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SCHÜTZENBERGER-LIKE PRODUCTS
IN NON-FREE MONOIDS (*)

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Abstract. – *The Schützenberger product has been introduced for studying languages that are subsets of a free monoid. However, is often retains its useful properties when the underlying monoid is not free. The paper investigates when this is the case. It uses for this purpose an alternative proof for the Schützenberger product, based on properties of a certain partition of a monoid. The same approach is then used for a similar investigation of the diamond product for traces.*

Keywords: *Schützenberger product, Straubing product, diamond product, equidivisible monoids, recognizable languages, traces, partially commutative monoids.*

Résumé. – *Le produit de Schützenberger a été introduit pour étudier les langages qui sont parties d'un monoïde libre. Cependant, certaines propriétés de ce produit s'étendent à d'autres cas que le monoïde libre. Cet article étudie ces extensions. On propose d'abord une autre preuve du résultat standard sur le produit de Schützenberger, basé sur les propriétés d'une certaine partition du monoïde. La même approche est utilisée pour étudier le produit diamant sur les traces.*

1. INTRODUCTION

When studying the recognizable subsets of a monoid M, one often faces this problem: given monoids that recognize subsets $L_1 \subseteq M$ and $L_2 \subseteq M$, construct a monoid that recognizes the product $L_1 L_2$. In [14] [1], Schützenberger presented a solution now known as the "Schützenberger product". In [17], Straubing generalized the Schützenberger's solution to $n \geq 2$ subsets $L_1, L_2, ..., L_n$. The Schützenberger product and its generalized form have many applications in the theory of formal languages. A survey of these applications can be found in [16] and [18].

In all applications of the Schützenberger product, M is a free monoid. This paper is an attempt to investigate what happens when M is not free. We use for this purpose an alternative proof of the Schützenberger product recongizing $L_1 L_2 ... L_n$. The proof is inspired by McKnight's "composite refinement" [6], [7]. It uses the observation that a subset being recognized by a finite monoid is equivalent to that subset being saturated by a finite quotient of M. We choose a partition of M that saturates the product $L_1 L_2 ... L_n$ and show that, under certain conditions, this partition is a quotient of M. The Schützenberger product is then obtained as a monoid isomorphic to that quotient.

The proof uses a general property of the minimal partition of a monoid saturating a given family of subsets (Theorem 3). Once this property has been established, the proof becomes quite short. One starts with the Levi's lemma and proves a kind of "Levi's lemma for quotients". The result follows then from the general theorem. This approach may be of some interest in itself, as it seems more transparent than the classical proofs in [14], [17]. It explains the form of the Schützenberger product as the necessary consequence of choosing a specific partition, and seems to offer some insight into the reasons for that choice. This insight may help create new tools working on the same principle.

The proof exhibits clearly the conditions under which the chosen partition is a quotient; these conditions are sufficient for the Schützenberger product to recognize $L_1 L_2 ... L_n$. We show in this way that, in particular, the Schützenberger product recognizes $L_1 L_2 ... L_n$ in any equidivisible monoid M, but also whenever the quotients recognizing $L_1, L_2, ..., L_n$ satisfy a certain weaker condition, called "refinability". This condition is sufficient, but not necessary. It is necessary only if $L_1 L_2 ... L_n$ is to be recognized using a specific homomorphism. This leaves out the results of Reutenauer [13] and Pin [11] about all subsets recognized by the Schützenberger product; these results are about all possible homomorphisms.

Some applicatons of the Schützenberger product use the fact that it recognizes subsets of the form $L_1 u_1 L_2 u_2 ... u_{n-1} L_n$ where $u_1, ..., u_{n-1} \in M$. Our procedure is easily modified to investigate also that case. As a result, we find that the Schützenberger product recognizes these subsets in any equidivisible monoid M, under condition that each of $u_1, ..., u_{n-1}$ is prime. Again, this condition is not necessary, and there is a weaker condition on the quotients recognizing $L_1, L_2, ..., L_n$. This condition is similar to, but different from, refinability. Again, it is necessary, but only for a specific homomorphism.

The approach used for the Schützenberger product is easily adapted to other, similar constructions. We use it to investigate the "diamond product" introduced recently in [4] by Gastin, Petit and Zielonka. The diamond product is the counterpart of the Schützenberger product for partially commutative monoids (the "trace monoids"). We find that the diamond product recognizes the product $L_1 L_2$ in any monoid M satisfying Levi's lemma for traces. Again, we establish a weaker condition on the quotients saturating L_1, L_2.

The product $Z \diamondsuit_n (M_1, ..., M_n)$ introduced by Weil in [19] can probably be treated in the same way, but this has not been attempted.

2. DEFINITIONS AND NOTATION

The reader is assumed to be familiar with the notion of a monoid. We consider a fixed monoid M. Its monoid operation is referred to as *product*. The product of elements a, $b \in M$ is written as ab. This product is extended to subsets A, $B \subseteq M$ by defining $AB = \{ab | a \in A \text{ and } b \in B\}$.

For monoids other than M, the monoid operation is denoted by $\circ$. The monoid in question is then always identified by the context.

A *partition* of M is a family of mutually disjoint nonempty subsets of M such that their union is M. The elements of a partition are called *classes*. A partition *saturates* a subset $A \subseteq M$ if A is a union of classes of the partition.

We say that a partition is a *quotient* of M to mean that it is a quotient of M by some congruence relation. One can easily see that a partition **A** is a quotient if and only if for every pair of classes A, $B \in \mathbf{A}$ there exists a class $C \in \mathbf{A}$ such that $AB \subseteq C$. A quotient of M is itself a monoid with respect to the operation $\circ$ defined by $A \circ B = C$ for C as above.

