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STRONGLY LOCALLY TESTABLE SEMIGROUPS WITH COMMUTING IDEMPOTENTS AND RELATED LANGUAGES

CARLA SELMI[1]

Abstract. If we consider words over the alphabet which is the set of all elements of a semigroup S, then such a word determines an element of S: the product of the letters of the word. S is strongly locally testable if whenever two words over the alphabet S have the same factors of a fixed length k, then the products of the letters of these words are equal. We had previously proved [19] that the syntactic semigroup of a rational language L is strongly locally testable if and only if L is both locally and piecewise testable. We characterize in this paper the variety of strongly locally testable semigroups with commuting idempotents and, using the theory of implicit operations on a variety of semigroups, we derive an elementary combinatorial description of the related variety of languages.

1. INTRODUCTION

Eilenberg's variety theorem, published in 1976, asserts that there exists a one-to-one correspondence between certain classes of recognizable languages (the varieties of languages) and certain classes of finite semigroups (the varieties of semigroups). The algebraic characterizations of star-free languages [17], locally testable languages [10] and piecewice testable languages [20], among others, are instances of this correspondence.

The theory of implicit operations, introduced by Reiterman [16] and developed by Almeida [1–5] (see also Almeida and Weil [6, 7], Weil [21] and Zeitoun [24]), allows us to solve some questions about varieties of finite semigroups. One can associate to a given variety of semigroups $\mathbf{V}$ and to a given alphabet A, a topological semigroup, denoted by $\widehat{F}_A(\mathbf{V})$, which is called the semigroup of implicit

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operations on $\mathbf{V}$. The semigroup $\widehat{F}_A(\mathbf{V})$ plays the role of the free object for the variety on the alphabet A, in a certain sense. Moreover, the family of languages on A^+ associated to $\mathbf{V}$ is characterized by the topological structure of $\widehat{F}_A(\mathbf{V})$.

We characterize in this paper the variety of strongly locally testable semigroups with commuting idempotents (denoted by $\mathbf{SLT} \cap \mathbf{E_{com}}$) and we use this algebraic characterization and the theory of implicit operations, to derive a combinatorial description of the related variety of languages. Strongly locally testable semigroups are a natural extension of locally testable semigroups, introduced by Zalcstein [22, 23]. The definition is the following: if we consider words over the alphabet which is the set of all elements of a semigroup S, then such a word determines an element of S: the product of the letters of the word. A semigroup S is locally testable if whenever two words over the alphabet S have the same factors of a fixed length k, then the products of the letters of these words are equal. The variety of languages associated to the variety of strongly locally testable semigroups is the class of languages that are both locally testable and piecewise testable [19].

Our main result is the following: a language L on the alphabet A is recognized by $\mathbf{SLT} \cap \mathbf{E_{com}}$ if and only if L belongs to the boolean algebra generated by the languages of the form $B_0^* a_1 B_2^* \ldots a_n B_n^*$ where $n \geq 0$, the a_i are letters of A, the B_i are nonempty, mutually disjoint subsets of A, and where a_i does not belongs to $B_{i-1} \cup B_i$. Note that this result connects with a number of descriptions of varieties of languages involving languages of the form $B_0^* a_1 B_2^* \ldots a_n B_n^*$ with various conditions on the letters a_i and the subsets B_i of A (*e.g.* piecewise testable languages (Simon [20]), $\mathcal{R}$-trivial languages (Eilenberg [11]), doth-depth two languages (Pin and Straubing [15]), Ash, Hall and Pin's result on commuting idempotents [8], over testable languages [19], *etc.*).

In Section 2 we recall the basic notions of the theory of varieties and implicit operations. In Section 3 we recall the notion of strongly locally testable semigroups and of over testable languages. In Section 4 we characterize the variety $\mathbf{SLT} \cap \mathbf{E_{com}}$. In Section 5 we exhibit a family of languages recognized by $\mathbf{SLT} \cap \mathbf{E_{com}}$. In Section 6 we describe the implicit operations on $\mathbf{SLT} \cap \mathbf{E_{com}}$. Finally, in Section 7 we prove our main result.

2. Preliminaries

We first review basic definitions from the theory of varieties and implicit operations. For further details, the reader is referred to [1] and [14].

2.1. Varieties of semigroups and varieties of languages

A variety of semigroups (sometimes called pseudo-variety) is any class of finite semigroups that is closed under taking subsemigroups, homomorphic images and finite direct products.

