \ldots, n_k} : (\Sigma \cup \{[_i,]_i \mid 1 \leq i \leq k\})^* \longrightarrow \Sigma^*$, by $a \longrightarrow a$, for $a \in \Sigma$, $[_i \longrightarrow u_i^{n_i}$, $]_i \longrightarrow v_i^{n_i}$, $1 \leq i \leq k$. Put $z = z_1 z_2 \ldots z_{2k}, z_j \in \{[_i,]_i \mid 1 \leq i \leq k\}$. Then $D \subseteq \Sigma^*$ is a ***k*-Dyck loop** if, for some u_i, v_i, w_i, z as above,

$$D = \{h_{n_1, \ldots, n_k}(w_0 z_1 w_1 z_2 w_2 \ldots z_{2k} w_{2k}) \mid n_i \geq 0\} \cdot \tag{1}$$

We shall call z an *underlying word* of D. (Clearly, z is not unique.) Also, h will stand for $h_{1,1, \ldots, 1}$ and will be called an *underlying morphism* of D. D is a **Dyck loop** iff it is a k-Dyck loop, for some k. Notice that if $l < k$, then any l-Dyck loop is also a k-Dyck loop.

We give below two examples of Dyck loops which will be used also later.

Example 2. For the underlying Dyck word $z = [_1]_1[_2[_3[_4]_4[_5]_5]_3[_6]_6[_7]_7]_2$, we construct the Dyck loop

$$L_1 = \{a^{2n_1}(ba)^{n_1}b(ab)^{3n_2}b^{n_3}b^{5n_4}b^{n_5}a^{4n_5}(aba)^{n_3}b^{3n_6}a^{n_7}b^{2n_7}b^{n_2} \mid n_i \geq 0\} \cdot \tag{2}$$

About the underlying morphism we mention only that the images of any of $]_4$ and $]_6$ are empty.

Example 3. The underlying Dyck word for the Dyck-loop

$$L_2 = \{(ab)^{2n_1}a(ba)^{3n_2}aab^{n_2}b^{2n_3}b^{n_4}aba^{n_4}a^{3n_3}a^{n_5}a^{2n_1} \mid n_i \geq 0\}, \tag{3}$$

is $z = [_1[_2]_2[_3[_4]_4]_3[_5]_5]_1$ and we assume $h(]_5) = \varepsilon$.

Bounded languages

The following result of Ginsburg and Spanier [6] will be essential for our purpose.

Theorem 4. *[6] The family of bounded context-free languages is the smallest family which contains all finite languages and is closed under the following operations: (i) union, (ii) catenation, (iii) $(x, y) \star L = \bigcup_{n \geq 0} x^n L y^n$, for x, y words.*

Remark 5. Clearly, Theorem 4 is still valid if, instead of finite languages, one starts from unary languages (that is, languages containing one word only). Moreover, if one starts from unary languages and uses only the operation (ii) and (iii) from Theorem 4, then what is obtained is always a Dyck loop. Conversely, any Dyck loop can be obtained in this way. Indeed, this is clear from the definition of the Dyck loops; the role of the production $S \longrightarrow SS$ is the same with the one of the catenation and the role of a production $S \longrightarrow [_i S]_i$ is the same with the one of the operation $\star$, in the sense that we use $(u_i, v_i) \star L$.

Therefore, we get from Theorems 1 and 4:

Theorem 6. *For a context-free language L, the following assertions are equivalent: (i) L is bounded, (ii) L is poly-slender, (iii) L is a finite union of Dyck loops.*

Proof. First, (i) and (ii) are equivalent by Theorem 1. Second, (iii) clearly implies any of (i) and (ii). Third, (i) implies (iii) follows from Remark 5 and the distributivity of the catenation and "$\star$" with respect to union. □

Characterization of k-poly-slenderness

We prove in this section our main result, that is, the characterization of the k-poly-slender context-free languages. In the case of 0-poly-slender languages, such a characterization was proved in [9] and [12]; using the above notations, the result of [9, 12] is written as

Theorem 7. *[9, 12] A context-free language is 0-poly-slender iff it is a finite union of 1-Dyck loops.*

In what follows, we shall generalize Theorem 7 to:

Theorem 8. *For any $k \geq 0$, a context-free language is k-poly-slender iff it is a finite union of $(k+1)$-Dyck loops.*

Before proving the theorem we need several notions and a lemma.

