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Fibrations and recursivity

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FIBRATIONS AND RECURSIVITY (¹)

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Abstract. – *We present here some facts about recursive function theory that we describe in an abstract form; for this we use here fibered categories. A first approach of this was made by P. Mulry by using of topos theory. Out of the results find here, we give the first steps towards the treatment of partiality in an abstract form.*

Résumé. – *Nous donnons ici quelques résultats sur les fonctions récursives au moyen des fibrations. Une première approche de cette formalisation avait été faite par P. Mulry en utilisant la théorie des topos. En utilisant les résultats trouvés ici, nous donnons les premières définitions pour une formalisation abstraite des fonctions partielles.*

0. INTRODUCTION

We present here some results concerning recursivity by use of category theory and more particularly fibrations. A first work on this subject was made by P. Mulry (10) by considering recursive topos. We study here particularly effective operations by introducing enumerated fibrations.

In chapter I we give some properties about the fibration of recursively enumerable sets and the fibration of partial recursive functions.

Using these examples, we introduce in chapter II a general definition of enumerated fibrations and effective operators. We give some properties about these fibrations and in particular some fixed point theorems.

In chapter III, we introduce the first definitions of what we mean by an abstract definition of partial functions. The definitions we give generalize the notion of presheaf over a locale defined by Fourman and Scott (4). Further results about this will be given in a forthcoming paper.

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1. NOTATIONS

Let B, S be categories and p a functor from B to S. If $u : J \to I$ is a morphism of S, a morphism $f : Y \to X$ in B is said to be cartesian over u if $p(f) = u$ and if, for any $g : Z \to X$ in B and $v : p(Z) \to J$ such that $u.v = p(g)$, there exists an unique $h : Z \to Y$ in B such that $p(h) = v$ and $f.h = g$.

(B, p) is said to be a fibration (or a fibered category) over S if for any $X \in B$ and any $u : J \to p(X)$ in S, there exists a cartesian morphism $f : Y \to X$ over u. We denote by $B^I = p^{-1}(I)$ the fibre over I.

We say that the fibration (B, p) is split if there exists a choice, for each pair (X, u) with $X \in B$ and $u : J \to p(X) = I$, of a cartesian morphism over u with domain denoted $u^*(X)$ (and then $u^* : p^{-1}(I) \to p^{-1}(J)$ is a functor), such that $(Id_I)^* = Id_{p^{-1}(I)}$ and $(u.v)^* = v^*.u^*$.

A fibration (B, p) over S is discrete if for any $I \in S$ B^I is a discrete category.

If (B, p) and (C, q) are two fibrations over S, a cartesian functor from (B, p) to (C, q) is a functor $G : B \to C$ such that $q.G = p$ and that sends each cartesian morphism of B onto a cartesian morphism of C. If the fibrations are split, a cartesian functor from (B, p) to (C, q) commute with $(\ )^*$.

If $K \in S$, we denote by $([K], \partial_0)$ the split discrete fibration over S such that $[K]^I$ is the set of morphisms of S from I to K; if $u : J \to I$ is in S, then $u^* : [K]^I \to [K]^J$ is the functor composition with $u.\partial_0$ is the domain functor from $[K]$ to S. Usuelly this fibration is denoted $(S/K, \partial_0)$; the notation $[K]$ will be appear clear in the next paragraph. It's the only change of notation that we use on fibration.

If (B, p) is a split fibration over S and if $X \in B^K$, then GX defined by $GX(u) = u^*(X)$, where $u : J \to K$, is a cartesian functor from $([K], \partial_0)$ to (B, p) and the correspondance $X \to GX$ is a bijection between B^K and the set of cartesian functor from $([K], \partial_0)$ to (B, p) and natural transformations between these functors.

A fibration (B, p) is said to be small if there is $K \in S$ such that $([K], \partial_0) \stackrel{\sim}{=} (B, p)$.

Let (B, p) be a fibration over S. We denote by $\mathrm{Fam}(B)$ the comma category (p, Id_S) and by $\mathrm{Fam}(p)$ the projection from $\mathrm{Fam}(B)$ to S. The objects of $\mathrm{Fam}(B)$ are the pairs (Y, u) where $Y \in B$ and $u : p(Y) \to I$ in S. We have $\mathrm{Fam}(p)(Y, u) = I$. In general, $(\mathrm{Fam}(B), \mathrm{Fam}(p))$ is not a

fibration (*see* (2)). We denote by Δ the diagonal functor from (B, p) to $(\mathrm{Fam}(B), \mathrm{Fam}(p))$.