A subset $A \subseteq M$ is *recognized* by a monoid M' if there exists a homomorphism $h : M \to M'$ such that $A = h^{-1}(h(A))$. This is equivalent to A being saturated by the quotient $M/\ker(h)$ consisting of the classes $h^{-1}(m)$ for $m \in M$. It is easy to see that A is recognized by any quotient saturating A, by any monoid isomorphic to that quotient, and by any monoid with a submonoid recognizing A.

A subset A is *recognizable* if it is recognized by a finite monoid.

The powerset of a set A is denoted by $\mathcal{P}(A)$, and the set of all elements of A that are not in B is denoted by $A - B$.
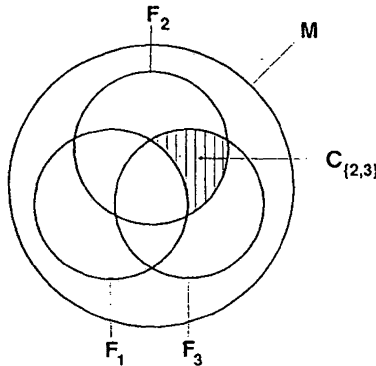
Figure 1.

3. THE MINIMAL PARTITION SATURATING A FAMILY OF SUBSETS

Given any family $\mathbf{F} = \{F_i | i \in I\}$ of nonempty subsets of M, one can always construct a partition that saturates every member of the family. The minimal such partition consists of all nonempty subsets

$$C_\rho = \bigcap_{i \in \rho} F_i \quad \cap \quad \bigcap_{i \notin \rho} (M - F_i) \tag{1}$$

for $\rho \in \mathcal{P}(I)$. An example for a family $\mathbf{F} = \{F_1, F_2, F_3\}$ is shown in Figure 1. The eight faces of the diagram represent the subsets C_ρ for the eight indices $\rho \in \mathcal{P}(\{1, 2, 3\})$. The subsets of the form (1) are sometimes [5] called the *constituents* of the Boolean algebra generated by the family $\mathbf{F}$.

The partition of M into the constituents C_ρ is in the following denoted by $\mathbf{C}(\mathbf{F})$.

Depending on the family $\mathbf{F}$, the partition $\mathbf{C}(\mathbf{F})$ is, or is not, a quotient of M. We shall use here a property of $\mathbf{F}$ that is sufficient for $\mathbf{C}(\mathbf{F})$ being a quotient. It is, however, not necessary; a simple property that would be both sufficient and necessary seems difficult to find.

DEFINITION 1: *We say that the family* $\mathbf{F}$ *is* decomposable *if for any member* $F_k \in \mathbf{F}$, *and any* $a, b \in M$ *such that* $ab \in F_k$, *there exist* $F_i \in \mathbf{F}$ *and* $F_j \in \mathbf{F}$ *such that* $a \in F_i$, $b \in F_j$, *and* $F_i F_j \subseteq F_k$.

If, instead of $F_i F_j \subseteq F_k$, *the subsets* F_i, F_j, F_k *satisfy another relation* $\Phi \subseteq (\mathbf{F} \times \mathbf{F} \times \mathbf{F})$ *implying* $F_i F_j \subseteq F_k$, *we say that* $\mathbf{F}$ *is* decomposable *according to* Φ. ∎

LEMMA 2: *If* **F** *is decomposable according to* Φ*, the following conditions are equivalent, for any classes* C_μ, $C_\nu \in \mathbf{C}\,(\mathbf{F})$ *and any member* $F_k \in \mathbf{F}$*:*

(a) $C_\mu\,C_\nu \subseteq F_k$.

(b) $C_\mu\,C_\nu \cap F_k \neq \phi$.

(c) *There exist* $i \in \mu$ *and* $j \in \nu$ *such that* $(F_i,\,F_j,\,F_k) \in \Phi$.

Proof: (a) $\Rightarrow$ (b) is trivial since C_μ and C_ν are by definition nonempty.

(b) $\Rightarrow$ (c): Suppose $C_\mu\,C_\nu \cap F_k \neq \phi$. Consider any $x \in C_\mu\,C_\nu \cap F_k$. We have $x = ab \in F_k$ where $a \in C_\mu$, $b \in C_\nu$. If **F** is decomposable according to Φ, there exist $F_i \in \mathbf{F}$ and $F_j \in \mathbf{F}$ such that $a \in F_i$, $b \in F_j$, and $(F_i,\,F_j,\,F_k) \in \Phi$. Suppose $i \notin \mu$. From the second term of (1) follows then $C_\mu \subseteq M - F_i$, which contradicts the fact that a is in both C_μ and F_i. Hence, $i \in \mu$. In a similar way, $j \in \nu$.

(c) $\Rightarrow$ (a): Suppose there exist $i \in \mu$ and $j \in \nu$ such that $(F_i,\,F_j,\,F_k) \in \Phi$. According to the first term of (1), $i \in \mu$ implies $C_\mu \subseteq F_i$ and $j \in \nu$ implies $C_\nu \subseteq F_j$. From this follows $C_\mu\,C_\nu \subseteq F_i\,F_j$. If **F** is decomposable according to Φ, $(F_i,\,F_j,\,F_k) \in \Phi$ implies $F_i\,F_j \subseteq F_k$. ∎

THEOREM 3: *The following two conditions are equivalent:*

(a) **F** *is decomposable according to* Φ*.*

(b) $\mathbf{C}\,(\mathbf{F})$ *is a quotient of M, with its monoid operation defined by* $C_\mu \circ C_\nu = C_\rho$ *where*

$$\rho = \{k\,|\,(F_i,\,F_j,\,F_k) \in \Phi \; for \; some \; i \in \mu \; and \; j \in \nu\}. \tag{2}$$

Proof: (a) $\Rightarrow$ (b): Suppose **F** is decomposable according to Φ. Consider any classes C_μ, $C_\nu \in \mathbf{C}\,(\mathbf{F})$. Their product $C_\mu\,C_\nu$, being a nonempty subset of M, must have a nonempty intersection with some class $C_\rho \in \mathbf{C}\,(\mathbf{F})$. According to (1) this means $C_\mu\,C_\nu \cap F_k \neq \phi$ for each $k \in \rho$, and $C_\mu\,C_\nu \cap (M - F_k) \neq \phi$ for each $k \notin \rho$. By Lemma 2 (a) and (b), we have $C_\mu\,C_\nu \subseteq F_k$ for $k \in \rho$, and $C_\mu\,C_\nu \subseteq (M - F_k)$ for $k \notin \rho$. From this and (1) follows $C_\mu\,C_\nu \subseteq C_\rho$, which shows that $\mathbf{C}\,(\mathbf{F})$ is a quotient.

