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THE PSEUDOVARIETY J IS HYPERDECIDABLE (*)

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Abstract. – *This article defines the notion of hyperdecidability for a class of finite semigroups, which is closely connected to the notion of decidability. It then proves that the pseudovariety* **J** *of* $\mathcal{J}$*-trivial semigroups is hyperdecidable.*

1. INTRODUCTION

The framework of this paper is the study of decision problems on semigroups. The main result states that the pseudovariety **J** of all finite $\mathcal{J}$-trivial semigroups is hyperdecidable. The notion of hyperdecidability is a strengthening of the notion of decidability (a precise definition will be given in Section 3). It was recently introduced by the first author [4] to establish the decidability of the membership problem in several instances.

Let us recall that the membership problem for a given class of semigroups $\mathcal{C}$ consists in deciding whether a finite semigroup belongs to $\mathcal{C}$. This is one of the main questions concerning some particular classes, the pseudovarieties of finite semigroups. A class of semigroups is said to be *decidable* if its membership problem is decidable.

Since pseudovarieties arise when studying combinatorial problems on languages, they are frequently given by means of simpler pseudovarieties and operators. It is not a trivial problem to determine whether such a pseudovariety is decidable or not. It is known that in general, most operators on pseudovarieties do not preserve decidability. For instance, Albert, Baldinger and Rhodes [1] proved that there exist decidable pseudovarieties

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$\mathbf{V}$ and $\mathbf{W}$ such that their join $\mathbf{V} \vee \mathbf{W}$ is not decidable. Most of the time, existing results only have *ad hoc* proofs which require a deep knowledge of the involved pseudovarieties.

The first author noticed that it could be more convenient to use a stronger property than decidability for solving such questions. The first hope is to have a property yielding more easily decidability results, that is, which is preserved by operators on pseudovarieties. The second one is to get a notion that is general enough to apply to most "usual" pseudovarieties.

Henckell [8] already defined such a property by introducing pointlike sets. A pseudovariety $\mathbf{V}$ is said to be *strongly decidable* if for every semigroup S, the set of $\mathbf{V}$-pointlike subsets of S is computable. Hyperdecidability is a property of pseudovarieties that implies strong decidability. (However, whether these notions are equivalent is not clear at present, although it seems very unlikely. See [5] for additional details.)

Proving that a pseudovariety is decidable may be straightforward while it may be arduous to prove that the same pseudovariety is strongly decidable or hyperdecidable. There are two famous and difficult results concerning strong decidability: Ash [7] proved that the pseudovariety of finite groups is hyperdecidable; Henckell [8] showed that the pseudovariety of finite group-free semigroups is strongly decidable.

The formal definition of hyperdecidability was drawn in [4] without having in mind Ash's paper, in which this property is not emphasized and isolated in full generality. Thus, this notion was investigated by an author and independently rediscovered by another one. Such a consideration suggests that this concept is not artificial, but that it rather emerges as an idea inherent in the study of the membership problem. Even if formulated recently, it seems already to be a key notion, and its understanding is quite an important stake. For instance, the applications stated in Section 5 are now easily provided by general results of [5], while they previously only had difficult and painstaking proofs. It is likely that in the future, the concept of hyperdecidability will provide tools to answer the membership problem.

The techniques and results used in this paper to show that $\mathbf{J}$ is hyperdecidable require a basic knowledge of implicit operations on $\mathbf{J}$. The reader is referred to [3] for details we do not give here. The proofs consist in an elementary interpretation of these operations on automata, and hence they seem to be very natural.

The paper is organized as follows. In Section 2, we set up the notation, and we recall the most general results used in the paper. Section 3 introduces the notion of hyperdecidability. The main result, stating that $\mathbf{J}$ is hyperdecidable, is shown in Section 4. Lastly, we give in Section 5 some applications of the main result.

2. PREREQUISITES

We assume the reader to be familiar with the theories of finite semigroups, pseudovarieties and implicit operations. We now make precise some notation.

2.1. Terminology and notation

We denote by A^* (resp. by A^+) the free monoid (resp. the free semigroup) generated by the finite alphabet A, and by ε the empty word. The cardinality of a finite set X is denoted by $|X|$. The length of a word u is denoted as usual by $|u|$, and the set of letters occurring in u, called its *content*, by $c(u)$.

Given a semigroup S, S^1 is the semigroup S itself if it is a monoid, or $S \cup \{1\}$ where $1 \notin S$ acts as a neutral element otherwise. If S_1 and S_2 are subsets of S, we denote by $S_1^{-1}S_2$ the set $\{s \in S \mid \exists\ s_1 \in S_1$ such that $s_1 s \in S_2\}$.

A *pseudovariety* is a class of finite semigroups closed under formation of finite direct product, subsemigroup and homomorphic image. The paper deals mainly with $\mathbf{J}$, the pseudovariety of $\mathcal{J}$-trivial semigroups. We recall in Section 2.3 the most important properties of $\mathbf{J}$.

2.2. Implicit operations

We just recall some definitions and statements (without their justifications) concerning implicit operations. See [2, 3] for details.

Let $\mathbf{V}$ be a pseudovariety and let $A = \{a_1, \ldots, a_n\}$ be an alphabet. An n-ary implicit operation on $\mathbf{V}$ is a collection $(\pi_S)_{S \in \mathbf{V}}$ where each π_S is a function from S^n into S such that, for S, T in $\mathbf{V}$, the following diagram commutes

$$\begin{array}{ccc} S^n & \xrightarrow{\pi_S} & S \\ {\scriptstyle\varphi^n}\downarrow & & \downarrow{\scriptstyle\varphi} \\ T^n & \xrightarrow[\pi_T]{} & T \end{array}$$

We represent the set of n-ary implicit operations on $\mathbf{V}$ by $\overline{\mathrm{F}}_n(\mathbf{V})$.

We associate to the word $u = a_{i_1} \cdots a_{i_k}$ of A^+ the collection of functions $(u_S)_{S \in \mathbf{V}}$ defined by $u_S(s_1, \ldots, s_n) = s_{i_1} \cdots s_{i_k}$. One can easily check that this defines in fact an implicit operation. Such an operation, induced by a word, is said to be *explicit*. We simply denote it by u, and we denote the set of n-ary explicit operations on $\mathbf{V}$ by $\mathrm{F}_n(\mathbf{V})$.

The multiplicative law on $\overline{\mathrm{F}}_n(\mathbf{V})$ defined by $(\pi_S)\cdot(\rho_S) = (\pi_S \cdot \rho_S)$ makes it a semigroup, and $\mathrm{F}_n(\mathbf{V})$ a subsemigroup of $\overline{\mathrm{F}}_n(\mathbf{V})$. We endow $\overline{\mathrm{F}}_n(\mathbf{V})$ with the initial topology for the evaluation morphisms

$$\begin{aligned} e_T : \quad \overline{\mathrm{F}}_n(\mathbf{V}) &\longrightarrow T^{T^n} \\ (\pi_S)_{S\in\mathbf{V}} &\longmapsto \pi_T \end{aligned}$$

where T runs in $\mathbf{V}$, and where each finite semigroup T^{T^n} is endowed with the discrete topology. This topology makes $\overline{\mathrm{F}}_n(\mathbf{V})$ a compact and 0-dimensional topological semigroup in which $\mathrm{F}_n(\mathbf{V})$ is dense.

Let us consider the morphism $\iota : A^+ \to \overline{\mathrm{F}}_n(\mathbf{V})$ defined by $\iota(a_i) = a_i$. The $\mathbf{V}$-*closure* of a language L of A^+ is by definition the topological closure of $\iota(L)$ in $\overline{\mathrm{F}}_n(\mathbf{V})$.

The pseudovariety $\mathbf{Sl}$ of finite idempotent and commutative semigroups plays an important role when extending the notion of content to implicit operations. Indeed, for pseudovarieties $\mathbf{V}$ containing $\mathbf{Sl}$, such as $\mathbf{J}$, there exists a unique continuous morphism from $\overline{\mathrm{F}}_n(\mathbf{V})$ into 2^A that coincides with the content function c on A^+. We still denote this morphism by c.

Given an implicit operation π on $\mathbf{V}$, it is easy to see that the sequence $(\pi^{k!})_{k\in\mathbb{N}}$ converges to an idempotent element of $\overline{\mathrm{F}}_n(\mathbf{V})$, denoted by π^ω. Note that in pseudovarieties containing $\mathbf{Sl}$, we have $c(\pi^{k!}) = c(\pi)$. By continuity of c, we therefore have $c(\pi^\omega) = c(\pi)$.

2.3. The pseudovariety J

The key notion when studying $\mathbf{J}$ is that of subword. Recall that a word $x = x_1 \cdots x_l$ is a *subword* of an implicit operation $\pi \in \overline{\mathrm{F}}_n(\mathbf{J})$ if π has a factorization of the form $\pi_0 x_1 \pi_1 \cdots x_l \pi_l$ where $\pi_i \in \overline{\mathrm{F}}_n(\mathbf{J})^1$. This definition coincides with the usual one on words when π is explicit.

For π and ρ in $\overline{\mathrm{F}}_n(\mathbf{J})$ and for each natural number ℓ, we write $\pi \sim_\ell \rho$ if π and ρ have the same subwords of length at most ℓ. The relation $\sim_\ell$ is a congruence, and $\sim_{\ell+1} \subseteq \sim_\ell$. The link between $\mathbf{J}$ and this family of congruences has been extensively studied. The fundamental result is due to Simon [12], who proved that a language over A is recognizable by a $\mathbf{J}$-trivial semigroup if and only if it is saturated by one of the congruences $\sim_\ell$.

We will use in Section 4.1.2 some combinatorial properties from Simon's paper. For now, we just state topological and structural results concerning the semigroup of implicit operations $\overline{F}_n(\mathbf{J})$. Both results are stated in [3, Theorem 8.2.8], and are closely related to Simon's result.