We denote by $\mathbf{J_1}$ the variety of idempotent and commutative semigroups, by $\mathbf{J}$ the variety of J-trivial semigroups, by $\mathbf{LJ_1}$ the variety of locally idempotent and locally commutative semigroups and by $\mathbf{E_{com}}$ the variety of semigroups with commuting idempotents.

Let $\mathbf{V}$ be a variety of semigroups. One associates to each finite alphabet A the class $A^+\mathcal{V}$ of languages of A^+ whose syntactic semigroup belongs to $\mathbf{V}$. This correspondence is a variety of languages. Thus, we have an application $\mathbf{V} \longrightarrow \mathcal{V}$, which maps each variety of semigroups to a variety of languages. Eilenberg's variety Theorem [11], asserts that for any variety of languages $\mathcal{V}$ there exists a unique variety of semigroups $\mathbf{V}$ such that $\mathbf{V} \longrightarrow \mathcal{V}$: $\mathbf{V}$ is the variety generated by the syntactic semigroups of languages belonging to $A^+\mathcal{V}$ for any alphabet A.

2.2. Implicit operations

Given a variety of semigroups it is in general a difficult problem to find a set of generators for the related variety of languages. A useful tool for solving this question, is the determination of free objects for that variety, when such objects exist. But, in general, a variety of semigroups does not have free objects. It turns out to be necessary to consider certain infinite compact semigroups. This is done in the framework of the theory of implicit operations.

We define the basics of the theory of implicit operations on a varieties of semigroups. For the proofs of the results started in this section, the reader is referred to Almeida [1].

Let $\mathbf{V}$ be a variety of semigroups, let $n \geq 1$ and let $A = \{a_1, \ldots , a_n\}$. An *n-ary implicit operation* π on $\mathbf{V}$ is a family $\pi = (\pi_S)$, indexed by the elements S of $\mathbf{V}$, of mappings from S^n into S, such that for each morphism ψ: $S \longrightarrow T$ between elements of $\mathbf{V}$, we have

$$(s_1, \ldots , s_n)\pi_S\psi = (s_1, \ldots , s_n)\psi^n\pi_T$$

for every $s_1, \ldots , s_n \in S$.

The set of all n-ary implicit operations on $\mathbf{V}$ is denoted by $\widehat{F}_A(\mathbf{V})$.

Example 2.1. Let $\mathbf{V}$ be a variety of semigroups. Let $S \in \mathbf{V}$ and $s \in S$. We denote by s^ω the unique idempotent of S which is a power of x. We denote x_S^ω: $S \longrightarrow S$ the map defined by $(s)x_S^\omega = s^\omega$. It is easy to verify that $x^\omega = x_S^\omega$ is an implicit operation on $\mathbf{V}$.

Let $a_i \in A$ and $S \in \mathbf{V}$. We denote $a_i\iota_S$: $S^n \longrightarrow S$ the map defined by $(s_1, \ldots , s_n)a_i\iota_S = s_i$. It is easy to verify that $a_i\iota = (a_i\iota_S)_{S\in\mathbf{V}}$ is an n-ary implicit operations on $\mathbf{V}$. The map ι: $A \longrightarrow \widehat{F}_A(\mathbf{V})$ extends to a morphism ι: $A^+ \longrightarrow \widehat{F}_A(\mathbf{V})$. We denote by $F_A(\mathbf{V})$ the semigroup $A^+\iota$ and we call it the set n-ary explicit operations on $\mathbf{V}$. An n-ary explicit operations on $\mathbf{V}$ is an implicit operations on $\mathbf{V}$ induced by a mot of A^+.

Let π and $\rho \in \widehat{F}_A(\mathbf{V})$ and $S \in \mathbf{V}$. Then, for every $s_1, \ldots s_n \in S$, we define

$$(s_1, \ldots , s_n)(\pi\rho)_S = (s_1, \ldots , s_n)\pi_S(s_1, \ldots , s_n)\rho_S.$$

This multiplication makes $\widehat{F}_A(\mathbf{V})$ a semigroup.

Let $\mathbf{Z} \subseteq \mathbf{V}$ be two varieties of semigroups. Then the map p: $\widehat{F}_A(\mathbf{V}) \longrightarrow \widehat{F}_A(\mathbf{Z})$ defined by $\pi p = (\pi_S)_{S \in \mathbf{Z}}$ for any $\pi \in \widehat{F}_A(\mathbf{V})$ is called *the natural projection.* The natural projection is a surjective semigroup morphism.