To verify (2), consider any C_μ, C_ν, C_ρ such that $C_\mu \circ C_\nu = C_\rho$, that is, $C_\mu\,C_\nu \subseteq C_\rho$.

Take any $k \in \rho$. According to the first term of (1), this means $C_\rho \subseteq F_k$, and thus $C_\mu\,C_\nu \subseteq F_k$. By Lemma 2 (a) and (c) there exist $i \in \mu$ and $j \in \nu$ such that $(F_i,\,F_j,\,F_k) \in \Phi$.

Take now any $i \notin \rho$. According to the second term of (1), this means $C_\rho \subseteq M - F_k$, and thus $C_\mu C_\nu \cap F_k = \phi$. By Lemma 2 (b) and (c) there do not exist $i \in \mu$ and $j \in \nu$ such that $(F_i, F_j, F_k) \in \Phi$.

(b) $\Rightarrow$ (a): Suppose $\mathbf{C}(\mathbf{F})$ is a quotient of M. Consider any $F_k \in \mathbf{F}$ and any $a, b \in M$ such that $ab \in F_k$. Let C_μ, C_ν be the classes of $\mathbf{C}(\mathbf{F})$ that contain, respectively, a and b. Let $C_\mu \circ C_\nu = C_\rho$; that means $ab \in C_\rho$. From $ab \in F_k$ and the second term of (1) follows $k \in \rho$.

Suppose the operation in $\mathbf{C}(\mathbf{F})$ is defined by (2). Then, there exist $i \in \mu$ and $j \in \nu$ such that $(F_i, F_j, F_k) \in \Phi$. According to the first term of (1), $i \in \mu$ means $C_\mu \subseteq F_i$ and thus $a \in F_i$. Similarly, $j \in \nu$ implies $b \in F_j$. ∎

Taking $F_i F_j \subseteq F_k$ as the relation Φ, we obtain:

COROLLARY 4: *For any decomposable family* $\mathbf{F}$, *the partition* $\mathbf{C}(\mathbf{F})$ *is a quotient of* M. ∎

Note that $\mathbf{F}$ being decomposable is not necessary for $\mathbf{C}(\mathbf{F})$ being a quotient. It is only necessary for $\mathbf{C}(\mathbf{F})$ being a quotient *with a specific monoid operation*.

4. THE SCHÜTZENBERGER-STRAUBING PRODUCT

Let $L_1, ..., L_n$ be $n \geq 2$ given subsets of M that are recognized, respectively, by monoids $M_1, ..., M_n$ using homomorphisms $h_1, ..., h_n$. We want to find a monoid recognizing all products of the form $L_p L_{p+1}...L_{r-1} L_r$ for $1 \leq p \leq r \leq n$. We shall do it by constructing a quotient of M that saturates all these products.

For $1 \leq i \leq n$, define $\mathbf{A}_i = M/\ker(h_i)$: since h_i recognizes L_i, the quotient $\mathbf{A}_i$ saturates L_i. Because products distribute over unions, it is sufficient to find a quotient that saturates all products of the form $A_p A_{p+1}...A_{r-1} A_r$, where $1 \leq p \leq r \leq n$ and $A_i \in \mathbf{A}_i$ for $p \leq i \leq r$. Let $\mathbf{F}$ be the family of all these products. We treat $\mathbf{F}$ as a family indexed by k-tuples $(A_p, A_{p+1}, ..., A_{r-1}, A_r)$, each k-tuple being the list of factors used in the product. In this way, we distinguish products of different factors even if they are identical sets. The symbols appearing in the k-tuples should be treated as labels for the classes $A_p, ..., A_r$, rather than the classes themselves. We do not make this distinction explicit in order not to complicate the notation. Each label is assumed to identify a class as a member of a specific quotient, so that the same subset will have different labels if it appears as a member in different quotients. We have thus:

$$\mathbf{F} = \{F_i | i \in I\}$$

where

$$I = \{(A_p, ..., A_r) | 1 \le p \le r \le n \text{ and } A_i \in \mathbf{A}_i \text{ for } p \le i \le r\} \tag{3}$$

and

$$F_{(A_p, ..., A_r)} = A_p...A_r.$$

A specific partition of M saturating all members of $\mathbf{F}$ is $\mathbf{C}(\mathbf{F})$. It consists of all nonempty subsets

$$C_\rho = (\bigcap_{(A_p, ..., A_r) \in \rho} A_p...A_r) \cap (\bigcap_{(A_p, ..., A_r) \notin \rho} M - A_p...A_r) \tag{4}$$

for $\rho \in \mathcal{P}(I)$. Our goal will be achieved if we can show that the partition $\mathbf{C}(\mathbf{F})$ is a quotient of M.