THEOREM 2.1 (Almeida [3]): *Every idempotent implicit operation on* $\mathbf{J}$ *is of the form* u^ω, *where* u *is explicit. More generally, every implicit operation* π *on* $\mathbf{J}$ *has a factorization* $\pi = \pi_1 \cdots \pi_k$ *such that:*

cf.1) Each factor π_i *is either explicit or of the form* u_i^ω *where* u_i *is explicit.*

cf.2) If π_i *and* π_{i+1} *are idempotent, the sets* $c(\pi_i)$ *and* $c(\pi_{i+1})$ *are incomparable.*

cf.3) Two consecutive factors π_i *and* π_{i+1} *are not both explicit.*

cf.4 If π_i *is explicit and* π_{i+1} *idempotent, then the last letter of* π_i *is not in* $c(\pi_{i+1})$. *If* π_i *is idempotent and* π_{i+1} *explicit, then the first letter of* π_{i+1} *is not in* $c(\pi_i)$.

Furthermore, if $\pi_1 \cdots \pi_k$ *is the factorization of* π *and if* $\rho_1 \cdots \rho_l$ *is the factorization of* ρ, *then the following conditions are equivalent:*

i. $\pi = \rho$.

ii. $k = l$, *and* $\pi_j = \rho_j$ *for* $1 \leq j \leq k$.

iii. π *and* ρ *have the same subwords.* ■

We say that the factorization of π satisfying the conditions cf.1) to cf.4) of the theorem is the *canonical* factorization of π. A description of reduction rules to obtain the canonical form of an implicit operation built from letters using multiplication and ω-powers is given in [3, Section 8.2] (page 226):

rr.1) Eliminate parentheses concerning the application of the operation of multiplication;

rr.2) Substitute any occurrence of t^ω by u^ω, where u is the product of the variables that occur in t, say in increasing order of the indices;

rr.3) Absorb in factors of the form u^ω any adjacent factors in which only variables of u occur.

A direct consequence of Theorem 2.1 is that if u and v are two words having the same content, then the implicit operations u^ω and v^ω of $\overline{F}_n(\mathbf{J})$ are equal. In the sequel, if $B = c(u)$, we will sometimes denote by B^ω this operation.

One can deduce from Theorem 2.1 a useful corollary.

COROLLARY 2.2: *Let* π *be an implicit operation of* $\overline{F}_n(\mathbf{J})$. *A sequence* $(\pi_i)_{i\in\mathbb{N}}$ *converges to* π *in* $\overline{F}_n(\mathbf{J})$ *if and only if for every natural* ℓ, *there exists* $N \in \mathbb{N}$ *such that* $i > N$ *implies* $\pi \sim_\ell \pi_i$. ■

2.4. Automata

We assume that the reader is familiar with the basic notions of the theory of finite automata. We refer the reader to [9, 10] for an introduction to this theory.

We denote by $\mathcal{A}$ a deterministic finite automaton, by Q its set of states, and by q_{ini} its initial state. A final state will be in general denoted by q_{fin}. Recall that a *path* in $\mathcal{A}$ is a sequence of consecutive transitions. The set of states reached from q after reading a word u is denoted by $q \cdot u$. When dealing with deterministic automata, this set is a singleton.

We will denote transitions between two states by a solid arrow: $p \xrightarrow{u} q$. A dashed arrow represents a path between two states which is not necessarily a transition: $p \overset{u}{\dashrightarrow} q$. We also use this convention in figures.

In the figures of this paper, the initial state will be pointed out by an arrow, and final states will be doubly circled.

The arrows of an automaton are usually labeled by letters. We will consider automata whose transitions are labeled by implicit operations on $\mathbf{J}$ instead. We adopt for this kind of automata the definitions and conventions we have just given for usual automata. We also extend the notion of recognizability: we say that an implicit operation $\pi \in \overline{\mathrm{F}}_n(\mathbf{J})$ is recognized by an automaton $\mathcal{A}$ if and only if there exists a factorization $\pi_1 \cdots \pi_k$ of π and a path $q_{ini} \xrightarrow{\pi_1} \cdots \xrightarrow{\pi_k} q_{fin}$ in $\mathcal{A}$ such that q_{ini} is the initial state and q_{fin} a final state. Such a path is said to be *successful*. The set of implicit operations that label successful paths is the *language recognized* by $\mathcal{A}$.

3. HYPERDECIDABILITY

We associate to a finite graph Γ the system of all equations of the form $xy = x'$ where $x \xrightarrow{y} x'$ is an edge of Γ. We denote this system by Σ_Γ. For $L \subseteq A^+$, we denote by $\bar{L}$ its $\mathbf{V}$-closure.

Recall that a semigroup pseudovariety $\mathbf{V}$ is *monoidal* if it is generated by all semigroups S^1 with $S \in \mathbf{V}$. The pseudovariety $\mathbf{J}$ is an example of a monoidal pseudovariety. In order to avoid some technical complications arising with semigroup pseudovarieties, we will define hyperdecidability for monoidal pseudovarieties only. A more general definition would not be difficult to state, but is not needed in this paper. We say that a monoidal pseudovariety $\mathbf{V}$ is *hyperdecidable* if the following problem is decidable.

Data: - A finite semigroup $S = \{s_1, \ldots, s_n\}$.

- The finite alphabet $A = \{a_1, \ldots, a_n\}$ together with the canonical morphism from A^+ onto S defined by $\varphi(a_i) = s_i$.
- A finite graph Γ, whose set of vertices is $X = \{x_1, \ldots, x_k\}$ and whose set of edges is $Y = \{y_1, \ldots, y_l\}$.
- Elements $t_1, \ldots, t_k, u_1, \ldots, u_l$ of S.

Problem: Do there exist $\pi_p \in \overline{\varphi^{-1}(t_p)}$ and $\rho_r \in \overline{\varphi^{-1}(u_r)}$ such that, for each identity $x_p y_r = x_q$ of Σ_Γ, we have $\pi_p \rho_r = \pi_q$?

4. THE PSEUDOVARIETY J IS HYPERDECIDABLE

We prove that the pseudovariety **J** is hyperdecidable: first, we give an algorithm to compute the **J**-closure of a given rational language (Section 4.1). Next, we use this algorithm to determine whether the system associated to a finite graph has a solution (Section 4.2).

4.1. The J-closure of a rational language

This section shows how to compute the **J**-closure $\bar{L}$ of a rational language L of A^+. For this purpose, we construct from an automaton $\mathcal{A}$ recognizing L a new automaton $\bar{\mathcal{A}}$ recognizing $\bar{L}$. As explained in Section 2.4, $\bar{\mathcal{A}}$ is labeled by implicit operations on **J** rather than by letters.

4.1.1. *Construction of* $\bar{\mathcal{A}}$

DEFINITION 4.1: Given a deterministic automaton $\mathcal{A}$ recognizing L, we define $\bar{\mathcal{A}}$ as follows.

c.1) Start from $\mathcal{A}$. The initial state of $\bar{\mathcal{A}}$ is the initial state of $\mathcal{A}$.

c.2) For each state q and each subset $B \subseteq A$ such that there exists a loop $q \xrightarrow{u} q$ with $c(u) = B$, add a new state q_B in $\bar{\mathcal{A}}$, a transition $q \xrightarrow{B^\omega} q_B$ and a transition $q_B \xrightarrow{\varepsilon} q$.

The "old" states of $\bar{\mathcal{A}}$, which are not of the form q_B, will be called *kernel states*.

c.3) The final states of $\bar{\mathcal{A}}$ are the kernel states that were final in $\mathcal{A}$. ∎

The terminology "kernel state" is only introduced for convenience in this paper.

For Step c.2), notice that given a state q and a subset B of A, it is decidable whether there exists a loop containing q labeled by a word of content B. One can use for instance a breadth-first traversal to visit states of $\mathcal{A}$, starting from q and using letters of B.

We now give an example of this construction.

EXAMPLE 4.2: Let $\mathcal{A}$ be the automaton of Figure 1.

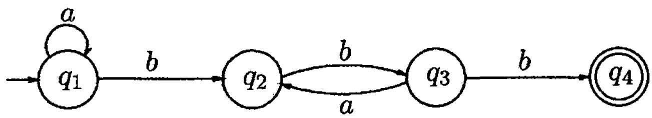


Figure 1. – An automaton recognizing $a^*bb(ab)^*b$

There is a loop of content $\{a\}$ around state q_1 and loops of content $\{a, b\}$ around states q_2 and q_3. After Step c.2), we therefore have three new states: $q_{1\,\{a\}}$, $q_{2\,\{a,b\}}$ and $q_{3\,\{a,b\}}$.

This yields the automaton of Figure 2 (we drop the braces around sets of letters).

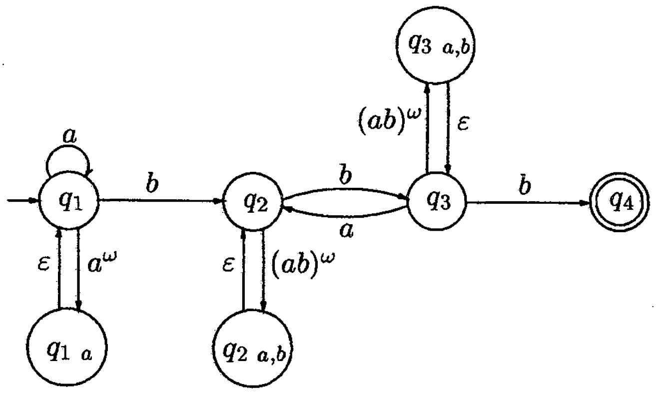


Figure 2. – The automaton $\bar{\mathcal{A}}$

We leave to the reader to check that this automaton recognizes the **J**-closure of the language $a^*bb(ab)^*b$ recognized by $\mathcal{A}$. In this example, it could easily be verified that this closure is $[(a^* \cup a^\omega)bb(ab)^*b] \cup (ab)^\omega$.

4.1.2. *The result*

The property we have just observed in Example 4.2 is general.