If S has finite products and if $K \in S$, we have a subcategory $\mathrm{Fam}_K(B)$ of $\mathrm{Fam}(B)$ whose objects are pairs $(Y, \pi_{2\,I})$ where $\pi_{2\,I} : K \times I \to I$; a morphism from $(Z, \pi_{2\,J})$ to $(Y, \pi_{2\,I})$ is a pair (f, u) where $u : J \to I$ in S and $f : Z \to Y$ in B such that $p(f) = K \times u$. If we denote by $\mathrm{Fam}_K(p)$ the restriction of $\mathrm{Fam}(p)$ to $\mathrm{Fam}_K(B)$, $(\mathrm{Fam}_K(B), \mathrm{Fam}_K(p))$ is now a fibration over S. In fact, $(\mathrm{Fam}_K(B), \mathrm{Fam}_K(p))$ is obtained by pullingback p along the functor $K \times$- from S to S and so is isomorphic to the fibration (B^K, p^K) where objects are pairs (Y, I) such that $p(Y) = K \times I$. If (B, p) and (C, q) are fibrations over S, we denote by $(B, p) \times (C, q)$ the fibration over S wich is the pullback of p and q. In particular, if the diagonal functor Δ from (B, p) to $(B, p) \times (B, p)$ has a right adjoint $\wedge$ wich is cartesian we say that (B, p) has finite products.

If (B, p) is a fibration over S and if $\mathfrak{T}$ is a class of objects of B, we say that $\mathfrak{T}$ is stable if for any $X \in \mathfrak{T}$ and any cartesian morphism $Y \to X$ we have $Y \in \mathfrak{T}$.

We denote by (Ω, Ω) the fibration over S whose objects are subobjects of $[I]$, $I \in S$, and whose morphisms from $(F >\to [J])$ to $(G >\to [I])$ are paires (t, u) where $u : J \to I$ is in S and $t : F \to G$ a cartesian functor such that the following diagram commute:

$$
\begin{array}{ccc}
G \,>\!\!\longrightarrow & [I] \\
\uparrow & & \uparrow \\
t \;\Big\uparrow & & \Big\uparrow\, [u] \\
\Big\vert & & \Big\vert \\
F \,>\!\!\longrightarrow & [J]
\end{array}
$$

where $[u]$ is the cartesian functor composition with u.

We have also the fibration (Ω_d, Ω_d): the objects of Ω_d are mono of S. A morphism from $m : J >\to I$ to $m' : J' >\to I'$ is a pair (u, v) of morphisms of S $u : I >\to I'$ and $v : J >\to J'$ such that $u.m = m'.v$. The functor Ω_d sends $m : J >\to I$ onto I.

2. THE MAIN EXAMPLES

We denote by S (respectively $S^{\perp}$) the category whose objects are finites products of $\mathbb{N}$ (the set of naturels numbers) and whose morphisms from $\mathbb{N}^p$ to $\mathbb{N}^q$ are uples $(u_1, \ldots, u_q)$ of totals (respectively partials) recursives maps from $\mathbb{N}^p$ to $\mathbb{N}$.

If u is a partial map from $\mathbb{N}^p$ to $\mathbb{N}$ et if $x \in \mathbb{N}^p$, we denote by $u(x) \downarrow$ the fact that u is defined on x.dom u is then the set of x such that $u(x) \downarrow$.

The constructions of fibrations over S or $S^{\perp}$ will be refered in the sequel by $\underline{ME}$.

We denote by $(\mathbb{P}, p)$ the fibration over S defined by the following way:

– the objects of $\mathbb{P}$ are pairs (A, I) where $I \in S$ and $A \subseteq I$.

– a morphism from (A, I) to (B, J) is given by $u : I \to J$ in S such that $A \subseteq u^{-1}(B)$; we denote by (u) such a morphism.

The functor $p : \mathbb{P} \to S$ is defined on objects by $p(A, I) = I$.

A morphism $(u) : (A, I) \to (B, J)$ is cartesian iff $A = u^{-1}(B)$.

We denote by $(\text{R\'e}, p)$ the full subfibration of $(\mathbb{P}, p)$ whose objects are pairs (A, I) where A is a recursively enumerable subset of I and by (Rec, p) the full subfibration of $(\text{R\'e}, p)$ whose objects are pairs (A, I) where A is a recursive subset of I. We denote by $\mathcal{V}$ the stable class of objects of $\mathbb{P}$ of the form (I, I).

We have an another interresting fibration over S which we denote by $(\mathbb{F}, q)$; it's defined by:

– objects of $\mathbb{F}$ are pairs (f, I) where f is a partial map from I to $\mathbb{N}$.