Example 2.2. Let $\mathbf{V}$ be a variety of semigroups. It is known that $\widehat{F}_A(\mathbf{J_1}) = \mathcal{P}(A)$, where $\mathcal{P}(A)$ is the set of all nonempty subsets on the alphabet A endowed with the multiplication defined by union.
Let $\mathbf{J_1} \subseteq \mathbf{V}$. We denote by c: $\widehat{F}_A(\mathbf{V}) \longrightarrow \widehat{F}_A(\mathbf{J_1}) = \mathcal{P}(A)$ the natural projection and we call it *content.*

Let $\mathbf{V}$ be a variety of semigroups. A *pseudo-identity* $\pi = \rho$ for $\mathbf{V}$ is a formal identity of implicit operations on $\mathbf{V}$. A semigroup $S \in \mathbf{V}$ verifies the pseudo-identity $\pi = \rho$, $\pi, \rho \in \widehat{F}_A(\mathbf{V})$, if $\pi_S = \rho_S$. The theorem of Reiterman [16], states that any variety of semigroups is defined by a set of pseudo-identities.

We will use in Section 7 the following important reformulation of Reiterman's theorem.

Theorem 2.3. *Let $\mathbf{V} \subseteq \mathbf{Z}$ be two varieties of semigroups. Then $\mathbf{V} \neq \mathbf{Z}$ if and only if there exists an alphabet A and $\pi, \rho \in \widehat{F}_A(\mathbf{Z})$ such that $\pi \neq \rho$ but $\pi_S = \rho_S$ for any $S \in \mathbf{V}$.*

3. Strongly locally testable semigroups

For each finite semigroup S, we let S^+ be the set of all finite sequences of elements of S.

Definition 3.1. Let S be a finite semigroup. Then S is strongly k-testable if for each pair of elements $(x_1, \dots, x_n), (y_1, \dots, y_m)$ of S^+, n, $m \geq k$, having the same set of k-factors, one has $x_1 \cdots x_n = y_1 \cdots y_m$. A semigroup is *strongly locally testable* if it is strongly k-testable for some $k \geq 1$.

We denote by $\mathbf{SLT}$ the class of strongly locally testable semigroups and by $\mathbf{SLT_k}$ the set of strongly k-testable semigroups. We have that $\mathbf{SLT} = \bigcup_{k=1}^{\infty} \mathbf{SLT_k}$.

Example 3.2. We have that $\mathbf{SLT_1} = \mathbf{J_1}$. Indeed, let $S \in \mathbf{SLT_1}$ and let $z \in S^+$. By definition, the product of the components of z is completely determined by the alphabet of z, so, $\mathbf{SLT_1} \subseteq \mathbf{J_1}$. Now, let $S \in \mathbf{J_1}$ and let $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_m)$, $n, m \geq 1$, be two sequences of elements of S having the same alphabet. Let $\{a_1, \dots, a_q\}$ be the common alphabet of x and y. Then,

$$\begin{array}{rcccl} x_1 \cdots x_n & = & a_1^{n_1} \cdots a_q^{n_q} & = & a_1 \cdots a_q \\ y_1 \cdots y_m & = & a_1^{m_1} \cdots a_q^{m_q} & = & a_1 \cdots a_q \end{array}$$

where n_i (m_i) is the number of occurrences of a_i in x (y). So, $\mathbf{J_1} \subseteq \mathbf{SLT_1}$.

The following theorem, proved in [19], contains a characterization of $\mathbf{SLT}$.

Theorem 3.3. $\mathbf{SLT} = \mathbf{J} \cap \mathbf{LJ_1}$ *and the variety* $\mathbf{J} \cap \mathbf{LJ_1}$ *is defined by the pseudo-identities* $(xy)^\omega = (yx)^\omega$ *and* $x^\omega y x^\omega = x^\omega y x^\omega y x^\omega$.

In particular this means that **SLT** constitutes a variety of semigroups. It follows from the algebraic characterizations of locally testable languages [10] and piecewice testable languages [20], that the variety of languages associated via Eilenberg's variety theorem to **SLT** is the variety of languages that are both locally and piecewice testable.

The following combinatorial description of locally and piecewise testable languages is given in [19].