PROPOSITION 4.3: *Let $L \subseteq A^+$ be a rational language and let $\mathcal{A}$ be a deterministic automaton recognizing L. Then, the automaton $\bar{\mathcal{A}}$ constructed in Definition 4.1 recognizes the* **J***-closure $\bar{L}$ of L.*

Proof: Let K be the language recognized by $\bar{\mathcal{A}}$. We have to show that $K = \bar{L}$. Let us begin with some simple remarks that follow directly from the construction of $\bar{\mathcal{A}}$.

FACT 4.4: Let $p \xrightarrow{\pi} q$ be a transition of $\bar{\mathcal{A}}$. Then

a. $\pi = B^\omega$ if and only if p is a kernel state and $q = p_B$.

b. π is a letter if and only if both p and q are kernel states.

c. $\pi = \varepsilon$ if and only if p is not a kernel state. In this case, q is a kernel state and $p = q_B$. ■

In order to show the inclusion $K \subseteq \bar{L}$, we study in the next lemmas two particular kinds of paths in $\bar{\mathcal{A}}$: those labeled by explicit operations and those labeled by products of idempotents.

LEMMA 4.5: *Let q, q' be states of $\bar{\mathcal{A}}$, let $u = u_1 \cdots u_k$ be a nonempty word and let $q \dashrightarrow^{u} q'$ be a path in $\bar{\mathcal{A}}$. If q is a kernel state, then so is q' and the path $q \dashrightarrow^{u} q'$ is present in $\mathcal{A}$.*

Proof: It is sufficient to prove that each state of the path is a kernel state. By Fact 4.4.c, there is no ε-transition starting from a kernel state. Moreover, Fact 4.4.b. implies that each state reached from a kernel state by reading a letter is also a kernel state, so a straightforward induction on $|u|$ gives the desired result. ■

The handling of ε-transitions now requires a definition. We say that a factorization $\pi = \pi_1 \cdots \pi_k$ is *compatible* with a pair (p, p') of states of an automaton $\mathcal{A}$ if there exists in $\mathcal{A}$ a path $p = p_0 \xrightarrow{\rho_1} \cdots \xrightarrow{\rho_l} p_l = p'$ such that the sequence obtained from $(\rho_1, \ldots, \rho_l)$ by removing each empty ρ_i is equal to $(\pi_1, \ldots, \pi_k)$. We call the sequence $(\rho_1, \ldots, \rho_l)$ a *π-sequence* with respect to (p, p'). The factorization $(\pi_1, \ldots, \pi_k)$ of π will be understood.

LEMMA 4.6: *Let π be in $\overline{\mathrm{F}}_n(\mathbf{J})$ and let $v_1^\omega \cdots v_k^\omega$ be a factorization of π compatible with a pair of kernel states (q, q') of $\bar{\mathcal{A}}$. Then $q = q'$ and there exists in $\mathcal{A}$ a path of the form*

$$q \dashrightarrow^{w_1} q \dashrightarrow^{w_2} \cdots \dashrightarrow^{w_{k-1}} q \dashrightarrow^{w_k} q$$

where $c(w_i) = c(v_i)$.

Proof: Set $B_i = c(v_i)$. Let $(\rho_1, \ldots, \rho_l)$ be a π-sequence with respect to (q, q'). State q is a kernel state, so $\rho_1 \neq \varepsilon$ by Fact 4.4.c. Therefore, $\rho_1 = v_1^\omega$ and $q \cdot \rho_1 = q_{B_1}$ (Fact 4.4.a). Since q' is a kernel state, $q_{B_1} \neq q'$ so that $l \geq 2$. By fact 4.4.c, the next transition is labeled $\rho_2 = \varepsilon$ and leads back to state q. Therefore, the first transitions are $q \xrightarrow{v_1^\omega} q_{B_1} \xrightarrow{\varepsilon} q$. An easy induction then gives $q = q'$ and the following path in $\bar{\mathcal{A}}$:

$$q \xrightarrow{v_1^\omega} q_{B_1} \xrightarrow{\varepsilon} q \xrightarrow{v_2^\omega} \cdots \xrightarrow{\varepsilon} q \xrightarrow{v_k^\omega} q_{B_k} \xrightarrow{\varepsilon} q$$

Now, Step c.2) of the construction of $\bar{\mathcal{A}}$ together with the presence of q_{B_i} shows that q is a state of $\mathcal{A}$ belonging to a loop labeled by a word w_i of content $B_i = c(v_i)$. ■

Let us now conclude the proof of the inclusion $K \subseteq \bar{L}$. Take π in K. By definition of K, there exists a factorization

$$\left[\prod_{s=0}^{k-1} u_{s,1} \cdots u_{s,i_s} \cdot v_{s,1}^\omega \cdots v_{s,j_s}^\omega\right] \cdot u_{k,1} \cdots u_{k,i_k}$$

of π which is compatible with a pair (q_{ini}, q_{fin}) of $\bar{\mathcal{A}}$ where q_{ini} is the initial state and q_{fin} a final state of $\bar{\mathcal{A}}$. Each $u_{s,i}$ is a letter and each $v_{s,i}^\omega$ is an idempotent. Set $u_s = u_{s,1} \cdots u_{s,i_s}$ and $\pi_s = v_{s,1}^\omega \cdots v_{s,j_s}^\omega$. By convention, π is explicit if $k = 0$. In the same way i_0 (resp. i_k) may be equal to zero, in which case u_0 (resp. u_k) is empty. On the other hand, $i_s \geq 1$ for $1 \leq s \leq k-1$.

Consider the path $\mathcal{P}$ associated with a π-sequence with respect to the pair (q_{ini}, q_{fin}). Let p_s be the state of $\mathcal{P}$ reached just before doing the transition labeled by $u_{s,1}$ and let q_s be the state of $\mathcal{P}$ reached just after doing the transition labeled by u_{s,i_s}. If $u_0 = \varepsilon$ (resp. if $u_k = \varepsilon$), p_0 (resp. q_k) is not defined. One can picture $\mathcal{P}$ by the following diagram:

$$q_{ini} = p_0 \overset{u_0}{\dashrightarrow} q_0 \overset{\pi_0}{\dashrightarrow} p_1 \cdots q_{k-1} \overset{\pi_{k-1}}{\dashrightarrow} p_k \overset{u_k}{\dashrightarrow} q_k = q_{fin}$$

where the first arrow (resp. the last arrow) is not present if u_0 (resp. u_k) is empty. In this case, $q_0 = q_{ini}$ (resp. $p_k = q_{fin}$).

For $1 \leq s \leq k-1$, u_s is not empty. By Fact 4.4.b, both p_s and q_s are kernel states. This is also true for p_0 and q_0 (resp. for p_k and q_k) if u_0 (resp. u_k) is not empty. If $u_0 = \varepsilon$, then q_0 is the initial state, so it is a kernel state. In the same way, if $u_k = \varepsilon$, then p_k is a final state, hence it is a kernel state. To sum up, each p_i and each q_i is a kernel state.

Therefore, one can apply Lemma 4.5: there exists a path

$$p_s \overset{u_s}{\dashrightarrow} q_s \qquad (\mathcal{P}_s)$$

in $\mathcal{A}$ (provided that p_0 is present for $s = 0$ and that q_k is present for $s = k$). One can also apply Lemma 4.6 between states q_s and p_{s+1}. This gives the equality $q_s = p_{s+1}$ and the path in $\mathcal{A}$

$$q_s \xrightarrow{w_{s,1}} q_s \xrightarrow{w_{s,2}} \cdots \xrightarrow{w_{s,j_s}} q_s = p_{s+1} \qquad (\mathcal{P}'_s)$$

with $c(w_{s,i}) = c(v_{s,i})$. Joining all paths $(\mathcal{P}_s)$ and $(\mathcal{P}'_s)$ together in the natural way, we get a path in $\mathcal{A}$ going from the initial state to a final state. Therefore, the language L recognized by $\mathcal{A}$ contains

$$\left[\prod_{s=0}^{k-1} u_s \cdot (w_{s,1}^* \cdots w_{s,j_s}^*)\right] u_k$$

It is obvious that u^ω lies in the J-closure of u^*. Therefore, $\bar{L}$ contains the implicit operation $\left[\prod_{s=0}^{k-1} u_s \cdot (w_{s,1}^\omega \cdots w_{s,j_s}^\omega)\right] u_k$. Now, $w_{s,i}^\omega$ and $v_{s,i}^\omega$ are equal in $\overline{\mathrm{F}}_n(\mathbf{J})$. Indeed, $c(w_{s,i}) = c(v_{s,i})$, and therefore we get the same canonical form for both operations (using the reduction rules of Section 2.3). Hence $\pi = \left[\prod_{s=0}^{k-1} u_s \pi_s\right] u_k$ belongs to $\bar{L}$, as required.

Conversely, let us prove the inclusion $\bar{L} \subseteq K$. We borrow some notation from Simon's article [12] and recall three combinatorial properties of the congruence $\sim_\ell$ (defined in Section 2.3). We write $x \; R_\ell \; y$ if $x \sim_\ell y$ and there exist u and v in A^* and a in A such that $x = uav$ and $y = uv$. Let R_ℓ^* denote the reflexive and transitive closure of R_ℓ.

Lemma 4.7: *Let u and v be in A^+ and let $\ell > 0$. Then, $u \sim_\ell uv$ if and only if there exist $u_1, \ldots, u_\ell$ in A^+ such that $u = u_1 \cdots u_\ell$ and $c(v) \subseteq c(u_\ell) \subseteq \cdots \subseteq c(u_1)$.* ■

Lemma 4.8: *For u and v in A^* and a in A, $uav \sim_\ell uv$ if and only if there exist p and p' such that $p + p' \geq \ell$, $u \sim_p ua$ and $v \sim_{p'} av$.* ■

Lemma 4.9: *For every x and y in A^*, $x \sim_\ell y$ if and only if there exists z in A^* such that $z \; R_\ell^* \; x$ and $z \; R_\ell^* \; y$.* ■

We shall need another definition in the sequel.