For three words u, w, v such that u and v are non-empty, we say that w **links** u with v, denoted $\mathsf{link}(u, w, v)$, iff $\rho(u)w = w\rho(v)$; it means that there are words p and q such that $\rho(u) = pq, \rho(v) = qp$, and $w \in (pq)^*p$. The idea is that w links u with v when in words of the form $u^n w v^m$ the period $|\rho(u)|$ is continued over w throughout v^m ($u^n w v^m$ is a prefix of $\rho(u)^\omega$), that is, one cannot really distinguish w there. Notice that, when w is empty, $\mathsf{link}(u, \varepsilon, v)$ means precisely that u and v are powers of the same word, in accordance with the intuitive meaning of link.

The following lemma will be a very useful tool in the proof.

Lemma 9. *Consider the words $x_i \in \Sigma^+, y_i \in \Sigma^*$, and some non-negative integers n_i, m_i. Denote $w = y_0 x_1^{n_1} y_1 x_2^{n_2} y_2 \ldots x_r^{n_r} y_r$, $w' = y_0 x_1^{m_1} y_1 x_2^{m_2} y_2 \ldots x_r^{m_r} y_r$ and assume that $\mathsf{link}(x_i, y_i, x_{i+1})$ holds for no i. Then there is a constant N_0, depending only on the lengths of the words x_i and y_i, such that, if all n_i, m_i are larger than N_0 and there is i with $n_i \neq m_i$, then $w \neq w'$.*

Proof. We prove that $N_0 = \frac{1}{\min|x_i|}(\max|y_i| + 4\max|x_i|)$ is good. Assume $w = w'$. Without loss of generality, we may assume $n_1 > m_1$. If $(n_1 - m_1)|x_1| \geq |y_1| + |x_1x_2|$, then, by Fine and Wilf's theorem and the choice of N_0, $\rho(x_1) \sim \rho(x_2)$ and, if $\rho(x_1) = pq, \rho(x_2) = qp$, then $y_1 \in (pq)^*p$. This means $\mathsf{link}(x_1, y_1, x_2)$, a contradiction. Otherwise, by the choice of N_0, we get, for some $s \geq 0$, $x_1^{n_1-m_1}y_1 = y_1\rho(x_2)^s$ thus again $\mathsf{link}(x_1, y_1, x_2)$, contradicting our assumption. The lemma is proved. □

Proof of Theorem 8. One implication is obvious. For the other, the basic idea is to do the construction in Theorem 4 in a certain way, that is (roughly), anytime "$\star$" is applied, the highest power of n in the complexity function $\#_L(n)$ associated with the language L is increased by one and, anytime catenation is applied to two infinite languages, the highest exponent in the complexity of the resulting language is the sum of the former two. Let us consider a context-free language L which is k-poly-slender. Then, by Theorem 6, L is a finite union of Dyck loops. If L contains no l-Dyck loop with $l > k + 1$, then we are done. Assume then there is such a loop, say D, and that it has the form in (1) with l instead of k. We may assume, with no loss of generality, that $u_iv_i \neq \varepsilon$, for any $1 \leq i \leq l$, as otherwise we have a Dyck loop of smaller order. We shall consider very much in the sequel the links made by w_i between the two adjacent images of h, $h(z_i)$ and $h(z_{i+1})$. But those images of h which are empty have no interest for the language (this is why link is undefined when the first or the third component is empty) and therefore, for any $1 \leq i \leq 2l - 1$, we define $\mathsf{next}(i)$ as the smallest $j \geq i + 1$ such that $h(z_j) \neq \varepsilon$. If there is no $\mathsf{link}(h(z_i), w_iw_{i+1}\dots w_{\mathsf{next}(i)-1}, h(z_{\mathsf{next}(i)}))$, then, by Lemma 9, any two different tuples of n_i's (n_i's are all assumed large enough – the complexity order is not affected) give different words of D. Therefore, $\#_D(n) \neq \mathcal{O}(n^{l-2})$, which contradicts the fact that L is k-poly-slender. Consequently, there are such links.