PROPOSITION 5: *If M is a free monoid, the partition $\mathbf{C}(\mathbf{F})$ is a quotient of M, with monoid operation defined by $C_\mu \circ C_\nu = C_\rho$ where*

$$\rho = \{(A_p, ..., A_{q-1}, (A'_q \circ A''_q), A_{q+1}, ..., A_r)$$
$$|(A_p, ..., A_{q-1}, A'_q) \in \mu \quad \text{and} \quad (A''_q, A_{q+1}, ..., A_r) \in \nu\}. \tag{5}$$

Proof: Consider any $F_k = A_p...A_r \in \mathbf{F}$. Suppose $a, b \in M$ are such that $ab \in F_k$. That means $ab = x_p...x_r$ for some $x_p \in A_p, ..., x_r \in A_r$. We recall a result about free monoids known as the "Levi's lemma": for any $a, b, c, d \in M$ satisfying $ab = cd$, there exists $z \in M$ satisfying either $c = az$, $zd = b$, or $cz = a$, $zb = d$. (*See*, for example, [8], p. 103. The original source is [9].) Using Levi's lemma, one can verify by induction that there exists an index q, $p \le q \le r$, and two elements $y, z \in M$ such that

$$\left. \begin{array}{l} yz = x_q, \\ a = x_p...x_{q-1} y, \\ b = zx_{q+1}...x_r. \end{array} \right\} \tag{6}$$

Let A'_q and A''_q be the classes of $\mathbf{A}_q$ that contain, respectively, y and z. We have thus

$$a \in A_p...A_{q-1} A'_q = F_i,$$
$$b \in A''_q A_{q+1}...A_r = F_j.$$

From $yz = x_q$ follows $A'_q \circ A''_q = A_q$. The indices i, j, k of the classes F_i, F_j, F_k satisfy the condition

$$\left. \begin{aligned} k &= (A_p, ..., A_{q-1}, (A'_q \circ A''_q), A_{q+1}, ..., A_r) \\ \text{where } &(A_p, ..., A_{q-1}, A'_q) = i \quad \text{and} \quad (A''_q, A_{q+1}, ..., A_r) = j. \end{aligned} \right\} \quad (7)$$

Define the relation Φ so that $(F_i, F_j, F_k) \in \Phi$ if and only if (7) is true. One can easily see that (7) implies $F_i F_j \subseteq F_k$. The family $\mathbf{F}$ is thus decomposable according to Φ. Substituting this Φ in (2), we obtain (5). The result follows by Theorem 3. ∎

The quotient $\mathbf{C}(\mathbf{F})$, treated as a monoid, recognizes the products $L_p L_{p+1}...L_{r-1} L_r$ using the natural homomorphism f from M to its quotient. Instead of $\mathbf{C}(\mathbf{F})$, we can take the set $\mathcal{P}(I)$ with the operation $\mu \circ \nu = \rho$ defined by (5). It is easily verified to be a monoid. The monoid $\mathcal{P}(I)$ recognizes the products $L_p L_{p+1}...L_{r-1} L_r$ using the composition of f and the isomorphism $g(C_\rho) = \rho$.

We note that the monoid $\mathbf{C}(\mathbf{F})$ is isomorphic to a subset of $\mathcal{P}(I)$, rather than the whole set $\mathcal{P}(I)$. This is so because (5) denotes an empty set for many among the indices $\rho \in \mathcal{P}(I)$. For example, $C_\rho = \phi$ if ρ contains both (A'_q) and (A''_q) where A'_q and A''_q are classes of the same partition $\mathbf{A}_q$. Also, $\mathbf{C}(\mathbf{F}) = \phi$ if ρ contains (A_q), but does not contain all k-tuples $(1_p, ..., 1_{q-1}, A_q, 1_{q+1}, ..., 1_r)$, where 1_i is the class of $\mathbf{A}_i$ containing the unit element of M.

If we want a monoid expressed in terms of the original monoids $M_1, ..., M_n$, we can replace each class A_i in $\mathcal{P}(I)$ and (5) by the element $h_i(A_i) = m_i \in M_i$ [2]. We obtain in this way a monoid $(S, \circ)$, where

$$\left. \begin{aligned} S = \mathcal{P}(\{(m_p, ..., m_r) | 1 \leq p \leq r \leq n \\ \text{and} \quad m_i \in M_i \text{ for } p \leq i \leq r\}), \end{aligned} \right\} \quad (8)$$

and

$$\left. \begin{aligned} \mu \circ \nu = \{ &(m_p, ..., m_{q-1}, (m'_q \circ m''_q), m_{q+1}, ..., m_r) \\ &|(m_p, ..., m_{q-1}, m'_q) \in \mu \\ &\text{and} \quad (m''_q, m_{q+1}, ..., m_r) \in \nu\} \end{aligned} \right\} \quad (9)$$

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[2] $h_i(A_i)$ is, strictly speaking, a singleton set $\{m_i\}$; by an abuse of notation, we identify it with the element m_i.

for μ, $\nu \in S$. It is isomorphic to $\mathcal{P}(I)$, and recognizes the products $L_p L_{p+1}...L_{r-1} L_r$ using the homomorphism

$$h(a) = \{(h_p(A_p), ..., h_r(A_r))|(A_p, ..., A_r) \in \rho\} \quad \text{for} \quad a \in C_\rho. \quad (10)$$

The "n-fold Schützenberger product" $\Diamond(M_1, ..., M_n)$ defined by Straubing in [17] is identical to the submonoid of $(S, \circ)$ consisting of only those elements ρ that contain exactly one k-tuple (m_i) for each i, $1 \leq i \leq n$. This does not affect our construction because the excluded elements correspond to empty classes of $\mathbf{C}(\mathbf{F})$ [3]. For a convenient computation of the product (9), an element $\rho \in S$ is represented in [17] as an $n \times n$ matrix whose (p, r)-th element consists of all the k-tuples $(m_p, ..., m_r)$ belonging to ρ. As a later improvement, the k-tuples $(m_p, ..., m_r)$ are extended with unit elements to n-tuples that can be multiplied component by component. (*See*, for example, [11], [12], [16], [18].)

The special case of $\Diamond(M_1, ..., M_n)$ for $n = 2$ is the original Schützenberger product $\Diamond(M_1, ..., M_2)$ introduced in [14] and used, for example, in [3], [13], [15]. (It also appears under the name "Boolean product" on p. 181 in [8].)