Definition 4.10: Let $x_1, \ldots, x_k, y_1, \ldots, y_{k-1}$ be words in A^* and let $m > 0$. Define a factorization $u_i v_i w_i$ of y_i as follows:

- If $x_{i+1} \neq \varepsilon$ or if $i = k - 1$, then $w_i = \varepsilon$. Otherwise, w_i is the largest suffix of y_i of content contained in $c(y_{i+1})$.

- If $x_i \neq \varepsilon$ or if $i = 1$, then $u_i = \varepsilon$. Otherwise, u_i is the largest prefix of y_i of content contained in $c(y_{i-1})$.

We say that the product $x_1 y_1 \cdots x_{k-1} y_{k-1} x_k$ is *m-normal* if the following conditions hold.

n.1) The first letter of x_i does not belong to $c(y_{i-1})$ and the last letter of x_i does not belong to $c(y_i)$.

n.2) If $x_i = \varepsilon$, then $c(y_{i-1})$ and $c(y_i)$ are incomparable.

n.3) The factor v_i of y_i is a product of m words of content $c(y_i)$. ■

The proof of the inclusion $\bar{L} \subseteq K$ consists in three steps:

- Take a canonical factorization $x_1 B_1^\omega \cdots x_{k-1} B_{k-1}^\omega x_k$ of an implicit operation π in $\bar{L}$. At first, an intuitive understanding of the J-closure suggests that for m large enough, there exists in L an m-normal product $x_1 y_1 \cdots x_{k-1} y_{k-1} x_k$ with $c(y_i) = B_i$. This fact is shown in Lemma 4.14.

- Next, if m is large enough and if such an m-normal product is recognized by $\mathcal{A}$, there should exist a factorization $y_i = r_i s_i t_i$, with $c(s_i) = c(y_i)$, and such that the path labeled s_i is a loop. Consequently, $x_1(r_1 s_1^* t_1) \cdots x_{k-1}(r_{k-1} s_{k-1}^* t_{k-1}) x_k$ is also recognized by $\mathcal{A}$. This is stated in Lemma 4.15.

- Lastly, this will imply that $x_1 B_1^\omega \cdots x_{k-1} B_{k-1}^\omega x_k$ is recognized by $\bar{\mathcal{A}}$.

Let us begin with some technical lemmas.

LEMMA 4.11: *Let $x_1 y_1 \cdots x_{k-1} y_{k-1} x_k$ be an m-normal factorization of $z \in A^*$ and let $l \in [1, k-1]$. The following assertions hold:*

(i) Let x'_{l+1} be a nonempty prefix of x_{l+1}. Then $c(x'_{l+1}) \not\subseteq c(y_l)$.

(ii) The inclusion $c(x_{l+1} y_{l+1}) \subseteq c(y_l)$ does not hold.

(iii) Let $a \in c(y_l)$ and assume that $y_l = y'_l y''_l$. Set $\tilde{y}_l = y'_l a y''_l$ and $\tilde{y}_i = y_i$ for $i \neq l$. Then the product $\tilde{z} = x_1 \tilde{y}_1 \cdots x_{k-1} \tilde{y}_{k-1} x_k$ is m-normal.

(iv) Let $y_l = u_l v_l w_l$ be the factorization of y_l defined in 4.10. Let $\tilde{y}_l$ be the word obtained from y_l by removing one letter in u_l (resp. in w_l) and let $\tilde{y}_i = y_i$ for $i \neq l$. Then the product $\tilde{z} = x_1 \tilde{y}_1 \cdots x_{k-1} \tilde{y}_{k-1} x_k$ is m-normal.

Proof: The first assertion follows from the fact that the first letter of x_{l+1} does not belong to $c(y_l)$.

For (ii), we therefore deduce that if $c(x_{l+1} y_{l+1}) \subseteq c(y_l)$, then x_{l+1} must be empty. Hence $c(y_l)$ and $c(y_{l+1})$ are comparable, in contradiction with n.2).

For (iii), $c(\tilde{y}_l) = c(y_l)$, so n.1) and n.2) that hold for z also hold for $\tilde{z}$. Let $y_i = u_i v_i w_i$ be the factorization of y_i defined in 4.10. Since $c(\tilde{y}_l) = c(y_l)$, the factorization of $\tilde{y}_i$ for $i \neq l$ is $\tilde{y}_i = u_i v_i w_i$. Suppose that a is inserted inside u_l, that is, $u_l = u'_l u''_l$, and $\tilde{y}_l = (u'_l a u''_l) v_l w_l$. If a does not belong to $c(y_{l-1})$, the factorization of $\tilde{y}_l$ is $u_l'\tilde{v}_l w_l$ with $\tilde{v}_l = au_l''v_l$. If a belongs to $c(y_{l-1})$, this factorization is $\tilde{y}_l = \tilde{u}_l v_l w_l$ with $\tilde{u}_l = u_l' a u_l''$. In both cases, we check that n.3) is satisfied. The proof is dual in case a is inserted inside w_l. In the remaining case, one can write $v_l = v_l'v_l''$ with $\tilde{v}_l = v_l'av_l''$. The factorization of $\tilde{y}_l$ is $u_l\tilde{v}_l w_l$, and n.3) is still satisfied.

The proof of (iv) is analogous. ■

LEMMA 4.12: *Let $z \in A^*$. Assume that z has a factorization*

$$z = x_1 y_1 \cdots x_{k-1} y_{k-1} x_k \tag{1}$$

Let $m \geq |x_1 \cdots x_k| + k - 1$, and suppose that the factorization (1) is m-normal. Let $\ell > 2m$, let $z = z'z''$ and let $a \in A$ such that $\tilde{z} = z'az'' \sim_\ell z$. Then $\tilde{z}$ has an m-normal factorization $x_1\tilde{y}_1 \cdots x_{k-1}\tilde{y}_{k-1}x_k$ where $c(\tilde{y}_i) = c(y_i)$.

Proof: We consider two different cases.

<u>1st case:</u> There exists $x_l = x_l'x_l''$ such that a is inserted between x_l' and x_l''. We deduce from Lemma 4.8 and (1) that there exist p and p' such that $p + p' \geq \ell$,

$$x_1 y_1 \cdots x_l' \sim_p x_1 y_1 \cdots x_l' a \tag{2}$$

and

$$x_l'' \cdots y_{k-1} x_k \sim_{p'} a x_l'' \cdots y_{k-1} x_k$$

Since $p + p' \geq \ell > 2m$, we have either $p > m$ or $p' > m$. By symmetry, one may assume for instance that $p > m$. We will show that $l \geq 1$, $x_l' = \varepsilon$ and $a \in c(y_{l-1})$. From Lemma 4.7, we get words $s_1, \ldots, s_p$ such that

$$x_1 y_1 \cdots x_l' = s_1 \cdots s_p \tag{3}$$

and

$$a \in c(s_p) \subseteq \cdots \subseteq c(s_1) \tag{4}$$

The function from $[1, p]$ to $[1, |x_1 \cdots x_k| + k - 1]$ which maps i to

$$\max\{|x_1 x_2 \cdots x_{j-1} x_j'| + j - 1 \text{ s.t. } x_1 y_1 \cdots x_{j-1} y_{j-1} x_j' \text{ is a prefix of } s_1 \cdots s_i,\ x_j' \text{ is a prefix of } x_j\}$$

is clearly order preserving in view of (3). Since $p > m \geq |x_1 \cdots x_k| + k - 1$, this function maps two integers of $[1, p]$ to the same image. Consequently, there exist naturals q, r such that s_q is a factor of y_r. Suppose that $r < l-1$. Then we would have

$$c(y_r) \supseteq c(s_q) \supseteq c(s_{q+1} \cdots s_p) \supseteq c(x_{r+1} y_{r+1} \cdots x_l') \tag{5}$$

Hence $c(y_r) \supseteq c(x_{r+1} y_{r+1})$ and we get a contradiction with Assertion (ii) of Lemma 4.11. So $r = l-1$. Now, x_l' must be empty, in view of (5) and Assertion (i) of Lemma 4.11. Set $\tilde{y}_j = y_j$ for $j = 1, \ldots, k-1$, $j \neq r$ and $\tilde{y}_r = y_r \cdot a$. Now, $a \in c(s_q) \subseteq c(y_r)$, so $c(\tilde{y}_r) = c(y_r)$. Therefore $x_1 \tilde{y}_1 \cdots x_{k-1} \tilde{y}_{k-1} x_k$ is an m-normal factorization of $\tilde{z}$ by Assertion (iii) of Lemma 4.11.

2$^{\text{nd}}$ case: There exists $y_l = y_l' y_l''$ such that a is inserted between y_l' and y_l''. If a belongs to $c(y_l)$, the factorization of $\tilde{z}$ is m-normal, in view of Assertion (iii) of Lemma 4.11. We may now assume that a is not in $c(y_l)$.

As in the first case, one may assume that there exists $p > m$ such that

$$x_1 y_1 \cdots y_l' \sim_p x_1 y_1 \cdots y_l' a$$

and Lemma 4.7 gives once again words $s_1, \ldots, s_p$ satisfying (4) and such that

$$x_1 y_1 \cdots y_l' = s_1 \cdots s_p$$

Since $p > m \geq |x_1 \cdots x_k| + k - 1$, an argument similar to that used in the first case shows that there exists q such that s_q is a factor of y_l' or of y_r for a given $r \leq l-1$. If s_q were a factor of y_l', (4) would give $a \in c(s_q) \subseteq c(y_l') \subseteq c(y_l)$, a case we excluded. So we may assume that s_q is a factor of some y_r. Therefore,

$$c(y_r) \supseteq c(s_q) \supseteq c(s_{q+1} \cdots s_p) \supseteq c(x_{r+1} y_{r+1} \cdots y_l') \cup \{a\} \tag{6}$$

By Assertions (i) and (ii) of Lemma 4.11, we have $r = l-1$ and $x_{r+1} = \varepsilon$. Let $y_i = u_i v_i w_i$ be the factorizations of y_i defined in 4.10. By definition, the first letter of v_l does not belong to $c(y_{l-1})$. Now, $c(y_l') \subseteq c(y_{l-1})$ by (6), hence y_l' is a prefix of u_l. Let $u_l = y_l' \tilde{u}_l$, $\tilde{v}_{l-1} = v_{l-1} w_{l-1} y_l' a$ and $\tilde{w}_{l-1} = \varepsilon$. Define $\tilde{y}_i = u_i v_i w_i$ for $i < l-1$ or $i > l$, $\tilde{y}_{l-1} = u_{l-1} v_{l-1} \tilde{w}_{l-1}$, and $\tilde{y}_l = \tilde{u}_l v_l w_l$. Note first that $c(\tilde{y}_i) \subseteq c(y_i)$. Since (1) is m-normal, we also have $c(y_i) = c(v_i)$, which gives $c(\tilde{y}_i) = c(y_i)$, so the product $x_1 \tilde{y}_1 \cdots x_{k-1} \tilde{y}_{k-1} x_k$ satisfies conditions n.1) and n.2) of Definition 4.10.