We then group together the linked powers and apply Lemma 9 to those. In order to make things clear, we define two relations on the set $\{i \mid 1 \leq i \leq 2l, h(z_i) \neq \varepsilon\}$. (Recall that the positions i with $h(z_i) = \varepsilon$ are ignored.) The first is $\mathsf{chain}(i, \mathsf{next}(i))$ iff $\mathsf{link}(h(z_i), w_iw_{i+1}\dots w_{\mathsf{next}(i)-1}, h(z_{\mathsf{next}(i)}))$. We say that i and j are in the same **chain** iff $\mathsf{chain}^\sharp(i, j)$, where $\mathsf{chain}^\sharp$ is the equivalence generated by chain. The second is $\mathsf{syst}(i, j)$ iff $\mathsf{chain}^\sharp(i, j)$ or $z_i = [_p, z_j =]_p$, for some $1 \leq p \leq l$. We say that i and j are in the same **system** iff $\mathsf{syst}^\sharp(i, j)$, where $\mathsf{syst}^\sharp$ is the equivalence generated by syst.

For the Dyck loop in (2), we have the following classes of $\mathsf{chain}^\sharp$: $\{1\}$, $\{2, 3\}$, $\{4, 5, 7\}$, $\{8\}$, $\{9\}$, $\{10\}$, $\{12\}$, $\{13, 14\}$ and the classes of $\mathsf{syst}^\sharp$ (unions of the former ones): $\{1, 2, 3, 12, 13, 14\}$, $\{4, 5, 7, 8, 9\}$, $\{10\}$.

We next group together all powers in the same chain. For (2) this means that we write it as

$$L_1 = \{a^{2n_1}b(ab)^{n_1+3n_2}b^{n_3+5n_4+n_5}a^{4n_5}(aba)^{n_3}b^{3n_6}a^{n_7}b^{n_2+2n_7} \mid n_i \geq 0\}.$$

Now, the application of Lemma 9 gives a different result in the sense that the powers in the lemma are no longer single powers of our Dyck loop but linear combinations of those in the same chain. This means for (2) that any new tuple $(2n_1, n_1 + 3n_2, n_3 + 5n_4 + n_5, 4n_5, n_3, 3n_6, n_7, n_2 + 2n_7)$ (recall that we assume the elements large enough) gives a new word. Incidentally, for (2), we get such a new tuple exactly when the tuple $(n_i)_{1\le i\le 7}$ is new, due to the fact that the three systems corresponding to the classes of $\mathsf{syst}^\sharp$, namely

$$\begin{cases} 2n_1 = 0 \\ n_1 + 3n_2 = 0 \\ n_7 = 0 \\ n_2 + 2n_7 = 0 \end{cases} \qquad \begin{cases} n_3 + 5n_4 + n_5 = 0 \\ 4n_5 = 0 \\ n_3 = 0 \end{cases} \qquad \{\ 3n_6 = 0 \tag{4}$$

have only trivial solutions. (It is worth noticing that the unknowns in the above systems are not the exponents but rather the differencs of these for two different descriptions of the same word.)

However, this need not be the case in general. As an example, consider the loop in (3). There are links and we have the classes of $\mathsf{chain}^\sharp$: $\{1,2\}$, $\{3,4,5\}$, $\{6,7,8,10\}$ and only one class of $\mathsf{syst}^\sharp$: $\{1,2,\ldots,8,10\}$. We write L_2 as

$$L_2 = \{a(ba)^{2n_1+3n_2}aab^{n_2+2n_3+n_4}aba^{2n_1+3n_3+n_4+n_5} \mid n_i \ge 0\}$$

and, because the matrix of the associated system has more columns than rows, the associated system has also non-trivial solutions.

Consider then, in the general case, the systems associated with the equivalence classes of $\mathsf{syst}^\sharp$, as seen in the above example. (Notice that each system has as many equations as the number of classes of $\mathsf{chain}^\sharp$ in the respective class of $\mathsf{syst}^\sharp$.) As we noticed, if all such systems have trivial solutions only, then we have the same situation as in the case of no links. Thus, the same contradiction with the k-poly-slenderness of L is obtained. Therefore, there are systems which have also non-trivial solutions.

Consider one class of $\mathsf{syst}^\sharp$, say $[i_0]_{\mathsf{syst}^\sharp}$ such that its associated system, say

$$\sum_{j=1}^{t} a_{ij}n_j = 0, \text{ for all } 1 \le i \le s, \tag{5}$$

has also non-trivial solutions. Here s is the number of chains (classes of $\mathsf{chain}^\sharp$) in $[i_0]_{\mathsf{syst}^\sharp}$ and t is the number of n_i's involved, that is, the number of pairs $[_i]_i$ involved.

Let us show first that $s \le t + 1$. We can view the class $[i_0]_{\mathsf{syst}^\sharp}$ as a graph with s vertices, which are the chaines, such that two vertices are connected by an edge only if there is a pair $[_i]_i$ connecting them according with the definition of syst ($[_i$ is in one chain and $]_i$ is in the other). As the graph is connected, there are at least $s-1$ edges, hence there are at least $s-1$ pairs $[_i]_i$, which means $t \ge s-1$.