– a morphism from (f, I) to (g, J) is given by $u : I \to J$ in S such that $f \triangleleft u^{-1} g$; this notation said that for $x \in I$, if $f(x) \downarrow$ then $g(u(x)) \downarrow$ and in this case they have the same value. We denote by $\langle u \rangle$ such a morphism.

The functor $q : \mathbb{F} \to S$ is defined on objects by $q(f, I) = I$.

A morphism $\langle u \rangle : (f, I) \to (g, J)$ is cartesian iff $f = u^{-1}(g)$.

We denote by (Fr, q) the full subfibration of $(\mathbb{F}, q)$ whose objects are pairs (f, I) where f is partial recursive. We denote by (Frt, q) the full subfibration of (Fr, q) whose objects are pairs (f, I) where f is total recursive.

In the same way we define fibrations $(\mathbb{P}^{\perp}, p^{\perp})$, $(\text{R\'e}^{\perp}, p^{\perp})$, $(\mathbb{F}^{\perp}, q^{\perp})$, $(\text{Fr}^{\perp}, q^{\perp})$ over $S^{\perp}$ where the objects are the same as above but the morphisms (u) are taken in $S^{\perp}$.

We give now some facts about these fibrations.

PROPOSITION 1: *The fibrations* $(\mathbb{P}, p)$, $(\text{Ré}, p)$, (Rec, p) *have finites products and so have the categories* $\mathbb{P}$, Ré, Rec. *The same fact is available for the fibrations with* $\perp$ *in exposant.*

Proof: If (A, I) and (B, I) are objects of these categories, one put $(A, I) \wedge (B, I) = (A \cap B, I)$. Morever, as S has finites products, the categories $\mathbb{P}$, Ré, Rec have finites products.

PROPOSITION 2: *The fibrations* $(\mathbb{P}, p)$, $(\text{Ré}, p)$, $(\mathbb{P}^{\perp}, p^{\perp})$ *and* $(\text{Ré}^{\perp}, p^{\perp})$ *have small sums: for each of these categories the diagonal functor* Δ *has a left adjoint.*

Proof: If $((A, I), u) \in \text{Fam}(\mathbb{P})^J$ where $u : I \rightarrow J$, we put $\amalg((A, I), u) = (u(A), I)$. It's clear that $\amalg$ defines a cartesian functor left adjoint to Δ. This functor restricts to Ré.

PROPOSITION 3: *There are embedding from* (Ω_d, Ω_d) *to* $(\text{Ré}, p)$ *and from* $(\text{Ré}, p)$ *to* (Ω, Ω).

Proof: The first embedding is the functor wich to any mono m send the image of m. For the secund, consider the functor G from $\mathbb{P}$ to Ω defined by:

− $G(A, I) = \hat{A} \rightarrow [I]$ where $\hat{A}(K) = \{ v : K \rightarrow I / v(K) \subseteq A \}$

− $(u) : (A, I) \rightarrow (A', J)$ in $\mathbb{P}$ is sended on $(\hat{u}, u)$ where $\hat{u} : \hat{A} \rightarrow \hat{A}'$ is defined by $\hat{u}(K)(v) = u.v$.

It's easy to see that G is cartesian.

Consider the restriction of G to Ré and show that G is an embedding.

First, G is injective on objects; if (A, I) and (A', I) are objects of Ré such that $G(A, I) = \hat{A}$, $G(A', I) = \hat{A}'$ and $\hat{A} = \hat{A}'$, take $u, u' : \mathbb{N} \rightarrow I$ such that $u(\mathbb{N}) = A$ and $u'(\mathbb{N}) = A'$. Then $u \in \hat{A}(\mathbb{N})$ and so $u \in \hat{A}'$; thus $u(\mathbb{N}) = A \subseteq A'$. The converse is same and thus $A = A'$.

Let (u), $(u') : (A, I) \rightarrow (B, J)$ in Ré such that $G((u)) = G((u'))$. By construction of G, it's clear that $u = u'$.

We have in particular the following inclusions that Mulry had discovered (10): $\text{Rec} \rightarrowtail \text{Ré} \rightarrowtail \Omega$.

PROPOSITION 4: *There exists a cartesian functor E from* $(\mathbb{F}, q)$ *to* $(\mathbb{P}, p)$ *such that the restriction of E to* Fr *defines a cartesian functor from* (Fr, q) *to* $(\text{Ré}, p)$ *such that $E^{-1}(\mathcal{V}) = \text{Frt}$.*

Proof: E is defined on objects of $\mathbb{F}$ by $E(f, I) = (\text{dom } f, I)$. If $\langle u \rangle : (f, I) \rightarrow (g, J)$ is a morphism of $\mathbb{F}$ then we put $E(\langle u \rangle) = (u)$.