Let us see that n.3) also holds. We have $c(v_{l-1} w_{l-1}) \subseteq c(v_{l-1}) = c(y_{l-1})$ since (1) is m-normal. Furthermore, we have by (6) $a \in c(y_l') \subseteq c(y_{l-1}) =$

$c(v_{l-1})$. Hence $c(\tilde{v}_{l-1}) = c(v_{l-1}w_{l-1}y'_l a) \subseteq c(v_{l-1})$. Since v_{l-1} is a product of m words of content $c(y_{l-1})$ and is a prefix of $\tilde{v}_{l-1}$, $\tilde{v}_{l-1}$ is also such a product. Next, $\tilde{w}_{l-1} = \varepsilon$ is the largest suffix of $\tilde{y}_{l-1}$ with content contained in $c(y_l)$, because the last letter of $\tilde{y}_{l-1}$ is a which does not belong to $c(y_l)$. For the same reason, the largest prefix of $\tilde{y}_l$ with content contained in $c(y_{l-1})$ is $\tilde{u}_l$. The remaining verifications are straightforward. ■

Lemma 4.12 has a dual version: instead of inserting, one can delete a letter. We state this result without giving the proof, which is very similar to the previous one.

LEMMA 4.13: *Let z in A^*. Assume that z has a factorization*

$$z = x_1 y_1 \cdots x_{k-1} y_{k-1} x_k$$

Let $m \geq |x_1 \cdots x_k| + k - 1$. Assume that the factorization (1) is m-normal. Let $\ell > 2m$, let $z = z'az''$ and let $a \in A$ such that $\tilde{z} = z'z'' \sim_\ell z$. Then $\tilde{z}$ has an m-normal factorization $x_1\tilde{y}_1 \cdots x_{k-1}\tilde{y}_{k-1}x_k$ where $c(\tilde{y}_i) = c(y_i)$. ■

We are now able to state the result announced in the first step of the proof outline of the inclusion $\bar{L} \subseteq K$ (page 10).

LEMMA 4.14: *Let $x_1 B_1^\omega \cdots x_{k-1} B_{k-1}^\omega x_k$ be the canonical factorization of an implicit operation π on $\overline{\mathrm{F}}_n(\mathbf{J})$ where each x_i is a (possibly empty) explicit operation. Let $m \geq |x_1 \cdots x_k| + k - 1$ and $\ell > 2m$, and let w be a word such that $w \sim_\ell \pi$. Then, w has an m-normal factorization $x_1 y_1 \cdots x_{k-1} y_{k-1} x_k$, such that, for $i = 1, \ldots, k-1$, $c(y_i) = B_i$.*

Proof: Let t_i be an arbitrary word of content B_i and let t be the word

$$t = x_1 t_1^\ell \cdots x_{k-1} t_{k-1}^\ell x_k \tag{7}$$

Notice that $\pi \sim_\ell t$, so $\pi \sim_\ell w$ implies $t \sim_\ell w$. From Lemma 4.9, we know that, since $t \sim_\ell w$, there exists $z \in A^*$ such that $z \; R_\ell^* \; t$ and $z \; R_\ell^* \; w$. By definition of R_ℓ^*, this means that there exist words $z_0, \ldots, z_i$ and $z'_0, \ldots, z'_{i'}$ such that

$$t = z_0 \sim_\ell z_1 \sim_\ell \cdots \sim_\ell z_i = z \tag{8}$$

$$z = z'_0 \sim_\ell z'_1 \sim_\ell \cdots \sim_\ell z'_{i'} = w \tag{9}$$

and such that z_{l+1} is obtained from z_l by inserting a letter, and z'_{l+1} is obtained from z'_l by deleting a letter. We first consider (8). Since the factorization of π is the canonical one, we start from an m-normal factorization (7) of t. Apply i times Lemma 4.12, successively between z_j and z_{j+1}, for $j = 0, 1, \ldots, i-1$: each z_j has an m-normal factorization of the form $x_1 y_{j,1} \cdots x_{k-1} y_{j,k-1} x_k$ with $c(y_{j,l}) = c(t_l)$. This holds in particular for $z_i = z$. Now, we use (9). Apply i' times Lemma 4.13, successively between z'_j and z'_{j+1}, for $j = 0, 1, \ldots, i'-1$: each z'_j has an m-normal factorization of the form $x_1 y'_{j,1} \cdots x_{k-1} y'_{j,k-1} x_k$ with $c(y'_{j,l}) = c(y'_{0,l}) = c(y_{i,l}) = c(t_l) = B_l$. This holds in particular for $w = z'_{i'}$. ■

LEMMA 4.15: *Let $u = x_1 y_1 \cdots x_{k-1} y_{k-1} x_k$ be an m-normal product labeling a path $\mathcal{P}$ in an automaton $\mathcal{A}$. Denote by Q the set of states of $\mathcal{A}$, and assume that $m > |Q|$. Then, each y_i admits a factorization $r_i s_i t_i$ with $c(s_i) = c(y_i)$, and such that the subpath of $\mathcal{P}$ labeled s_i in $\mathcal{A}$ is a loop.*

Proof: Definition 4.10 implies that each y_i is a product of m factors of content $c(y_i)$. Thus we may set

$$y_i = z_{i,1} \cdots z_{i,m}, \qquad c(z_{i,j}) = c(y_i)$$

Let q_{ini} be the initial state of $\mathcal{A}$. Fix an index $i \leq k-1$ and set

$$q_i = q_{ini} \cdot (x_1 y_1 \cdots x_i)$$

and

$$q_{i,j} = q_i \cdot (z_{i,1} \cdots z_{i,j}), \qquad j = 1, \ldots, m$$

Since $m > |Q|$, there exist j, k such that $j < k$ and $q_{i,j} = q_{i,k}$. The path between $q_{i,j}$ and $q_{i,k}$ is labeled $z_i = z_{i,j+1} \cdots z_{i,k}$, which has content $c(y_i)$. In other terms, there exists a loop in $\mathcal{A}$ labeled by a word of content $c(y_i)$ between states $q_{i,j}$ and $q_{i,k}$, as depicted in Figure 3.

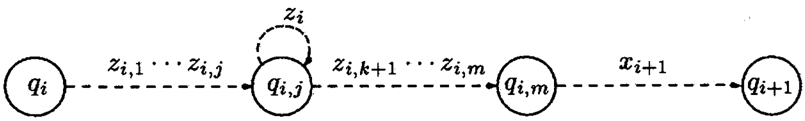


Figure 3. – The path between q_i and q_{i+1}

The result is obtained by choosing $r_i = z_{i,1} \cdots z_{i,j}$, $s_i = z_i$ and $t_i = z_{i,k+1} \cdots z_{i,m}$. ■

In order to prove the inclusion $\bar{L} \subseteq K$, take a canonical factorization $x_1 B_1^\omega \cdots x_{k-1} B_{k-1}^\omega x_k$ of $\pi \in \bar{L}$, and let $(u_i)_{i \in \mathbb{N}}$ be a sequence of words of

L converging to π. Fix m such that $m > |Q|$ and $m \geq |x_1 \cdots x_k| + k - 1$, and let $\ell > 2m$. By Corollary 2.2, there exists a member u of the sequence $(u_i)_{i \in \mathbb{N}}$ such that $\pi \sim_\ell u$. Thus, by Lemma 4.14, u has an m-normal factorization of the form

$$u = x_1 y_1 \cdots x_{k-1} y_{k-1} x_k, \qquad c(y_i) = B_i \tag{10}$$

We now apply Lemma 4.15. Let $\mathcal{P}$ be the successful path labeled u in $\mathcal{A}$. Each y_i has a factorization $r_i s_i t_i$, where $c(s_i) = B_i$, such that the subpath of $\mathcal{P}$ labeled s_i is a loop, say around state q_i. Therefore, there exists a state $q_{i\,B_i}$ in $\bar{\mathcal{A}}$ and transitions $q_i \xrightarrow{B_i^\omega} q_{i\,B_i} \xrightarrow{\varepsilon} q_i$ built at Step c.2) of Definition 4.1. The path in $\bar{\mathcal{A}}$ obtained from $\mathcal{P}$ by replacing each loop $q_i \dashrightarrow^{s_i} q_i$ by the transitions $q_i \xrightarrow{B_i^\omega} q_{i\,B_i} \xrightarrow{\varepsilon} q_i$ is successful. It is labeled $x_1(r_1 B_1^\omega t_1) \cdots x_{k-1}(r_{k-1} B_{k-1}^\omega t_{k-1}) x_k$. Now, $c(r_i t_i) \subseteq B_i$, whence $r_i B_i^\omega t_i = B_i^\omega$ in view of the reduction rules giving the canonical form of an implicit operation on $\mathbf{J}$. So $x_1 B_1^\omega \cdots x_{k-1} B_{k-1}^\omega x_k$ is recognized by $\bar{\mathcal{A}}$, that is, π belongs to K.

4.2. The algorithm

In this section, we prove the main result of the paper.