Theorem 3.4. *Let* $L \subseteq A^+$. *Then* L *is locally and piecewise testable if and only if it is a boolean combination of languages of the form*

$$L = u_0 B_1^+ u_1 \cdots u_{n-1} B_n^+ u_n$$

where $u_i \in A^*$ *for* $0 \leq i \leq n$, $B_i \subseteq A$ *for* $1 \leq i \leq n$, $B_i \cap B_j = \emptyset$ *if* $i \neq j$ *and the last letter of* u_{i-1} *and the first letter of* u_i *dont belong to* B_i *for* $1 \leq i \leq n$.

The notion of strongly locally testable language was introduced and studied by Beauquier and Pin [9]. A language L is strongly locally testable if the membership of a word in L is determined by the set of its factors of length k, for some k. Strongly locally testable languages are not characterized by a property of their syntactic semigroups, so they do not constitute a variety of languages.

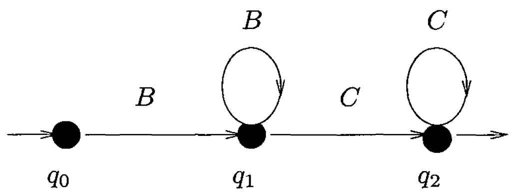
Locally and piecewise testable languages are strongly locally testable [19] and hence, since the family of strongly locally testable languages does not form a variety of languages, they constitute a strict subclass of the strongly locally testable languages.

4. $\mathbf{SLT} \cap \mathbf{E_{com}}$

In this section we give the pseudo-identities defining the variety $\mathbf{SLT} \cap \mathbf{E_{com}}$ formed by all strongly locally testable semigroups with commuting idempotents.

First, we exhibit an example of strongly locally testable semigroup S that does not belong to $\mathbf{E_{com}}$. This proves that $\mathbf{SLT} \cap \mathbf{E_{com}}$ is a strict subvariety of **SLT**.

Example 4.1. Let A be an alphabet and let B and C be nonempty subsets of A such that $B \cap C \neq \emptyset$. We consider the language $L = B^+C^+$. L is an elementary language on A^+. So, by Theorem 3.4, its syntactic semigroup S is in **SLT**. It is easy to check that the minimal automaton for L is the automaton $\mathcal{A}$.



Let η: $A^+ \longrightarrow S$ be the transition morphism of $\mathcal{A}$. Then, for any $a \in B \cup C$, $a\eta$ is an idempotent of S. Let $b \in B$ and let $c \in C$. The domain of $(bc)\eta$ is $\{q_0\}$ and its image is $\{q_2\}$. But the domain of $(cb)\eta$ is the empty set. So $(bc)\eta \neq (cb)\eta$. Then, $S \in \mathbf{SLT}$ but $S \notin \mathbf{E_{com}}$.

Proposition 4.2. *The variety* $\mathbf{SLT} \cap \mathbf{E_{com}}$ *is defined by the pseudo-identities* $(xy)^\omega = (yx)^\omega$, $x^\omega y x^\omega = x^\omega y x^\omega y x^\omega$ *and* $x^\omega y^\omega = y^\omega x^\omega$.

Proof. The variety $\mathbf{E_{com}}$ is defined by $x^\omega y^\omega = y^\omega x^\omega$. The proposition follows by Theorem 3.3. □

The rest of the paper will be devoted to obtaining a combinatorial description of the languages whose syntactic semigroup belongs to $\mathbf{SLT} \cap \mathbf{E_{com}}$.

5. A family of languages

In this section we exhibit a family of languages whose syntactic semigroups belong to $\mathbf{SLT} \cap \mathbf{E_{com}}$.

Let A be an alphabet. An *elementary* language on A^+ is a language of the form

$$L = B_0^* a_1 B_2^* \dots a_n B_n^*$$

where $a_i \in A$ $(1 \leq i \leq n)$, $B_i \subseteq A$ $(0 \leq i \leq n)$, $B_i \cap B_j = \emptyset$ if $i \neq j$ and $a_i \notin B_{i-1} \cup B_i$ $(1 \leq i \leq n)$.