It's clear that E becomes a cartesian functor from $(\mathbb{F}, q)$ to $(\mathbb{P}, p)$. The others affirmations are obvious.
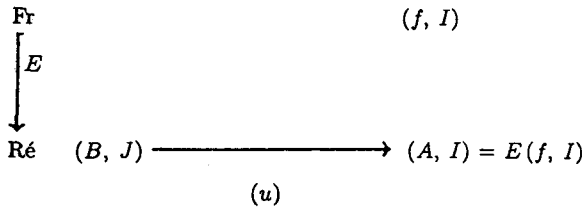
In fact we have more about E and this will be of great importance for paragraph 4.

THEOREM 5: (Fr, E) *is a fibration over* Ré *and* $(\mathbb{F}, E)$ *is a fibration over* $\mathbb{P}$.

Proof: We give the proof in the first case.

We remark that every morphism of Fr is cartesian.

Let consider now the above situation where $(f, I) \in$ Fr:

Fr (f, I)

$\downarrow E$

Ré $(B, J) \xrightarrow{\quad\quad\quad\quad\quad} (A, I) = E(f, I)$

(u)

We want a cartesian morphism over (u); let $h = f.u$ and $g = h\lceil B$ the restriction of h to B. So we have $\langle u \rangle : (g, J) \to (f, I)$ such that $E(\langle u \rangle) = (u)$.

Finally the last fact about these examples is enumeration.

THEOREM 6: *There exists, for any* $K \in S$, *a cartesian functor* $\varphi_K : ([\mathbb{N}], \partial_0) \to (Fr^K, q^K)$ *such that for any* $I \in S$ *and any* $f : ([I], \partial_0) \to (Fr^K, q^K)$ *cartesian there exists a morphism* $u : I \to \mathbb{N}$ *in* S *such that the following diagram commute:*

φ_K

$[\mathbb{N}] \xrightarrow{\quad\quad\quad\quad\quad} Fr^K$

$[u] \uparrow \qquad \nearrow f$

$[I]$
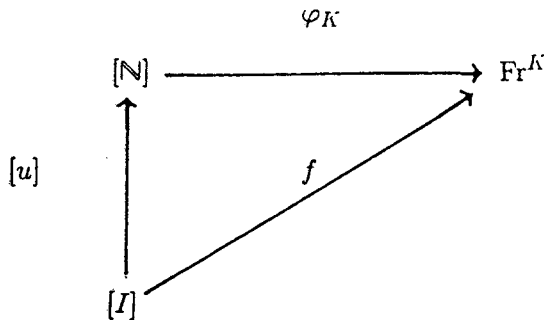
Proof: Let φ_K be an universal partial recursive function $\mathbb{N} \times K \to \mathbb{N}$. We know (by S-m-n) that for any partial recursive function $g : I \times K \to \mathbb{N}$ there exists a totale recursive function $u : I \to \mathbb{N}$ such that $\varphi_K(u(x), y) = g(x, y)$ where $x \in I$ and $y \in K$.

We translate this property in the following way; such an element φ_K is the same thing as to give an object of Fr^K in the fibre over $\mathbb{N}$. So φ_K define a cartesian functor $([\mathbb{N}], \partial_0) \to \mathrm{Fr}^K$ wich we denote again by φ_K. It's easy to see that this functor has the claimed property. In particular, if g is a partial recursive function $K \to \mathbb{N}$, there exists $e \in \mathbb{N}$ such that for $y \in K$ $g(y) = \varphi_K(e, y)$; we call e an index of g.

When we see the diagram above, we may compare it with the notion of enumerated set of Ershof. This is the reason that we take the notation $[\mathbb{N}]$ in place of $S/\mathbb{N}$.

We see that we have the same property for $(\mathrm{Ré}, p)$ by using of an universel recursively enumerable subset of $\omega \times K$.

What are cartesian functors between these fibrations? If we consider a cartesian functor Φ from (Fr^I, q^I) to (Fr^J, q^J), with $I, J \in S$, we consider not only a functionnal from the fibre Fr^I to the fibre Fr^J but also a functionnal from the fibre $(\mathrm{Fr}^I)^K$ to the fibre $(\mathrm{Fr}^J)^K$ for any $K \in S$, and these functionnals must to commute with the u^* for any morphism u in S. For understand this fact, consider an example;

let $\Phi : \mathrm{Fr}^{\mathbb{N}^p} \times \mathrm{Fr}^{\mathbb{N}^2 \times \mathbb{N}^p} \to \mathrm{Fr}^{\mathbb{N} \times \mathbb{N}^p}$ wich sends (g, h) to f defined by

$$f(n, y) = \begin{vmatrix} g(y) & \text{if } n = 0 \\ h(y, n-1, f(n-1, y)) & \text{if } n \neq 0 \end{vmatrix}$$

for $n \in \mathbb{N}$ and $y \in \mathbb{N}^p$.