THEOREM 4.16: *The pseudovariety* **J** *is hyperdecidable.*

Proof: We will need in the sequel two closure properties: given two automata recognizing two languages L_1 and L_2 of $\overline{\mathrm{F}}_n(\mathrm{J})$, Lemma 4.22 shows that one can construct an automaton recognizing $L_1 \cap L_2$, and Corollary 4.24 gives the construction of an automaton recognizing $L_1^{-1} L_2$. The proofs of these results are based on their counterparts for standard automata: they require an element of $L_1 \cap L_2$ to admit a factorization labeling a successful path in each of the automata recognizing L_1 and L_2. With our automata, a natural candidate for this common factorization is obtained from the canonical form by splitting each explicit operation in a product of letters.

The definition of $\bar{\mathcal{A}}$ was suitable for proving that it recognizes $\bar{L}$. It was convenient for this proof to have only one transition starting from state q_B, so that a given factorization of an implicit operation could label at most one path. But this has also a major drawback: it may happen that π is recognized by $\bar{\mathcal{A}}$, yet the product obtained from its canonical factorization by splitting each explicit operation in a product of letters is compatible with no pair (q_{ini}, q_{fin}) of $\bar{\mathcal{A}}$. For instance, the automaton $\bar{\mathcal{A}}$ of Figure 2 recognizes

$b \cdot (ab)^\omega \cdot \varepsilon \cdot b \cdot b = (ab)^\omega$, but there is no path labeled by ε and $(ab)^\omega$ going from q_1 to q_4.

To get a nicer situation, we will slightly modify $\bar{\mathcal{A}}$ without changing the language it recognizes. Let $\breve{\mathcal{A}}$ be the automaton obtained from $\bar{\mathcal{A}}$ by adding the following steps to its construction:

c.4) For each state q_B and each kernel state q' of $\bar{\mathcal{A}}$, add an arrow $q_B \longrightarrow q'$ labeled ε if there is a path in $\mathcal{A}$ from q to q' whose label has content contained in B.

c.5) For each state q_B and each kernel state q' of $\bar{\mathcal{A}}$, add an arrow $q' \longrightarrow q_B$ labeled B^ω if there is a path in $\mathcal{A}$ from q' to q whose label has content contained in B.

Example 4.17: We go back to Example 4.2. Step c.4) adds some new ε-transitions, as shown in Figure 4. Step c.5) adds some other transitions labeled B^ω. For instance, there is a transition from q_1 to q_2 labeled b in Figure 2, and b belongs to $\{a, b\}$, so we must add a transition from q_1 to $q_{2\,\{a,b\}}$ labeled $(ab)^\omega$ in $\bar{\mathcal{A}}$. We finally get the automaton of Figure 5. One can check in this example that $\breve{\mathcal{A}}$ recognizes the same language as $\bar{\mathcal{A}}$. Moreover, in $\breve{\mathcal{A}}$, $(ab)^\omega$ is compatible with (q_1,q_4).

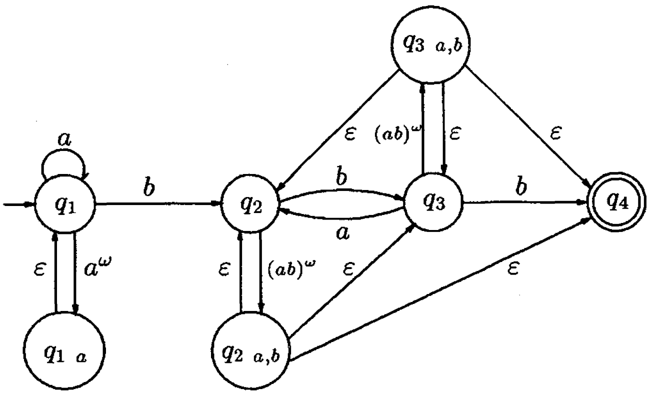


Figure 4. – After Step c.4)

These two requirements expected for $\breve{\mathcal{A}}$ are stated in lemmas 4.18 and 4.20.

Lemma 4.18: *The automata $\breve{\mathcal{A}}$ and $\bar{\mathcal{A}}$ recognize the same language.*

Proof: It is clear that every implicit operation recognized by $\bar{\mathcal{A}}$ is also recognized by $\breve{\mathcal{A}}$. We claim that the converse also holds. We will prove

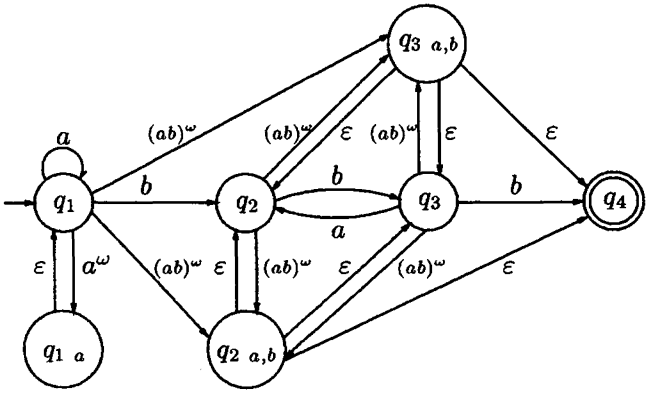


Figure 5. – The automaton $\breve{\mathcal{A}}$

for instance that each implicit operation recognized by the automaton $\mathcal{A}'$ obtained after Step c.4) is also recognized by $\mathcal{A}$. The proof that each operation recognized by $\breve{\mathcal{A}}$ is also recognized by $\mathcal{A}'$ would be similar.

Let $q_B \xrightarrow{\varepsilon} q'$ be a transition of $\mathcal{A}'$ added by Step c.4). By construction, q_B is linked to the kernel state q in $\bar{\mathcal{A}}$. If a successful path in $\mathcal{A}'$ uses the new transition $q_B \xrightarrow{\varepsilon} q'$, then, there is a state p preceding q_B in this path, because q_B cannot be initial. Necessarily, $p = q$ (see Fact 4.4), and there is only one transition from q to q_B, labeled B^ω, which occurs also in $\bar{\mathcal{A}}$. We thus have a subpath of our successful path in $\mathcal{A}'$:

$$q \xrightarrow{B^\omega} q_B \xrightarrow{\varepsilon} q'$$

What we have to show is that there exists a path in $\bar{\mathcal{A}}$ labeled B^ω between states q and q'. If $q = q'$, then there is nothing to do. Otherwise, the transition $q_B \xrightarrow{\varepsilon} q'$ comes from c.4), so there is a path $q \xrightarrow{u} q'$ in $\mathcal{A}$ with $c(u) \subseteq B$. Therefore, the path

$$q \xrightarrow{B^\omega} q_B \xrightarrow{\varepsilon} q \dashrightarrow{u} q'$$

in $\bar{\mathcal{A}}$ is labeled $B^\omega u = B^\omega$, as required. ■

DEFINITION 4.19: Say that an automaton $\mathcal{A}$ satisfies $\mathcal{P}_{\mathcal{J}}$ if the following holds: for each implicit operation π labeling a path from p to q in $\mathcal{A}$, the factorization obtained from the canonical form of π by splitting each explicit operation in a product of letters is compatible with the pair (p, q). ■

LEMMA 4.20: *The automaton $\breve{\mathcal{A}}$ satisfies $\mathcal{P}_{\mathcal{J}}$.*

Proof: Let $\pi_1 \cdots \pi_l$ be a factorization of π compatible with (p, q) in $\breve{\mathcal{A}}$. We show by induction on l that the canonical form of π is also compatible with (p, q). If $l = 1$, then $\pi_1 \cdots \pi_l$ consists either in a single letter or an idempotent, hence it is canonical and there is nothing to do.

Assume now that the factorization $\pi_1 \cdots \pi_l$ is not canonical. The canonical form may be obtained from $\pi_1 \cdots \pi_l$ by applying repeatedly Rule rr.3) stated in Section 2.3. Thus, we have in $\pi_1 \cdots \pi_l$ two adjacent factors π_i and π_{i+1}, one of them being idempotent and containing each letter of the other one. We are led to Cases 1) to 4) below. In each of them, we shall use the following observation, similar to 4.4.

FACT 4.21: Let $p \xrightarrow{\pi} q$ be a transition of $\breve{\mathcal{A}}$. Then

a. $\pi = B^\omega$ if and only if p is a kernel state. In this case, q is of the form r_B.

b. π is a letter if and only if both p and q are kernel states.

c. $\pi = \varepsilon$ if and only if p is not a kernel state. In this case, q is a kernel state. ■

1) $\pi_i = a$ and $\pi_{i+1} = B^\omega$ with $a \in B$.

In view of Fact 4.21, we have in $\breve{\mathcal{A}}$ a path $p \xrightarrow{a} q \xrightarrow{B^\omega} r_B$ where p and q are kernel states. If $q \neq r$, then the transition $q \xrightarrow{B^\omega} r_B$ comes from Step c.5), so there is in $\mathcal{A}$ a path $q \overset{u}{\dashrightarrow} r$ with $c(u) \subseteq B$. Hence $p \overset{au}{\dashrightarrow} r$ is a path in $\mathcal{A}$, which also exists when $q = r$ (take $u = \varepsilon$). Since $c(au) \subseteq B$, there is a transition $p \xrightarrow{B^\omega} r_B$ in $\breve{\mathcal{A}}$, added in Step c.5), and we can use it instead of the original transitions $p \xrightarrow{a} q \xrightarrow{B^\omega} r_B$.

2) $\pi_i = C^\omega$ and $\pi_{i+1} = B^\omega$ with $C \subseteq B$.

Again by 4.21, we have in $\breve{\mathcal{A}}$ the path $p \xrightarrow{C^\omega} q_C \xrightarrow{\varepsilon} r \xrightarrow{B^\omega} s_B$. By construction of $\breve{\mathcal{A}}$, there exist in $\mathcal{A}$ two paths $p \overset{u}{\dashrightarrow} q$ and $q \overset{v}{\dashrightarrow} r$ with $c(u)$, $c(v) \subseteq C$. Therefore, $c(uv) \subseteq B$. Considering the path $p \overset{uv}{\dashrightarrow} r \xrightarrow{B^\omega} s_B$ and using Case 1) $|uv|$ times, we obtain in $\breve{\mathcal{A}}$ a transition $p \xrightarrow{B^\omega} s_B$, that can be used instead of $p \xrightarrow{C^\omega} q_C \xrightarrow{\varepsilon} r \xrightarrow{B^\omega} s_B$.