So far, we have provided an alternative proof of the known fact that in a free monoid M, $\Diamond(M_1, M_n)$ recognizes all products $L_p L_{p+1}...L_{r-1} L_r$ of subsets $L_1, ..., L_n$ that are recognized, respectively, by $M_1, ..., M_n$. One can argue whether this proof is simpler than the original one in [14] and [17]. However, it is useful because it extends easily to the case when M is not a free monoid.

We note first that in the proof of Proposition 5 we have used only Levi's lemma, and not the fact that M is free. The proof remains thus valid for any monoid M that satisfies Levi's lemma. The monoids satisfying Levi's lemma are called *equidivisible*. Their class is larger than that of free monoids (*see* [7] for an comprehensive study). This gives:

COROLLARY 6: $\Diamond(M_1, ..., M_n)$ *recognizes the products* $L_p L_{p+1}...L_{r-1} L_r$ *in any equidivisible monoid M*. ∎

We note next that the proof of Proposition 5 depends, at the end, on a certain property of the ordered collection of quotients $(\mathbf{A}_1, ..., \mathbf{A}_n)$, obtained as a consequence of Levi's lemma. We call it "refinability":

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[3] It seems, however, to be important for the results obtained in [11], [13], [16].

DEFINITION 7: *An ordered collection of quotients* $(\mathbf{A}_1, ..., \mathbf{A}_n)$ *is called refinable if for any product* $A_p...A_r \in \mathbf{F}$ *and any* $a, b \in M$ *such that* $ab \in A_p...A_r$, *there exists an index* $q, p \leq q \leq r$, *and two members* $A_q', A_q'' \in \mathbf{A}_q$ *such that*:

$$\left. \begin{aligned} A_q' \circ A_q'' &= A_q, \\ a &\in A_p...A_{q-1} A_q', \\ b &\in A_q'' A_{q+1}...A_r. \end{aligned} \right\} \tag{11}$$

■

Our observation can be now formulated as follows:

COROLLARY .8: $\Diamond (M_1, ..., M_n)$ *recognizes the products* $L_p L_{p+1}...L_{r-1} L_r$ *if the sequence of quotients* $(\mathbf{A}_1, ..., \mathbf{A}_n)$ *is refinable.* ■

(The term "refinable" is intended to indicate that each product $A_p...A_r$ can be refined to agree with any factorization of its element. Refinability has a certain resemblance to the generalized form (6) of Levi's lemma. One can regard it as a kind of "Levi's lemma for quotients".)

We have shown, as a part of the proof of Proposition 5, that in an equidivisible monoid M every ordered collection of quotients is refinable. Conversely, if *every* ordered collection of quotients of M is refinable, M is necessarily equidivisible: the partition $\mathbf{S}$ of M into singleton subsets is a quotient, and refinability of $(\mathbf{S}, \mathbf{S})$ is identical to Levi's lemma. However, a specific collection of quotients may be refinable without M being equidivisible, as shown by the following example:

Example 9: Suppose $M = \{1, s, s^2, s^3\}$ where $s^4 = s^2$. The monoid M is not equidivisible: the element $z \in M$ required by the definition does not exist for $a = b = s$, $c = d = s^2$. It would have to satisfy $zs^2 = s$ or $s^2 z = s$, which is not possible.

Let $B_0 = \{1, s^2\}$ and $B_1 = \{s, s^3\}$. One can easily see that $\{B_0, B_1\}$ is a quotient of M. Suppose that $\mathbf{A}_1 = \mathbf{A}_2 = \{B_0, B_1\}$. By checking all possible cases, one can verify that the collection of quotients $(\mathbf{A}_1, \mathbf{A}_2)$ is refinable. (There are 16 trivial cases of $ab \in B_i$, and 16 trivial cases of $ab \in B_i B_j$ where $a \in B_i$ and $b \in B_j$. The remaining 15 cases follow a simple pattern.) ■

In general, it is sufficient to require refinability of only those collections of quotients that are involved in a specific problem. For example, when studying recognizable subsets, we deal only with finite quotients. We obtain in this way:

COROLLARY 10: *A product of two recognizable subsets is recognizable in every monoid where every pair of finite quotients is refinable.* ■

The results of Corollary 8 and Corollary 10 are, unfortunately, not reversible. For example, a product of two subsets may be recognized by a monoid other than the Schützenberger product. We can only establish a rather weak necessary condition:

PROPOSITION   11:   *If* $\Diamond\,(M_1, ..., M_n)$ *recognizes   the   products* $L_p\,L_{p+1}...L_{r-1}\,L_r$ *using the homomorphism h defined by* (10), *the collection of quotients* $(\mathbf{A}_1, ..., \mathbf{A}_n)$ *is refinable.*

Proof: If h is a homomorphism, the partition $\mathbf{C}\,(\mathbf{F})$ is a quotient of M (namely, $M/\ker\,(h)$). Using (9) and (5) one can verify that the monoid operation in $\mathbf{C}\,(\mathbf{F})$ is defined by (5). The formula (5) is equivalent to (2) for the relation Φ defined by (7). From Theorem 3 follows that $\mathbf{F}$ is decomposable according to that relation Φ, which is easily seen equivalent to $(\mathbf{A}_1, ..., \mathbf{A}_n)$ being refinable. ■

The homomorphism (10) appears in the standard proofs [14], [17] and various applications. We note, however, that Reutenauer [13] and Pin [11] obtained important results using other homomorphisms. They are outside the scope of this paper.