We remark that if $u : \mathbb{N}^q \to \mathbb{N}^p$ is in S, then we have:

$$f(n, u(z)) = \begin{vmatrix} g(u(z)) & \text{if } n = 0 \\ h(u(z), n-1, f(n-1, u(z))) & \text{if } n \neq 0 \end{vmatrix}$$

for $n \in \mathbb{N}$ and $z \in \mathbb{N}^q$, and so that $u^*(f) = u^*(g, h)$.

We may then consider the operation of primitive recursion as a cartesian functor from $(\mathrm{Fr}, q) \times (\mathrm{Fr}^{\mathbb{N}^2}, q^{\mathbb{N}^2})$ to $(\mathrm{Fr}^{\mathbb{N}}, q^{\mathbb{N}})$. We resume in this simple fact construction by primitive recursion for functions of any arguments.

3. ω–FIBRATIONS AND EFFECTIVITY

The definitions that we give here about enumeration were given in (7).

We fix a category S with finite products and an object ω of S; all the fibrations in this section are over S and are split. When it'snt confusing we omit projection of fibration; for instance $([K], \partial_0)$ is denoted $[K]$.

DEFINITION: *A fibration (B, p) is said to be strongly ω-enumereted if for any $K \in S$ there exists a cartesian functor $W_K : [\omega] \to B^K$ such that for any $X \in (B^K)^I$, where $I \in S$, there exists a morphism $u : I \to \omega$ in S such that the following diagram commute:*

$$W_K$$

$$[\omega] \xrightarrow{\hspace{3cm}} B^K$$

$$[u] \nwarrow \qquad \nearrow X$$

$$[I]$$
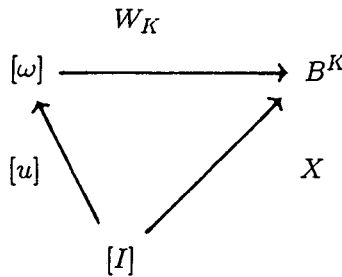
In the same way, we say that (B, p) is weakly ω-enumered if for any $K \in S$ there exists a cartesian functor $W_K : [\omega] \to B^K$ such that for any $X \in B^K$, there exists $e : 1 \to \omega$ in S such that $W_K.[e] = X$.

Obviously, any (B, p) wich is strongly ω-enumered is weakly ω-enumered.

With these notations, we denote by $W = \{ W_K / K \in S \}$; this is a class of objects of B and we say that (B, p, W) is a (strong or weak) ω-fibration.

If (B, p, W) and (C, q, V) are ω-fibrations, an effective operator at $(K_1, ..., K_n, J)$, where $K_1, ..., K_n, J \in S$, from (B, p, W) to (C, q, V) is a cartesian functor Φ from $(B^{K_1}, p^{K_1}) \times ... \times (B^{K_n}, p^{K_n})$ to (C^J, q^J) such that there exists a morphism $u : \omega^n \to \omega$ in S satisfying $\Phi.W = V_J.[u]$, where $W = (W_K, ..., W_K) : [\omega]^n \to (B^{K_1}, p^{K_1}) \times ... \times (B^{K_n}, p^{K_n})$. The results on effective operators are given in the sequel for $n = 1$.

Example: Take $\underline{ME}$; we have shown that (Fr, q, φ) and $(\text{Ré}, p, W)$ are $\mathbb{N}$-fibrations over S. An effective operator Φ at (K, J) from $(\text{Ré}, p, W)$ to itself is an effective operator (in the usuel sense of recursivity) from Ré^K to Ré^J; if $X \in \text{Ré}^K$ with index e then $\Phi(X) \in \text{Ré}^J$ with index $u(e)$.

PROPOSITION 1: *If (B, p, W) is a strong (weak) ω-fibration then for any $J \in S$ there exists a class W^J of objects of B^J such that (B^J, p^J, W^J) be a strong (weak) ω-fibration.*

Proof: For $K \in S$ we put $W_K^J : [\omega] \to (B^J)^K = W_{J \times K}$.