In Cases 1) and 2), the factorization obtained from $\pi_1 \cdots \pi_l$ by removing π_i has $l - 1$ factors and is compatible with (p, q).

3) $\pi_i = B^\omega$ and $\pi_{i+1} = a$ with $a \in B$.

By Fact 4.21, we have in $\breve{\mathcal{A}}$ a path $p \xrightarrow{B^\omega} q_B \xrightarrow{\varepsilon} r \xrightarrow{a} s$. If $q \neq r$, the transition $q_B \xrightarrow{\varepsilon} r$ comes from Step c.4), so there is a path $q \overset{u}{\dashrightarrow} r$ with $c(u) \subseteq B$. Therefore, we obtain the path $q \overset{ua}{\dashrightarrow} s$ with $c(ua) \subseteq B$, which also exists when $q = r$ (take $u = \varepsilon$). By Step c.4), there is in $\check{\mathcal{A}}$ an ε-transition from q_B to s. Therefore, we can replace the path $p \xrightarrow{B^\omega} q_B \xrightarrow{\varepsilon} r \xrightarrow{a} s$ by $p \xrightarrow{B^\omega} q_B \xrightarrow{\varepsilon} s$.

4) $\pi_i = B^\omega$ and $\pi_{i+1} = C^\omega$ with $C \subseteq B$.

Fact 4.21 shows that we have in $\check{\mathcal{A}}$ a path $p \xrightarrow{B^\omega} q_B \xrightarrow{\varepsilon} r \xrightarrow{C^\omega} s_C \xrightarrow{\varepsilon} t$. As in Case 2), there are in $\mathcal{A}$ paths $q \overset{u}{\dashrightarrow} r$, $r \overset{v}{\dashrightarrow} s$ and $s \overset{w}{\dashrightarrow} t$ with $c(u) \subseteq B$ and $c(v), c(w) \subseteq C \subseteq B$, yielding the path $p \xrightarrow{B^\omega} q_B \xrightarrow{\varepsilon} q \overset{uvw}{\dashrightarrow} t$. Using Case 3) $|uvw|$ times, we obtain in $\check{\mathcal{A}}$ the transitions $p \xrightarrow{B^\omega} q_B \xrightarrow{\varepsilon} t$.

In Cases 3) and 4) the factorization obtained from $\pi_1 \cdots \pi_l$ by removing π_{i+1} has $l-1$ factors and is compatible with (p, q). This completes the induction. ■

We now build from $\check{\mathcal{A}}$ the ε-free automaton recognizing the same language with the classical algorithm: for each letter or idempotent operation π, we denote by $\delta(q, \pi)$ the set of states q' of $\check{\mathcal{A}}$ such that π is compatible with (q, q'). This set can be easily computed. The ε-free automaton has the same set of states as $\check{\mathcal{A}}$, the same initial and final states, and its transitions are defined by $q \cdot \pi = \delta(q, \pi)$. Denote by $\hat{\mathcal{A}}$ the automaton obtained by determinizing this ε-free automaton (viewing idempotent operations as new letters). Then $\hat{\mathcal{A}}$ is a deterministic ε-free automaton that recognizes the same language as $\check{\mathcal{A}}$. (See [9] for these proofs.) More precisely, the statements obtained by replacing $\check{\mathcal{A}}$ by $\hat{\mathcal{A}}$ in Lemmas 4.18 and 4.20 hold. We will now deal with such automata, that are convenient for obtaining closure properties.

LEMMA 4.22: *Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be automata recognizing L_1, $L_2 \subseteq \overline{\mathrm{F}}_n(\mathbf{J})$ respectively. Assume that $\mathcal{A}_1$ and $\mathcal{A}_2$ satisfy $\mathcal{P}_{\mathcal{J}}$. Then one can construct an automaton $\mathcal{A}_{L_1 \cap L_2}$ satisfying $\mathcal{P}_{\mathcal{J}}$ that recognizes $L_1 \cap L_2$.*

Proof: It suffices to check that the usual construction giving the automaton $\mathcal{A}_{L_1 \cap L_2}$ that recognizes the intersection of two rational languages works: if Q_1 (resp. Q_2) is the set of states of $\mathcal{A}_1$ (resp. of $\mathcal{A}_2$), then the set of states of $\mathcal{A}_{L_1 \cap L_2}$ is $Q_1 \times Q_2$. If F_1 (resp. F_2) denotes the set of final states of $\mathcal{A}_1$ (resp. of $\mathcal{A}_2$), then the set of final states of $\mathcal{A}_{L_1 \cap L_2}$ is $F_1 \times F_2$. Finally, $(p_1, p_2) \xrightarrow{\pi} (q_1, q_2)$ is a transition of $\mathcal{A}_{L_1 \cap L_2}$ if $p_1 \xrightarrow{\pi} q_1$ is a transition of $\mathcal{A}_1$ and $p_2 \xrightarrow{\pi} q_2$ is a transition of $\mathcal{A}_2$. The proof that $\mathcal{A}_{L_1 \cap L_2}$ recognizes $L_1 \cap L_2$ is then classical. It is based on the fact that if π belongs to $L_1 \cap L_2$, then there is a factorization that labels a successful path in both $\mathcal{A}_1$ and $\mathcal{A}_2$. In the present case, this property directly comes from $\mathcal{P}_{\mathcal{J}}$, which is satisfied by $\mathcal{A}_1$ as well as by $\mathcal{A}_2$. Let us verify that $\mathcal{A}_{L_1 \cap L_2}$ also satisfies $\mathcal{P}_{\mathcal{J}}$: assume that π is compatible with a pair $((p_1, p_2), (q_1, q_2))$ of states of $\mathcal{A}_{L_1 \cap L_2}$. By construction, π is compatible with (p_1, q_1) in $\mathcal{A}_1$ and with (p_2, q_2) in $\mathcal{A}_2$. By $\mathcal{P}_{\mathcal{J}}$, the factorization obtained from the canonical form of π is compatible with (p_1, q_1) in $\mathcal{A}_1$ and with (p_2, q_2) in $\mathcal{A}_2$, hence it is compatible with $((p_1, p_2), (q_1, q_2))$ in $\mathcal{A}_{L_1 \cap L_2}$. ■

The generalization to a finite number of languages is straightforward.

COROLLARY 4.23: *Let $\mathcal{A}_1, \ldots, \mathcal{A}_p$ be automata that recognize $L_1, \ldots, L_p \subseteq \overline{\mathrm{F}}_n(\mathbf{J})$ respectively, such that each $\mathcal{A}_i$ satisfies $\mathcal{P}_{\mathcal{J}}$. Then one can construct an automaton satisfying $\mathcal{P}_{\mathcal{J}}$ that recognizes $\bigcap_{i=1}^{p} L_i$.* ■

Now that we can compute intersections, it is possible to compute left quotients as well.

COROLLARY 4.24: *Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be automata recognizing L_1, $L_2 \subseteq \overline{\mathrm{F}}_n(\mathbf{J})$. Assume that $\mathcal{A}_1$ and $\mathcal{A}_2$ satisfy $\mathcal{P}_{\mathcal{J}}$. Then one can construct an automaton $\mathcal{A}_{L_1^{-1}L_2}$ that recognizes $L_1^{-1}L_2$.*

Proof: Once again, we just check that the classical construction works. The automaton $\mathcal{A}_{L_1^{-1}L_2}$ recognizing $L_1^{-1}L_2$ is obtained from $\mathcal{A}_2$ by replacing the initial state q_{ini} of $\mathcal{A}_2$ by a new set of initial states. All that remains to prove is that we can determine these states.

For a given state q of $\mathcal{A}_2$, denote by $\mathcal{A}_{2,q}$ the automaton obtained from $\mathcal{A}_2$ by taking $\{q\}$ for set of final states, and by $L_{2,q}$ the language recognized by $\mathcal{A}_{2,q}$. A state q is initial in $\mathcal{A}_{L_1^{-1}L_2}$ when there exists a path from q_{ini} to q labeled by a word of L_1, that is, if $L_1 \cap L_{2,q}$ is not empty. Now, the property $\mathcal{P}_{\mathcal{J}}$ does not depend on final states; since $\mathcal{A}_2$ satisfies $\mathcal{P}_{\mathcal{J}}$, so does $\mathcal{A}_{2,q}$. Therefore, by Lemma 4.22, we can compute $L_1 \cap L_{2,q}$. Since emptiness of a given rational language can be decided, we can determine the set of initial states of $\mathcal{A}_{L_1^{-1}L_2}$. ■

Let us now conclude the proof of Theorem 4.16. Consider the problem stated in Section 3: let $S = \{s_1, \ldots, s_n\}$ be a finite semigroup, let $A = \{a_1, \ldots, a_n\}$ be an alphabet, and let φ be the canonical morphism from A^+ onto S defined by $\varphi(a_i) = s_i$. Finally, fix a finite graph Γ, whose set of vertices is $X = \{x_1, \ldots, x_k\}$ and whose set of edges is $Y = \{y_1, \ldots, y_l\}$, and fix elements $u_1, \ldots, u_k, v_1, \ldots, v_l$ of S.

Assume first that Γ is strongly connected and has at least one edge. We have for each i, i' a path $x_i \overset{y}{\dashrightarrow} x_{i'} \overset{y'}{\dashrightarrow} x_i$ where y, y' belong to Y^+. Therefore, there exists $\rho, \rho' \in \{\rho_1, \ldots, \rho_l\}^+$ such that $\pi_i \rho = \pi_{i'}$ and $\pi_{i'} \rho' = \pi_i$, hence $\pi_i \rho \rho' = \pi_i$. The reduction rules giving the construction of the canonical form (Section 2.3) show that the canonical factorization of π_i is of the form πB^ω and that $c(\rho\rho') \subseteq B$. Therefore, $\pi_{i'} = \pi_i \rho = \pi_i$. Furthermore, each y_j labels an edge in Γ, so that $\pi B^\omega \rho_j = \pi B^\omega$ and $c(\rho_j) \subseteq B$. Thus, a necessary condition for having a positive answer to the problem is: there exists a nonempty subset B of A and an implicit operation π in $\overline{\mathrm{F}}_n(\mathbf{J})$ such that for each $i = 1, \dots, k$, $\overline{\varphi^{-1}(u_i)}$ contains πB^ω and for each $j = 1, \dots, l$, $\overline{\varphi^{-1}(v_j)}$ contains an element of content B. Now, this condition is obviously sufficient.