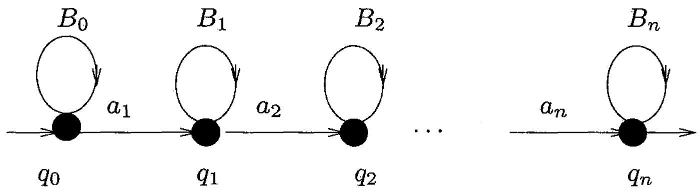
Remark 5.1. Since the sets B_i can be empty, the languages of the form

$$L = u_0 B_1^* u_1 \dots B_n^* u_n$$

where $u_0, u_n \in A^*$, $u_i \in A^+$ $(1 \leq i \leq n-1)$, $B_i \cap B_j = \emptyset$ if $i \neq j$ and the last letter of u_{i-1} and the first letter of u_i do not belong to B_i $(1 \leq i \leq n)$, are elementary.

We denote by $A^+\mathcal{W}$ the boolean algebra generated by all elementary languages on A^+. We prove in this section that, if $L \in A^+\mathcal{W}$ then the syntactic semigroup $S(L)$ belongs to $\mathbf{SLT} \cap \mathbf{E_{com}}$.

Remark 5.2. Let $L = B_0^* a_1 B_2^* \dots a_n B_n^*$ be an elementary language on A^+ and let $\mathcal{A}$ be the following automaton:



The automaton $\mathcal{A}$ recognizes L and, since $a_i \notin B_{i-1} \cup B_i$ $(1 \leq i \leq n)$, $\mathcal{A}$ is a deterministic and codeterministic automaton. So $\mathcal{A}$ is the minimal automaton of L.

Moreover, let η: $A^+ \longrightarrow S$ be the transition morphism of the automaton $\mathcal{A}$. Then $\mathcal{A}$ verifies the following properties:

1. if $q_j = q_i(x\eta)$ with $x \in A^+$, then $i \leq j$;
2. loop alphabets of distinct states are pairwise disjoint;
3. $a_i \notin B_{i-1} \cup B_i$ for any $1 \leq i \leq n$.

In the rest of the paper we will use the notation $(u)alph$ for the set of letters which occur in $u \in A^+$.

Lemma 5.3. *Let $L = B_0^* a_1 B_1^* \dots a_n B_n^*$ be an elementary language on A^+. Let $\mathcal{A}$ be its minimal automaton and let η: $A^+ \longrightarrow S$ be the transition morphism of $\mathcal{A}$. Let $x\eta \in E(S)$ for some $x \in A^+$. If the domain of $x\eta$ is nonempty, then there exists a unique $0 \leq i \leq n$ such that $(x)alph \subseteq B_i$, and the domain of $x\eta$ and its image are exactly $\{q_i\}$.*

Proof. The automaton $\mathcal{A}$ is the automaton represented in Remark 5.2. Let $x\eta \in E(S)$ and let q be a state of $\mathcal{A}$ belonging to the domain of $x\eta$. Then $q.x\eta = q.(x^2\eta)$. So, there exists $0 \leq i \leq n$ such that $q.x\eta = q_i$ and $(x)alph \subseteq B_i$. But, by statement 3 of Remark 5.2, $a_i \notin B_i$ and so $q = q_i$. Conversely $q_i.x\eta = q_i$. Moreover, by statement 2, the alphabets B_j are pairwise disjoint, therefore there exists a unique $0 \leq i \leq n$ such that $(x)alph \subseteq B_i$. So, the domain of $x\eta$ and its image are exactly $\{q_i\}$. □

Proposition 5.4. *Let $L \in A^+\mathcal{W}$. Then $S(L) \in$* $\mathbf{SLT} \cap \mathbf{E_{com}}$.

Proof. Since $A^+\mathcal{W}$ is a boolean algebra and since $\mathbf{SLT} \cap \mathbf{E_{com}}$ is a variety of semigroups, it is sufficient to prove the proposition for the elementary languages on A^+.

Let $L = B_0^* a_1 B_2^* \dots a_n B_n^*$ be an elementary language on A^+. Let $\mathcal{A}$ be the automaton represented in Remark 5.2 and let η: $A^+ \longrightarrow S$ be the transition morphism of $\mathcal{A}$. By Remark 5.2, $\mathcal{A}$ is the minimal automaton of L and hence S is the syntactic semigroup of L. Therefore, we prove $S \in \mathbf{SLT} \cap \mathbf{E_{com}}$.

By Proposition 4.2, it suffices to show that S verifies the pseudo-identities defining $\mathbf{SLT} \cap \mathbf{E_{com}}$.