PROPOSITION 2: *Let (B, p, W) be a strong (weak) ω-fibration, (C, q) a fibration and G a cartesian functor from (B, p) to (C, q) surjective on the objects. Then there is a class of objects V of C such that (C, q, V) is a strong (weak) ω-fibration.*

Proof: Let $K \in S$ and $G^K : (B^K, p^K) \to (C^K, q^K)$. Put $V_K = G^K.W_K$. It's easy to see that the class $V = \{ V_K / K \in S \}$ is such that (C, q, V) is a strong (weak) ω-fibration if (B, p, W) is a strong (weak) ω-fibration.

Example: Take $\underline{ME}$: we define two new fibrations over S.

1) The fibration $\left(\prod_n ; \Pi_n \right)$: the objects of $\prod_n$ are pairs (A, I) where $I \in S$ and A is a $\prod_n$ subset of I. A morphism from (A, I) to (B, J) is given by $u : I \to J$ in S such that $A \subseteq u^{-1}(B).\Pi_n$ sends (A, I) onto I.

In the same way we have the fibration $\left(\sum_n, \Sigma_n \right)$ if we replace $\prod_n$ subsets by $\sum_n$ subsets of I. By induction on n we prove that $\left(\prod_n ; \Pi_n \right)$ and $\left(\sum_n, \Sigma_n \right)$ are strongly ω-enumerated. $\left(\sum_1, \Sigma_1 \right)$ is (Ré, p).

If we consider the cartesian functor $\neg$ from (Ré, p) to $\left(\prod_1, \Pi_1 \right)$ defined on object by $\neg(A, I) = (\neg A, I)$, this functor is surjective on objects.

If we consider the cartesian functor G from $\left(\prod_1^{\mathbb{N}}, \Pi_1^{\mathbb{N}} \right)$ to $\left(\sum_2, \Sigma_2 \right)$ defined on objects by $g(A, I \times \mathbb{N}) = (B, I)$ where, for $x \in I$, we have $x \in B$ iff there $n \in \mathbb{N}$ such that $(x, n) \in A$; this functor is surjective on objects.

2) The fibration (Fr_h, q_h) where $h \in \mathbb{N}$ fixed: the category Fr_h is the full subcategory of Fr where the objects are pairs (f, I) such that codomain $f \subseteq \{ 0, 1, ..., h \}.q_h$ is the restriction of q to Fr_h.

The functor G from (Fr, q) to (Fr_h, q_h) defined by $G(f, i) = (g, I)$ where g is defined by $g(x) = f(x)$ if $f(x) \leq h$ and else undefined, is cartesian and surjective on objects. So (Fr_h, q_h) is strongly $\mathbb{N}$-enumerated.

PROPOSITION 3: *If (B, p, W) and (C, q, V) are strong ω-fibrations then cartesian functors from (B^K, p^K) to (C^J, q^J) are exactly effective operators at (K, J) from (B, p, W) to (C, q, V).*

Proof: Let Φ cartesian from (B^K, p^K) to (C^J, q^J) and consider $\Phi.W_K$. As (C, q, V) is a stong ω-fibration, there exists $u : \omega \to \omega$ in S such that $\Phi.W_K = V_J.[u]$.

THEOREM 4 *(Fixpoint)*: *Let* (B, p, W) *be a weak ω-fibration. For any* $K \in S$, *any effective operator* Φ *at* (K, K) *from* (B, p, W) *to itself has a fixed point; so there exists* $X \in B^K$ *such that* $\Phi(X) = X$.

Proof: The proof is an adaptation of the welknown one in recursivity. For simplification we don't write the projection of fibration; these projections are clear. For instance, a cartesian functor G from (B, p) to (C, q) is denoted by $G : B \to C$.
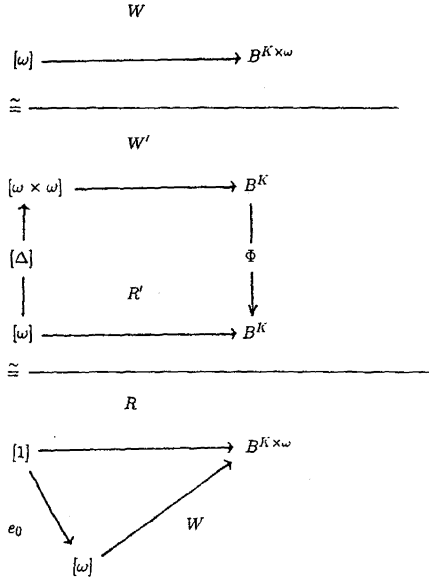
Let $\lambda : [\omega \times \omega] \to [\omega] \times [\omega]$ et $\gamma : B^{K \times \omega} \to (B^K)^\omega$ the canonical isomorphisms and let $ev : (B^K)^\omega \times [\omega] \to B^K$ the evaluation morphism.