5. ANOTHER APPLICATION OF THE SCHÜTZENBERGER-STRAUBING PRODUCT

One can also arrive at the n-fold Schützenberger product in a slightly different way. As before, let $L_1, ..., L_n$ be $n \geq 2$ given subsets of M, recognized by monoids $M_1, ..., M_n$ using homomorphisms $h_1, ..., h_n$. In addition, let $u_1, ..., u_{n-1}$ be $n-1$ given elements of M. We want to find a monoid recognizing all products of the form $L_p\,u_p\,L_{p+1}\,u_{p+1}...L_{r-1}\,u_{r-1}\,L_r$ for $1 \leq p \leq r \leq n$. In a similar way as before, this is reduced to finding a quotient saturating all products of the form $A_p\,u_p\,A_{p+1}\,u_{p+1}...A_{r-1}\,u_{r-1}\,A_r$, where $1 \leq p \leq r \leq n$ and $\mathbf{A}_i = M/\ker\,(h_i)$ for $p \leq i \leq r$. Let now $\mathbf{F}$ be the family of all these products. Since $u_1, ..., u_{n-1}$ are fixed, each member of $\mathbf{F}$ is identified by the classes $A_p, ..., A_r$ appearing in it; $\mathbf{F}$ can be thus considered as a family indexed by the same set as before. We have now:

$$\mathbf{F} = \{F_i | i \in I\}$$

where

$$I = \{(A_p, ..., A_r) | 1 \le p \le r \le n \quad \text{and} \quad A_i \in \mathbf{A}_i \text{ for } p \le i \le r\} \tag{12}$$

and

$$F_{(A_p,...,A_r)} = A_p \, u_p ... A_{r-1} \, u_{r-1} \, A_r.$$

A specific partition of M saturating all members of $\mathbf{F}$ is $\mathbf{C}(\mathbf{F})$. It consists of all nonempty subsets

$$\begin{aligned}
C_\rho = (\bigcap_{(A_p,...,A_r) \in \rho} A_p \, u_p \, A_{p+1} ... u_{r-1} \, A_r) \\
\cap (\bigcap_{(A_p,...,A_r) \notin \rho} M - A_p \, u_p \, A_{p+1} ... u_{r-1} \, A_r)
\end{aligned} \tag{13}$$

for $\rho \in \mathcal{P}(I)$. Using the insights gained in the preceding section, we start by modifying the definition of refinability:

DEFINITION 12: *Given* $u_1, ..., u_{n-1} \in M$, *we say that an ordered collection of quotients* $(\mathbf{A}_1, ..., \mathbf{A}_n)$ *is* refinable under the marking $(u_1, ..., u_{n-1})$ *if for any product* $A_p \, u_p \, A_{p+1} ... u_{r-1} \, A_r \in \mathbf{F}$ *and any* $a, b \in M$ *such that* $ab \in A_p \, u_p \, A_{p+1} ... u_{r-1} \, A_r$, *there exists an index* q, $p \le q \le r$, *and two members* $A'_q, A''_q \in \mathbf{A}_q$ *such that:*

$$\left. \begin{aligned}
A'_q \circ A''_q &= A_q, \\
a &\in A_p \, u_p \, A_{p+1} ... u_{q-1} \, A'_q, \\
b &\in A''_q \, u_q \, A_{p+1} ... u_{r-1} \, A_r.
\end{aligned} \right\} \tag{14}$$

PROPOSITION 13: *The following two conditions are equivalent:*

(a) *The collection of quotients* $(\mathbf{A}_1, ..., \mathbf{A}_n)$ *is refinable under the marking* $(u_1, ..., u_{n-1})$.

(b) *The partition* $\mathbf{C}(\mathbf{F})$ *is a quotient of* M *with monoid operation defined by* (5).

Proof: One can easily verify that $(A_1, ..., A_n)$ is refinable under marking $(u_1, ..., u_{n-1})$ if and only if $\mathbf{F}$ is decomposable according to the relation Φ defined by (7). The equivalence of (a) and (b) follows from Theorem 3. ∎

PROPOSITION 14: *If* M *is equidivisible, and each of* $u_1, ..., u_{i-1}$ *is prime (that is, does not have any other factors than 1 and itself), then every*

ordered collection of quotients $(\mathbf{A}_1, ..., \mathbf{A}_n)$ *is refinable under the marking*
$(u_1, ..., u_{n-1})$.

Proof: Assume M is equidivisible. Consider any product
$A_p\, u_p\, A_{p+1}...u_{r-1}\, A_r\; \in\; \mathbf{F}$ and $a, b\; \in\; M$ such that $ab\; \in$
$A_p\, u_p\, A_{p+1}...u_{r-1}\, A_r$. That means $ab\; =\; x_p\, u_p\, x_{p+1}...u_{r-1}\, x_r$ for some
$x_p \in A_p, ..., x_r \in A_r$. Using Levi's lemma and induction, one can verify
that there exists an index $q, p \leq q \leq r$, and two elements $y, z \in M$
such that either

$$\left.\begin{array}{l} yz = x_q, \\[4pt] a = x_p\, u_p...x_{q-1}\, u_{q-1}\, y, \\[4pt] b = z u_q\, x_{q+1}...u_{r-1}\, x_r, \end{array}\right\} \tag{15}$$

or

$$\left.\begin{array}{l} yz = u_q, \\[4pt] a = x_p\, u_p...u_{q-1}\, x_q\, y, \\[4pt] b = z x_{q+1}\, u_{q+1}...u_{r-1} x_r. \end{array}\right\} \tag{16}$$

In case (15), let A'_q, A''_q be the classes of $\mathbf{A}_q$ that contain, respectively,
y and z. Then, $a \in A_p\, u_p...A_{q-1}\, u_{q-1}\, A'_q$ and $b \in A''_q\, u_q\, A_{q+1}...u_{r-1}\, A_r$.
From $yz = x_q$ follows $A'_q \circ A''_q = A_q$.