Let k be such that $(xy)^k\eta$ and $(yx)^k\eta$ are idempotents in S. By Lemma 5.3, there exist $0 \leq i, j \leq n$ such that $(xy)alph \subseteq B_i$, $(yx)alph \subseteq B_j$, and the domain of $(xy)^k\eta$ and its image are exactly $\{q_i\}$. But $(xy)alph = (yx)alph$, so $i = j$ and $(xy)^k\eta = (yx)^k\eta$.

Let $x, y \in A^+$ and let $(x\eta)^k$ be the idempotent power of $x\eta$ in S. By Lemma 5.3, there exists a unique $0 \leq i \leq n$ such that $(x)alph \subseteq B_i$, the domain of $(x\eta)^k$ and its image are exactly $\{q_i\}$. Let now $y \in A^+$. If $(y)alph \not\subseteq B_i$ then the domain of $(x\eta)^k y\eta (x\eta)^k$ is the emptyset. Otherwise, $(y)alph \subseteq B_i$. In either case, we get $(x\eta)^k y\eta (x\eta)^k = (x\eta)^k y\eta (x\eta)^k y\eta (x\eta)^k$.

Let $x, y \in A^+$ and let $(x\eta)^k$ and $(y\eta)^k$ be the idempotent powers of $x\eta$ and $y\eta$ in S respectively. By Lemma 5.3, there exist $0 \leq i, j \leq n$ such that $(x)alph \subseteq B_i$ and the domain of $(y\eta)^k$ and its image are exactly $\{q_j\}$. By the hypothesis made on L, $B_i \cap B_j = \emptyset$ if $i \neq j$. So, if $i \neq j$, the domains of $(x\eta)^k(y\eta)^k$ and of $(y\eta)^k(x\eta)^k$

are empty. Otherwise, $i = j$, and hence the domains of $(x\eta)^k(y\eta)^k$ and $(y\eta)^k(x\eta)^k$ and their images are exactly $\{q_i\}$. $\square$

6. The implicit operations on $\mathbf{SLT} \cap \mathbf{E_{com}}$

The variety $\mathbf{SLT} \cap \mathbf{E_{com}}$ does not have free objects. It turns out to be necessary to consider the semigroup of implicit operations on the variety $\mathbf{SLT} \cap \mathbf{E_{com}}$. We will use in Section 7 the properties of $\widehat{F}_A(\mathbf{SLT} \cap \mathbf{E_{com}})$ to find a combinatorial characterization of the languages recognized by $\mathbf{SLT} \cap \mathbf{E_{com}}$.

6.1. $\widehat{F}_A(SLT \cap E_{com})$: a normal form

We give in this section a normal form for the elements of $\widehat{F}_A(\mathbf{SLT} \cap \mathbf{E_{com}})$.

Let ϑ: $\widehat{F}_A(\mathbf{SLT}) \longrightarrow \widehat{F}_A(\mathbf{SLT} \cap \mathbf{E_{com}})$ be the natural projection. Since $\mathbf{J_1} = \mathbf{SLT_1} \cap \mathbf{E_{com}} \subseteq \mathbf{SLT} \cap \mathbf{E_{com}} \subset \mathbf{SLT}$, we can define the content morphism for $\widehat{F}_A(\mathbf{SLT} \cap \mathbf{E_{com}})$ and for $\widehat{F}_A(\mathbf{SLT})$, which we denote by $\overline{c}$ and c respectively.

Now we give the description of $\widehat{F}_A(\mathbf{SLT})$, which we will use in the sequel [19].

Theorem 6.1. *The idempotents of $\widehat{F}_A(\mathbf{SLT})$ are entirely determined by their content. Each element of $\widehat{F}_A(\mathbf{SLT})$ can be written in a unique normal form as a product $\pi = u_0v_1u_1 \dots v_nu_n$, where $n \geq 0$, $u_i \in A^*$ (i.e. u_i is explicit), the v_i are idempotent elements of $\widehat{F}_A(\mathbf{SLT})$ such that $(v_l)c \cap (v_m)c = \emptyset$ if $l \neq m$, and the first and the last letter of u_i do not belong to $(v_i)c$ and $(v_{i+1})c$ respectively.*

Proposition 6.2. *The idempotents of $\widehat{F}_A(\mathbf{SLT} \cap \mathbf{E_{com}})$ are determined by their content.*

Proof. The proposition follows by surjectivity of the morphism ϑ, by the identity $\pi\vartheta\overline{c} = \pi c$, for any $\pi \in \widehat{F}_A(\mathbf{SLT})$, and by Theorem 6.1. $\square$

Let $B \subseteq A, B \neq \emptyset$. We denote by $\tilde{B}$ the unique idempotent of $\widehat{F}_A(\mathbf{SLT} \cap \mathbf{E_{com}})$ whose content is B. By Theorem 6.1, we have the following proposition.