Assume that $s = t+1$ (as, *e.g.*, in the first system in (4)). The matrix of the system $A = (a_{ij})_{\substack{1\le i\le s\\ 1\le j\le t}}$, has one or two non-zero elements in each column and at least one in each row. Therefore, there is a row with exactly one non-zero element. The matrix obtained by eliminating the row and the column corresponding to this element has the same properties as A. Inductively, we obtain that rank(A)=t and hence the system (5) has only trivial solutions, a contradiction. Therefore, $s \le t$ and rank(A) $< t$. If we denote the columns of A by $A_i, 1 \le i \le t$, then there exists a non-trivial linear dependence of A_i's, say

$$\sum_{i=1}^{r} c_i A_{j_i} = \sum_{i=1}^{s} d_i A_{k_i},$$

where c_i's and d_i's are positive integers. Recall that we are interested in the values of the vector $\sum_{i=1}^{t} n_i A_i$, for all $n_i \ge 0$. Denote $J = \{j_i \mid 1 \le i \le r\}$, $K = \{k_i \mid 1 \le i \le s\}$; we may assume $J \cap K = \emptyset$. We claim that the following equality holds:

$$\left\{\sum_{i=1}^{t} n_i A_i \mid n_i \ge 0\right\} = \bigcup_{q=1}^{r} \left\{\sum_{i=1}^{r} q_i A_{j_i} + \sum_{\substack{i=1\\ i\notin J}}^{t} m_i A_i + \sum_{\substack{i=1\\ i\ne q}}^{r} m_{j_i}(c_i A_{j_i}) \mid m_i \ge 0, 0 \le q_i \le c_i - 1\right\}. \quad (6)$$

The inclusion "$\supseteq$" is obvious. Let us prove the "$\subseteq$" part. Consider some fixed $(n_1, n_2, \dots, n_t)$ and put, for any $1 \le i \le r$, $n_{j_i} = n'_{j_i} c_i + q_i$, $0 \le q_i \le c_i - 1$. Take $q, 1 \le q \le r$, such that $n'_{j_q} = \min\{n'_{j_i} \mid 1 \le i \le r\}$. We have then

$$\begin{aligned}\sum_{i=1}^{t} n_i A_i &= \sum_{\substack{i=1\\ i\notin J}}^{t} n_i A_i + \sum_{i=1}^{r}((n'_{j_i} - n'_{j_q})c_i + q_i)A_{j_i} + n'_{j_q}\sum_{i=1}^{r} c_i A_{j_i}\\ &= \sum_{i=1}^{r} q_i A_{j_i} + \sum_{\substack{i=1\\ i\notin J\cup K}}^{t} n_i A_i + \sum_{i=1}^{s}(n_{k_i} + n'_{j_q} d_i)A_{k_i} + \sum_{\substack{i=1\\ i\ne q}}^{r}(n'_{j_i} - n'_{j_q})(c_i A_{j_i})\end{aligned}$$

and the inclusion is proved. Now, in any set in the right-hand side of (6) we have essentially combinations of $t-1$ vectors instead of t as we had initially, that is, in the left-hand side of (6). But, one A_i less means exactly one pair of parentheses less in the underlying Dyck word z. That means, the l-Dyck loop we started with can be written as a finite union of $(l-1)$-Dyck loops just because $l > k+1$. (Precisely, we proved that D can be decomposed as a union of $r\prod_{i=1}^{r} c_i$ $(l-1)$-Dyck loops.)

The above reasoning can be iterated, if needed, and therefore concludes the proof. □

We next give some examples of the construction in the proof of Theorem 8.