In case (16), since u_q is prime, we must have either $y = 1$, $z = u_q$ or
$y = u_q$, $z = 1$. The first case is identical to (15) with $y = x_q$, $z = 1$ and the
second to (15) for $q + 1$ with $y = 1$, $z = x_{q+1}$. ∎

Since the index set I and the monoid operation in $\mathbf{C}\,(\mathbf{F})$ are the same as
before, expressing the result in terms of the monoids $M_1, ..., M_n$ gives (8)
and (9), and establishes the same connection to the Schützenberger product
$\Diamond\,(M_1, ..., M_n)$ as before. From Proposition 13 and Proposition 14 follows:

COROLLARY 15: $\Diamond\,(M_1, ..., M_n)$ *recognizes all products*
$L_p\, u_p\, L_{p+1}\, u_{p+1}...L_{r-1}\, u_{r-1}\, L_r$ *with prime* $u_p, ..., u_{r-1}$ *in every*
equidivisible monoid M and, in general, whenever the collection of quotients
$(\mathbf{A}_1, ..., \mathbf{A}_n)$ *is refinable under the marking* $u_1, ..., u_{n-1}$. ∎

Formulating a result analogous to Proposition 11 is left to the reader.

6. THE DIAMOND PRODUCT OF GASTIN-PETIT-ZIELONKA

A monoid M is called *partially commutative* if it contains one or more
pairs of elements $u, v \in M$ such that $uv = vu$. Free, finitely generated,

partially commutative monoids are often called *trace monoids* and their elements are called *traces*. For their definition and a detailed discussion, the reader is referred to the monograph [2]. Trace monoids are not equidivisible; an example of an equation not satisfying the Levi's lemma is $uv = vu$ where u, v are generators. One uses instead a property known as the "Levi's lemma for traces": for any a, b, c, $d \in M$ satisfying $ab = cd$, there exist r, u, v, $s \in M$ such that $a = ru$, $b = vs$, $c = rv$, $d = us$, $uv = vu$ (*see* [1], Prop. 1.3, [2], p. 21). The class of monoids satisfying the Levi's lemma for traces is larger than the class of trace monoids.

For any partially commutative monoid M, there exists a quotient **U** having the property that for any given pair of classes U_i, $U_j \in$ **U**, either $uv = vu$ for all $(u, v) \in U_i \times U_j$, or $uv \neq vu$ for all $(u, v) \in U_i \times U_j$. For trace monoids, a quotient **U** with the stated property is obtained by classifying the elements of the monoid according to their generators: each class of **U** consists of elements with the same set of generators. The set of all pairs $(U_i, U_j) \in$ **U** $\times$ **U** having the property that $uv = vu$ for all $(u, v) \in U_i \times U_j$ is in the following denoted by *Comm*. One can easily verify that if $(U_i, U_j) \in Comm$, then for any pair of subsets $X \subseteq U_i$ and $Y \subseteq U_j$ holds $XY = YX$.

In the following, we assume M to be a partially commutative monoid with given **U**. Let L_1, L_2 be two given subsets of M, recognized, respectively, by monoids M_1 and M_2 using homomorphisms h_1 and h_2. We want to find a monoid recognizing the product $L_1 L_2$. As before, we shall do it by constructing a quotient of M that saturates $L_1 L_2$.

Define **A** $= M/\ker(h_1)$ and **B** $= M/\ker(h_2)$; the quotients **A** and **B** saturate, respectively, the subsets L_1 and L_2. The subset L_1 can be represented as a union of subsets of the form $(A \cap U)$ where $A \in$ **A** and $U \in$ **U**. Similarly, L_2 can be represented as a union of subsets of the form $(B \cap U)$ where $B \in$ **B** and $U \in$ **U**. Because products distribute over unions, it is sufficient to construct a quotient that saturates all products of the form $(A \cap U_1)(B \cap U_2)$ where $A \in$ **A**, $B \in$ **A**, $U_1 \in$ **U**, and $U_2 \in$ **U**. Let **F** be the family of all these products. We treat **F** as a family indexed by quadruples (A, U_1, B, U_2):

$$\mathbf{F} = \{F_i | i \in I\}$$

where

$$I = \{(A, U_1, B, U_2) | A \in \mathbf{A}, \ B \in \mathbf{B}, \ U_1, U_2 \in \mathbf{U}\} \qquad (17)$$

and

$$F_{(A, U_1, B, U_2)} = (A \cap U_1)(B \cap U_2).$$

A specific partition saturating all members of $\mathbf{F}$ is $\mathbf{C}(\mathbf{F})$. It consists of all nonempty subsets

$$C_\rho = (\bigcap_{(A, U_1, B, U_2) \in \rho} (A \cap U_1)(B \cap U_2))$$

$$\cap (\bigcap_{(A, U_1, B, U_2) \notin \rho} M - (A \cap U_1)(B \cap U_2)) \tag{18}$$

for $\rho \in \mathcal{P}(I)$. We proceed as before:

DEFINITION 16: *We say that a pair of quotients is* commutatively refinable *if for every product* $(A \cap U_1)(B \cap U_2) \in \mathbf{F}$ *and elements* $a, b \in M$ *such that* $ab \in (A \cap U_1)(B \cap U_2)$, *there exist* $A', A'' \in \mathbf{A}$, $B', B'' \in \mathbf{B}$, $U_1', U_1'', U_2', U_2'' \in \mathbf{U}$, *such that:*

$$\left.\begin{array}{l} A' \circ A'' = A, \quad U_1' \circ U_1'' = U_1, \\ B' \circ B'' = B, \quad U_2' \circ U_2'' = U_2, \\ (U_2', U_1'') \in Comm, \\ a \in (A' \cap U_1')(B' \cap U_2'), \\ b \in (A'' \cap U_1'')(B'' \cap U_2''). \end{array}\right\} \tag{19}$$

PROPOSITION 17: *The following two conditions are equivalent:*

(a) *The pair of quotients* $(\mathbf{A}, \mathbf{B})$ *is commutatively refinable.*