Proposition 6.3. *Let $\pi \in \widehat{F}_A(\mathbf{SLT} \cap \mathbf{E_{com}})$. Then, $\pi = u_0\tilde{B}_1u_1 \dots \tilde{B}_nu_n$, where $u_i \in A^*$ $(0 \leq i \leq m)$, $B_i \subseteq A$, $B_i \neq \emptyset$ $(1 \leq i \leq m)$, $B_i \cap B_j \neq \emptyset$ if $i \neq j$ and the last letter of u_{i-1} and the first letter of u_i do not belong to B_i $(1 \leq i \leq n)$.*

We will use the following important property of the product of the idempotents of $\widehat{F}_A(\mathbf{SLT} \cap \mathbf{E_{com}})$ to derive a normal form for the elements of $\widehat{F}_A(\mathbf{SLT} \cap \mathbf{E_{com}})$.

Proposition 6.4. *Let $\tilde{B}, \tilde{C}$ be two idempotents of $\widehat{F}_A(\mathbf{SLT} \cap \mathbf{E_{com}})$. Then $\tilde{B}\tilde{C} = \tilde{D}$, where $D = B \cup C$.*

Proof. Since $\tilde{B}\tilde{C}$ is an idempotent of $\widehat{F}_A(\mathbf{SLT} \cap \mathbf{E_{com}})$ whose content is $D = B \cup C$, by Proposition 6.2, $\tilde{B}\tilde{C} = \tilde{D}$. $\square$

Let $\pi \in \widehat{F}_A(\mathbf{SLT} \cap \mathbf{E_{com}})$. We say that π is *in normal form* if

$$\pi = u_0\tilde{B}_1u_1 \ldots \tilde{B}_nu_n$$

where $u_0, u_n \in A^*$, $u_i \in A^+$ $(1 \leq i \leq n-1)$, $B_i \cap B_j = \emptyset$ if $i \neq j$ and the last letter of u_{i-1} and the first letter of u_i do not belong to B_i $(1 \leq i \leq m)$.

By Proposition 6.3, we have the following proposition.

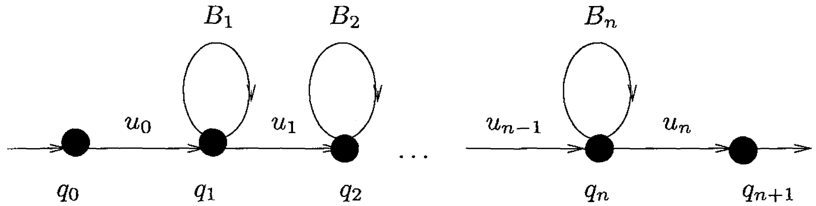
Proposition 6.5. *Any element $\pi \in \widehat{F}_A(\mathbf{SLT} \cap \mathbf{E_{com}})$ can be written in normal form.*

Now, we show the uniqueness of the normal form for the elements of $\widehat{F}_A(\mathbf{SLT} \cap \mathbf{E_{com}})$. We denote by $\mathbf{W}$ the variety generated by all semigroups of the form $S(L)$, L belonging $A^+\mathcal{W}$, for any alphabet A. By Proposition 5.4, $\mathbf{W} \subseteq \mathbf{SLT} \cap \mathbf{E_{com}}$.

Proposition 6.6. *Let $\pi = u_1\tilde{B}_1u_2 \ldots \tilde{B}_nu_n$, $\rho = v_1\tilde{C}_1v_2 \ldots \tilde{C}_mv_m$ be two elements of $\widehat{F}_A(\mathbf{SLT} \cap \mathbf{E_{com}})$ in normal form.*

1. *If $\pi_S = \rho_S$ for any $S \in \mathbf{W}$ then $m = n$, $u_i = v_i$ $(0 \leq i \leq n)$ and $B_i = C_i$ $(1 \leq i \leq n)$;*
2. *$\pi = \rho$ if and only if $m = n$, $u_i = v_i$ $(0 \leq i \leq n)$.*