Example 10. Consider the 5-Dyck loop in (3) which we have already discussed in the proof of Theorem 8. The matrix of the unique associated system is

$$A = \begin{pmatrix} 2 & 3 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 2 & 0 & 3 & 1 & 1 \end{pmatrix}.$$

We have the following simple relation between the columns $A_i, 1 \le i \le 5$, of A: $A_3 = 2A_4 + A_5$. It implies that

$$\left\{ \sum_{i=1}^{5} n_i A_i \mid n_i \ge 0 \right\} = \left\{ \sum_{\substack{i=1 \\ i \ne 3}}^{5} m_i A_i \mid m_i \ge 0 \right\}$$

which allows us to write L_2 as

$$L_2 = \{a(ba)^{2m_1+3m_2} aab^{m_2+m_3} aba^{2m_1+m_3+m_4} \mid m_i \ge 0\}.$$

The pair of parentheses $[_3]_3$ disappeared. But L_2 can be still reduced as the new matrix, say $B = (B_1\, B_2\, B_3\, B_4) = (A_1\, A_2\, A_4\, A_5)$ has still more columns than rows. After noticing the linear combination $3B_1 + 2B_3 = 2B_2 + 8B_4$, we get, according with the proof of Theorem 8,

$$\begin{aligned} \left\{ \sum_{i=1}^{4} m_i B_i \mid m_i \ge 0 \right\} &= \bigcup_{q_1=0}^{2} \bigcup_{q_3=0}^{1} \{q_1 B_1 + q_3 B_3 + p_2 B_2 \\ &\quad + p_3(2B_3) + p_4 B_4 \mid p_{2,3,4} \ge 0\} \\ &\cup \bigcup_{q_1=0}^{2} \bigcup_{q_3=0}^{1} \{q_1 B_1 + q_3 B_3 + p_1(3B_1) \\ &\quad + p_2 B_2 + p_4 B_4 \mid p_{1,2,4} \ge 0\}. \end{aligned}$$

Therefore, we obtain the following decomposition of L_2 as a finite union of 3-Dyck loops:

$$L_2 = \bigcup_{i=0}^{2} \bigcup_{j=0}^{1} L'_{i,j} \cup \bigcup_{i=0}^{2} \bigcup_{j=0}^{1} L''_{i,j}$$

where, for any $0 \le i \le 2, 0 \le j \le 1$,

$$\begin{aligned} L'_{i,j} &= \{a(ba)^{2i}(ba)^{3p_2} aab^j b^{p_2+2p_3} aba^{2i+j} a^{2p_3+p_4} \mid p_{2,3,4} \ge 0\}, \\ L''_{i,j} &= \{a(ba)^{2i}(ba)^{6p_1+3p_2} aab^j b^{p_2} aba^{2i+j} a^{6p_1+p_4} \mid p_{1,2,4} \ge 0\}, \end{aligned}$$

which cannot be reduced further as $\#_{L_2}(n) = \Theta(n^2)$.

Example 11. Consider the 4-Dyck loop

$$L_3 = \{a^{n_1}a^{n_2}ba^{n_2}a^{n_3}ba^{n_3}a^{n_4}ba^{n_4}a^{n_1} \mid n_i \geq 0\}.$$

As above, we reduce it to

$$\begin{aligned} L_3 &= \{a^{m_1}a^{m_2}ba^{m_2}ba^{m_4}ba^{m_4}a^{m_1} \mid m_{1,2,4} \geq 0\} \\ &\cup \{a^{m_2}ba^{m_2}a^{m_3}ba^{m_3}a^{m_4}ba^{m_4} \mid m_{2,3,4} \geq 0\}. \end{aligned}$$

We would like to notice that the 3-Dyck loop

$$L_4 = \{a^{n_1}a^{n_2}ba^{n_2}a^{n_3}ba^{n_3}a^{n_1} \mid n_i \geq 0\},$$

which is actually very similar with L_2 cannot be reduced. The (essential) difference is that the columns of the matrix corresponding to L_4 are linearly independent whereas the ones of L_3 are not.

Consequences

The first of the consequences of Theorem 8 is the characterization of k-poly-slender regular languages. We say that the k-Dyck loop D in (1) is **degenerate** if, for any $1 \leq i \leq k$, at most one of the words u_i and v_i is non-empty.

We get then immediately from Theorem 8 the following result which has been proved in [20].

Corollary 12. *For any $k \geq 0$, a regular language is k-poly-slender iff it is a finite union of degenerate $(k+1)$-Dyck loops.*

Consider next, for a language L, the following smoothening of the complexity function $\#_L(n)$:

$$\overline{\#}_L(n) = \max_{1 \leq i \leq n} \#_L(i).$$

We have then from Theorem 8:

Corollary 13. *For any poly-slender context-free language L, there exists a $k \geq 0$ such that $\overline{\#}_L(n) = \Theta(n^k)$.*

This means there is nothing in between integer powers, *e.g.*, there is no context-free language the complexity of which is of order $\Theta(n^{3.4})$, $\Theta(\log n)$, $\Theta(n \log\log n)$, etc.

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