(b) *The partition* $\mathbf{C}(\mathbf{F})$ *is a quotient of M with monoid operation defined by* $C_\mu \circ C_\nu = C_\rho$ *where*

$$\left.\begin{array}{c} \rho = \{(A' \circ A'', U_1' \circ U_1'', B' \circ B'', U_2' \circ U_2'') \\ |(A', U_1', B', U_2') \in \mu, (A'', U_1'', B'', U_2'') \in v \\ \text{and } (U_1'', U_2') \in Comm\}. \end{array}\right\} \tag{20}$$

Proof: Define the relation Φ so that $(F_i, F_j, F_k) \in \Phi$ if and only if the indices of F_i, F_j, F_k satisfy the following condition:

$$\left.\begin{array}{l} k = (A' \circ A'', U_1' \circ U_1'', B' \circ B'', U_2' \circ U_2'') \\ \text{where } (A', U_1', B', U_2') = i, (A'', U_1'', B'', U_2'') = j \\ \text{and} \quad (U_2', U_1'') \in Comm. \end{array}\right\} \tag{21}$$

For any F_i, F_j, F_k satisfying Φ holds:

$$F_i\, F_j = (A' \cap U_1')\, (B' \cap U_2')\, (A'' \cap U_1'')\, (B'' \cap U_2'')$$
$$= (A' \cap U_1')\, (A'' \cap U_1'')\, (B' \cap U_2')\, (B'' \cap U_2'')$$
$$\subseteq (A' \circ A'' \cap U_1' \circ U_1'')\, (B' \circ B'' \cap U_2' \circ U_2'') = F_k.$$

One can easily verify that $(\mathbf{A}, \mathbf{B})$ is commutatively refinable if and only if $\mathbf{F}$ is decomposable according to Φ. The formula (20) is obtained by substituting this Φ in (2). The equivalence of (a) and (b) follows from Theorem 3. ■

PROPOSITION 18: *If M satisfies Levi's lemma for traces, every pair of quotients $(\mathbf{A}, \mathbf{B})$ is commutatively refinable.*

Proof: Consider any product $(A \cap U_1)\, (B \cap U_2) \in \mathbf{F}$, and any $a, b \in M$ such that $ab \in (A \cap U_1)\, (B \cap U_2)$. That means $ab = cd$ for some $c \in (A \cap U_1)$ and $d \in (B \cap U_2)$. According to Levi's lemma for traces, there exist $r, u, v, s \in M$ such that $a = ru$, $b = vs$, $c = rv$, $d = us$, $uv = vu$. Let A', A'' be the classes of $\mathbf{A}$ that contain, respectively, r and v. Let B', B'' be the classes of $\mathbf{B}$ that contain u and s, and let U_1', U_2', U_1'', U_2'' be the classes of $\mathbf{U}$ that contain, respectively, r, u, v, and s. We have then $a \in (A' \cap U_1')\, (B' \cap U_2')$ and $b \in (A'' \cap U_1'')\, (B'' \cap U_2'')$. From $c = rv$ follows $A' \circ A'' = A$, from $d = us$ follows $B' \circ B'' = B$, and from $uv = vu$ follows $(U_2', U_1'') \in \mathrm{Comm}$. ■

As in the case of the Schützenberger product, we can replace $\mathbf{C}\,(\mathbf{F})$ by the set $\mathcal{P}\,(I)$ with the operation $\mu \circ v = \rho$ defined by (20), and then replace A by $h_1\,(A)$ and B by $h_2\,(B)$ in $\mathcal{P}\,(I)$ and (20). The result is a monoid $(T, \circ)$ that consists of the set

$$T = \mathcal{P}\,(\{m_1, U_1, m_2, U_2\} | m_1 \in M_1$$
$$\text{and} \quad m_2 \in M_2 \quad \text{and} \quad U_1, U_2 \in \mathbf{U}\}), \tag{22}$$

with the operation defined by

$$\mu \circ v = \{(m_1' \circ m_1'', U_1' \circ U_1'', m_2' \circ m_2'', U_2' \circ U_2'')$$
$$| (m_1', U_1', m_2', U_2') \in \mu, \ (m_1'', U_1'', m_2'', U_2'') \in v \tag{23}$$
$$\text{and} \quad (U_2', U_1'') \in \mathrm{Comm}\}$$

for $\mu, v \in T$. If $\mathbf{C}\,(\mathbf{F})$ is a quotient of M, it is isomorphic to a subset of $(T, \circ)$; the monoid $(T, \circ)$ recognizes then all the products $(A \cap U_1)\, (B \cap U_2)$,

and therefore also the product $L_1 L_2$. As a consequence of Proposition 17 and Proposition 18, we obtain:

COROLLARY 19: *The monoid* $(T, \circ)$ *recognizes* $L_1 L_2$ *in any partially commutative monoid satisfying the Levi's lemma for traces, and, in general, whenever the pair of quotients* $(\mathbf{A}, \mathbf{B})$ *is commutatively refinable.* ∎

COROLLARY 20: *A product of two recognizable subsets is recognizable in any partially commutative monoid where* $\mathbf{U}$ *is finite and every pair of finite quotients is commutatively refinable.* ∎

With $\mathbf{U}$ defined as classification according to the set of generators , $(T, \circ)$ is identical to the "diamond product" $\diamondsuit_I (M_1, M_2)$ introduced by Gastin, Petit and Zielonka in [4].

7. APPLICATION TO SEMIGROUPS

The assumption of M being a monoid (rather than a semigroup) is needed for a purely technical reason. The unit element of M allows in the Levi's lemma for the special case of $a = c$, $b = d$. In the Levi's lemma for traces, it allows for the special cases of $a = r$, $d = s$ and $c = r$, $b = s$. If these special cases are included explicitly and propagated through all results derived from these lemmas, the whole discussion can be modified so that it applies to any semigroup M, with all quotients being semigroups rather than monoids.

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