Proof. We suppose that $\pi_S = \rho_S$ for any $S \in \mathbf{W}$. Let $L = u_0B_1^*u_1 \ldots B_n^*u_n$. Since π is in normal form, by Remark 5.1, L is an elementary language. So, by Proposition 5.4, $S \in \mathbf{W}$. Let $\mathcal{A}$ be the minimal automaton and let η: $A^+ \longrightarrow S$ be its transition morphism. $\mathcal{A}$ is the following automaton:



Let k be such that s^k is an idempotent of S for any $s \in S$. For any $1 \leq i \leq n$, we choose $w_i \in A^+$ such that $(w_i)alph = B_i$. Then, by Proposition 6.2, $(w_i^k)_S = (w_i^\omega)_S = (\tilde{B}_i)_S$. So, $\pi_S = (u_0w_1^ku_1 \ldots w_n^ku_n)_S$. We choose likewise, for any $1 \leq j \leq m$, a word $z_j \in A^+$ such that $(z_j)alph = C_j$. Then $\rho_S = (v_0z_1^kv_1 \ldots z_m^kv_m)_S$.

By hypothesis, $\pi_S = \rho_S$. It follows by definition of π_S and ρ_S, that $(u_0w_1^ku_1 \ldots w_n^ku_n)\eta = (v_0z_1^kv_1 \ldots z_m^kv_m)\eta$. By Lemma 5.3, the domain of the transition generated by $(u_0w_1^ku_1 \ldots w_n^ku_n)\eta$ is $\{q_0\}$ and its image is $\{q_{n+1}\}$. But $v_0z_1^kv_1 \ldots z_m^kv_m$ is the label of a path from $\{q_0\}$ to $\{q_{n+1}\}$. By Lemma 5.3, there exists $1 \leq i \leq n$ such that $(z_1)alph = C_1 \subseteq B_i$, $q_0(v_0\eta) = q_i$ and $q_0(v_0z_1^k)\eta = q_i$. So u_0 is a prefix of v_0. Symmetrically, we can prove that v_0 is a prefix of u_0, so $u_0 = v_0$. This fact implies that $i = 1$ and $C_1 \subseteq B_1$. Symmetrically, we have also $B_1 \subseteq C_1$. Hence $B_1 = C_1$. Repeating the same argument we obtain $n = m$, $u_i = v_i$ $(0 \le i \le n)$ and $B_i = C_i$ $(0 \le i \le n)$.

For item 2, let $\pi = \rho$. By Proposition 5.4, we have $\pi_S = \rho_S$ for all $S \in \mathbf{W}$. Then, by item 1 of this proposition, $m = n$, $u_i = v_i$ $(0 \le i \le n)$ and $B_i = C_i$ $(1 \le i \le n)$. □

7. A combinatorial description

We give in this section a combinatorial characterization of the languages recognized by $\mathbf{SLT} \cap \mathbf{E_{com}}$.

By Proposition 5.4, $\mathbf{W} \subseteq \mathbf{SLT} \cap \mathbf{E_{com}}$. By Theorem 2.3, to show that $\mathbf{W} = \mathbf{SLT} \cap \mathbf{E_{com}}$, it is sufficient to prove the following proposition.

Proposition 7.1. *Let $\pi, \rho \in \widehat{F}_A(\mathbf{SLT} \cap \mathbf{E_{com}})$ such that $\pi_S = \rho_S$ for any $S \in \mathbf{W}$. Then, $\pi = \rho$.*

Proof. Let $\pi = u_1 \tilde{B}_1 u_2 \ldots \tilde{B}_n u_n$ and let $\rho = v_1 \tilde{C}_1 v_2 \ldots \tilde{C}_m v_m$ be in normal form. By Proposition 6.6, if $\pi_S = \rho_S$ for any $S \in \mathbf{W}$, then $m = n$, $u_i = v_i$ $(0 \le i \le n)$ and $B_i = C_i$ $(1 \le i \le n)$ and hence $\pi = \rho$. □

The next theorem is a corollary of Proposition 7.1.

Theorem 7.2. $\mathbf{W} = \mathbf{SLT} \cap \mathbf{E_{com}}$.

So, by Eilenberg's theorem, we have the following theorem.

Theorem 7.3. *Let $L \subseteq A^+$. Then L is recognized by $\mathbf{SLT} \cap \mathbf{E_{com}}$ if and only if $L \in A^+\mathcal{W}$.*

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