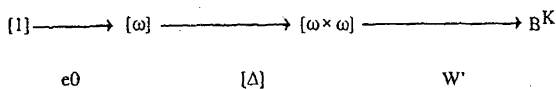Put $W = W_{K \times \omega}$ et $W' = ev.((\gamma.W) \times \mathrm{Id}_{[\omega]}).\lambda$.

Let $R' = \Phi.W'.[\Delta]$ where Δ is the diagonal morphism $\omega \to \omega \times \omega$ in S.

Denote by R the unique morphism from $[1]$ to $B^{K \times \omega}$ such that $R' = ev.((\gamma.R) \times \mathrm{Id}_{[\omega]}).\mu$, where $\mu : [\omega] \to [1] \times [\omega]$ is the canonical isomorphism. As B est weakly ω-enumered, there exists $e_0 : 1 \to \omega$ in S such that $W.e_0 = R$. Let $P = W'.[\Delta].e_0$.

We may see this construction by the following diagrams:

$$W$$
$$[\omega] \xrightarrow{\hspace{4cm}} B^{K \times \omega}$$
$$\cong$$

$$W'$$
$$[\omega \times \omega] \xrightarrow{\hspace{2cm}} B^K$$
$$\uparrow{[\Delta]} \qquad\qquad \downarrow{\Phi}$$
$$[\omega] \xrightarrow{\hspace{2cm}} B^K$$
$$\cong \qquad\qquad R'$$

$$R$$
$$[1] \xrightarrow{\hspace{3cm}} B^{K \times \omega}$$
$$e_0 \searrow \qquad \nearrow W$$
$$[\omega]$$

We then have :

$$[1] \xrightarrow{\hspace{1cm}} [\omega] \xrightarrow{\hspace{2cm}} [\omega \times \omega] \xrightarrow{\hspace{2.5cm}} B^K$$
$$e0 \qquad\qquad [\Delta] \qquad\qquad W'$$

Put $X = W'.[\Delta].e_0$ and evaluate $\Phi(X)$:

$$\Phi.X = \Phi.W.[\Delta].e_0 = R'.e_0.$$

But $R'.e_0 = ev.((\gamma.R) \times \mathrm{Id}_{[\omega]}).\mu.e_0 = ev.((\gamma.W.e_0) \times \mathrm{Id}_{[\omega]}).\mu.e_0$.

We have $X = W.[\Delta].e_0 = ev.((\gamma.W) \times \mathrm{Id}_{[\omega]}).\lambda.[\Delta].e_0$.

A categorical argument shows that $((\gamma.W) \times \mathrm{Id}_{[\omega]}).\lambda.[\Delta].e_0 = ((\gamma.W.e_0) \times \mathrm{Id}_{[\omega]}).\mu.e_0$. We deduce that $\Phi.X = X$.

The element finded here is not unique and is not necessary the leasted.

In the sequel we suppose that in S there are two morphisms $0 : 1 \to \omega$ and $s : \omega \to \omega$ such that for any morphisms in S, $f : 1 \to \omega$ and $g : \omega \to \omega$ there exists a morphism denoted $[f, g] : \omega \to \omega$ such that $[f, g].0 = f$ and $[f, g].s = g$.

In our example $\underline{ME}$ this says that we have the definition by cases for the total recursive functions.

PROPOSITION 5: *Let* (B, p, W) *be a strong ω-fibration, $X \in B^K$ and $Y \in B^{K \times \omega}$. There exists $Z \in B^{K \times \omega}$ such that $Z.[0] = X$ and $Z.[s] = Y$. In other words the following diagram commute:*
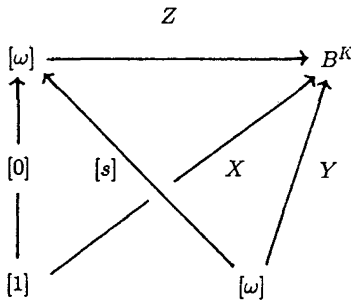
Proof: As (B, p, W) is a strong ω-fibration, for X and Y there exists respectively $e : 1 \to \omega$ and $u : \omega \to \omega$ in S such that $X = W_K.[e]$ and $Y = W_K.u$. If we take $v = [e, u]$, then $Z = W_K.v$ verify the desired properties.

Unfortunetly, such a Z is not unique; this fact don't depends of the enumeration but is a property on functions. In the sequel we suppose that we have unicity of this object and we denote it by $[X, Y]$: moreover, we suppose that $[-,-]$ defines a cartesian functor from $B^K \times B^{K \times \omega}$ to $B^{K \times \omega}$ which to (X, Y) sends $[X, Y]$.