As remarked at the end of Section 2.2, the content morphism is continuous on $\mathbf{J}$ since $\mathbf{Sl}$ is a subpseudovariety of $\mathbf{J}$. Hence, there exists ρ_j in $\overline{\varphi^{-1}(v_j)}$ such that $c(\rho_j) \subseteq B$ if and only if $\varphi^{-1}(v_j)$ includes a word of content contained in B. The necessary and sufficient condition for having a positive answer can therefore be formulated as follows: there exists a nonempty subset B of A such that

$$K_B = \overline{\mathrm{F}}_n(\mathbf{J})B^\omega \cap \bigcap_{i=1}^{k} \overline{\varphi^{-1}(u_i)} \text{ is not empty} \qquad (\mathcal{C}_1)$$

and

$$L_{j,B} = \{w_j \in \varphi^{-1}(v_j) \mid c(w_j) \subseteq B\} \text{ is not empty } (1 \leq j \leq l) \qquad (\mathcal{C}_2)$$

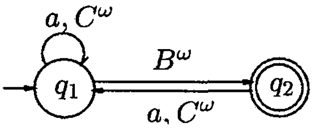
We say that K_B and $L_{j,B}$ are the *B-test languages* for Γ (the remaining data are understood). We have to show that we can decide whether there exists a nonempty $B \subseteq A$ such that the B-test languages are not empty. The nonempty sets $B \subseteq A$ such that $(\mathcal{C}_2)$ is satisfied can be easily determined. Indeed, for a rational language L, denote by $\mathrm{C}(L)$ the set of all possible contents of a word of L, that is, $\mathrm{C}(L) = \{c(w) \mid w \in L\}$. One can compute this set for any rational language, using the following rules:

$$\begin{aligned} \mathrm{C}(\{a\}) &= \{\{a\}\} \\ \mathrm{C}(L \cup L') &= \mathrm{C}(L) \cup \mathrm{C}(L') \\ \mathrm{C}(L \cdot L') &= \{c \cup c' \mid c \in \mathrm{C}(L) \text{ and } c' \in \mathrm{C}(L')\} \\ \mathrm{C}(L^+) &= \{c_1 \cup \dots \cup c_n \mid n \geq 1,\ c_1, \dots, c_n \in C(L)\} \end{aligned}$$

Now, $(\mathcal{C}_2)$ is satisfied if and only if B belongs to $\bigcap_{j=1}^{l} \mathrm{C}(\varphi^{-1}(v_j))$.

For each such B, one can then compute K_B, and thus check $(\mathcal{C}_1)$. Indeed,

1) We have an automaton recognizing $\overline{\mathrm{F}}_n(\mathbf{J})B^\omega$, given in Figure 6.



where a runs in A and C in $2^A \setminus \{B\}$

Figure 6. – An automaton recognizing $\overline{\mathrm{F}}_n(\mathbf{J})B^\omega$

Note that this automaton satisfies $\mathcal{P}_{\mathcal{J}}$.

2) We know by Proposition 4.3 an automaton $\mathcal{A}_i$ recognizing $\overline{\varphi^{-1}(u_i)}$, and the construction of $\hat{\mathcal{A}}$ gives an automaton recognizing the same language and satisfying $\mathcal{P}_{\mathcal{J}}$.

Thus, one can construct an automaton recognizing K_B by Corollary 4.23, and test whether one of the K_B's is empty or not. This concludes the proof when Γ is strongly connected and not trivial.

Let now Γ be any finite graph. The previous considerations may be applied in each strongly connected component $\Gamma^{(p)}$ of Γ to obtain necessary conditions. For the strongly connected component $\Gamma^{(p)}$, denote by $K_B^{(p)}$ and $L_{j,B}^{(p)}$ its B-test languages. We already know that there is a finite number of computable subsets B of A such that the corresponding $K_B^{(p)}$ and $L_{j,B}^{(p)}$ are nonempty. Moreover, for such a subset, we can compute $K_B^{(p)}$. Since they are finite in number, we can therefore fix in each strongly connected component $\Gamma^{(p)}$ one language of the form $K_{B_p}^{(p)}$, and try to find out whether this choice yields a positive answer for the problem.

Each edge y_r from $x_q \in \Gamma^{(q)}$ to $x_p \in \Gamma^{(p)}$ just requires the existence of $\rho_r \in \overline{\varphi^{-1}(v_r)}$ such that, $\pi_q \rho_r = \pi_p$. In other terms, we have to check whether $(K_{B_q}^{(q)})^{-1} K_{B_p}^{(p)}$ is not empty. Now, we can compute each of these left quotients by Corollary 4.24. Hence we can decide whether the problem has a solution.

The result is proved when each strongly connected component of Γ has at least one edge. To conclude, we establish a reduction to such a graph which works for any monoidal pseudovariety. Let $\mathcal{S} = (S, A, \varphi, \Gamma, (t_1, \ldots, t_k), (u_1, \ldots, u_l))$ be the data of the problem given in Section 3. We build new data $\dot{\mathcal{S}} = (\dot{S}, \dot{A}, \dot{\varphi}, \dot{\Gamma}, (t_1, \ldots, t_k), (u_1, \ldots, u_l, v_1, \ldots, v_k))$, such that each strongly connected component of $\dot{\Gamma}$ has at least one edge, and such that the problem has the same answer on both data.

Let $\dot{S} = S \uplus \{\mathbf{1}\}$, where $\mathbf{1}$ is a new neutral element. Let $\dot{A} = A \uplus \{a\}$ where $a \notin A$, and let $\dot{\varphi} : \dot{A}^+ \to \dot{S}$ be defined by $\dot{\varphi}|_A = \varphi|_A$ and $\dot{\varphi}(a) = \mathbf{1}$.

The graph $\dot{\Gamma}$ is obtained from Γ by adding a loop at each vertex, and $v_1 = v_2 = \cdots = v_k = \mathbf{1}$ are the elements of $\dot{S}$ associated with these loops.

Define two continuous homomorphisms $\chi : \bar{F}_A(\mathbf{J}) \to \bar{F}_{\dot{A}}(\mathbf{J})$ and $\xi : \bar{F}_{\dot{A}}(\mathbf{J}) \to (\bar{F}_A(\mathbf{J}))^1$ by $\chi(b) = a^\omega b a^\omega (b \in A)$, and $\xi(b) = b (b \in A)$, $\xi(a) = 1$. Note that the existence of ξ requires that $\mathbf{J}$ be monoidal.

Suppose first that the problem has a solution $(\dot{\pi}_1, \dots, \dot{\pi}_k)$, $(\dot{\rho}_1, \dots, \dot{\rho}_l, \dot{\sigma}_1, \dots, \dot{\sigma}_k)$ on $\dot{S}$. Then it is easy to verify that $(\xi(\dot{\pi}_1), \dots, \xi(\dot{\pi}_k))$, $(\xi(\dot{\rho}_1), \dots, \xi(\dot{\rho}_l))$ is a solution for S. Conversely, if the problem has a solution $(\pi_1, \dots, \pi_k)$, $(\rho_1, \dots, \rho_l)$ on S, one checks that $(\chi(\pi_1, \dots, \chi(\pi_k))$, $(\chi(\rho_1), \dots, \chi(\rho_l), a^\omega, \dots, a^\omega)$ is a solution for $\dot{S}$. This concludes the proof of Theorem 4.16. ■

5. CONSEQUENCES

This section states briefly some applications of Theorem 4.16. We first recall some basic definitions. We say that a pseudovariety $\mathbf{V}$ is *order-computable* if $\mathrm{F}_n(\mathbf{V})$ is finite and there is an algorithm to compute this semigroup for each integer n. The join $\mathbf{V} \vee \mathbf{W}$ of two pseudovarieties $\mathbf{V}$ and $\mathbf{W}$ is the smallest pseudovariety containing both $\mathbf{V}$ and $\mathbf{W}$. The semidirect product $\mathbf{V} * \mathbf{W}$ is the smallest pseudovariety containing all semidirect products $S * T$ with $S \in \mathbf{V}$ and $T \in \mathbf{W}$.

The first proposition follows from the general results stated in [4].

PROPOSITION 5.1: *Let $\mathbf{V}$ be an order-computable pseudovariety. Then the join $\mathbf{J} \vee \mathbf{V}$ is hyperdecidable.* ■

For instance, the pseudovariety $\mathbf{B}$ of finite bands is order-computable. Therefore, Proposition 5.1 implies that $\mathbf{J} \vee \mathbf{B}$ is decidable, a result proved by hand in [13].

Silva and the first author [6] proved a theorem to state a similar result involving semidirect products.

PROPOSITION 5.2: *Let $\mathbf{V}$ be an order-computable pseudovariety. Then the semidirect product $\mathbf{J} * \mathbf{V}$ is hyperdecidable.* ■

The last application answers a question proposed by Rhodes [11] concerning the decidability of $\mathbf{J} \vee \mathbf{G}$, where $\mathbf{G}$ denotes the pseudovariety of finite groups. Its proof needs further developments and will be established in a forthcomming paper of Azevedo and the authors. Recall that a semigroup is completely regular if all its $\mathcal{H}$-classes are groups.

THEOREM 5.3: *Let* $\mathbf{V}$ *be a pseudovariety of completely regular semigroups. If* $\mathbf{V}$ *is strongly decidable, then* $\mathbf{J} \vee \mathbf{V}$ *is decidable. In particular* $\mathbf{J} \vee \mathbf{G}$ *is decidable.* ■

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