PROPOSITION 6: *Let (B, p, W) be a strong ω-fibration, $K \in S$ and Φ an effective operator at (K, K) from (B, p, W) to itself. For $X \in B^K$ there exists $Y \in B^{K \times \omega}$ such that $Y.[0] = X$ and $Y.[s] = \Phi.Y$. So the following diagram commute:*
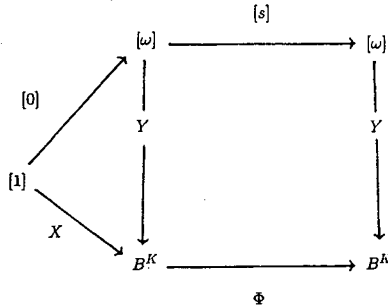
Proof: Let Ψ be the cartesian functor from $B^{K \times \omega}$ to itself defined by $\Psi(Z) = [X, \Phi.Z]$. Let Y be a fixed point of Ψ; then $Y = [X, \Phi.Y]$ and Y satisfies the properties of proposition. We remark that $[\omega]$ is a weak natural number object. Intuitively, Y represents all the iterated of X by Φ. We denote $Y = \mathrm{It}(X, \Phi)$. We see that iteration depends only of the set of codes ω of the elements of B. We conclude this chapter by the generalized least fixed point theorem.

For this consider the following conditions:

Let (B, p, W) be a strong ω-fibration, Φ an effective operator at (K, K) from (B, p, W) to itself such that there exists a cartesian functor $\mathrm{It}(\Phi)$ from (B^K, p^K) to $(B^{K \times \omega}, p^{K \times \omega})$ defined on objects by $\mathrm{It}(\Phi)(X) = \mathrm{It}(X, \Phi)$. We suppose morever that (B, p) has an initial object and that $\Delta : (B, p) \rightarrow (B^\omega, p^\omega)$ has a left adjoint II wich is cartesian and commute with Φ. Finally, suppose that the cartesian functor $\mathrm{It}(\Phi)$ defined above is faithfull. We have then:

THEOREM 7: *Φ admits a least fixed point; there exists $X \in B^K$ such that $\Phi(X) \cong X$ and for any $Y \in B^K$ such that $\Phi(Y) \cong Y$ there exists a morphism $X \rightarrow Y$ in B^K.*

Proof: Let $U \in B^K$ be an initial object and let $Z = \mathrm{It}(U, \Phi)$. Put $X = \mathrm{II}(Z, \pi_1)$ where π_1 is the first projection $K \times \omega \rightarrow K$. So $X \in B^K$: we show that it's the leasted fixpoint object. We have $\Phi(X) = \Phi(\mathrm{II}(Z, \pi_1)) = \mathrm{II}(\Phi(Z), \pi_1) = \mathrm{II}(s^* Z, \pi_1)$. But $X = \mathrm{II}(Z, \pi_1)$;

we must then show that $\text{II}(s^* Z, \pi_1) \cong \text{II}(Z, \pi_1)$. We remark first that $\text{It}(\Phi)(\Phi(U)) = s^* Z$ because $s^* Z.0 = Z.s.0 = \Phi(U)$ and $\Phi(s^* Z) = s^* \Phi(Z) = s^*(Z.s) = s^* Z.s$.

We have then $Z = \text{It}(\Phi)(U) \to \text{It}(\Phi)(\Phi(U)) = s^* Z$. We have also $s^* Z \to Z$ a cartesian morphism about $\omega \times s$.

For any $Y \in B^K$ we have the bijection:

$$\cong \frac{(Z, \pi_1) \to \Delta Y}{(s^* Z, \pi_1) \to \Delta Y}$$

and by adjunction:

$$\cong \frac{\text{II}(Z, \pi_1) \to Y}{\text{II}(s^* Z, \pi_1) \to Y}$$

In particular, we have that we want.

If Y is such that $\Phi(Y) \cong Y$ one has $U \to Y$ and so $\text{It}(U, \Phi) \to \text{It}(Y, \Phi)$. By adjunction we have $X \to Y$.

In particular, if each fibre B^I is an order, with the same hypothesis except faithfullness we have:

COROLLARY: Φ *has a least fixed point; there exists* $X \in B^K$ *such that* $\Phi(X) = X$ *and for any* $Y \in B^K$ *such that* $\Phi(Y) = Y$ *we have* $X \subseteq Y$.

4. ABSTRACT APPROACH OF PARTIALITY

Let's again have a look at the main example ME and more precisely theorem 6. The observations that we describe for it may be made for other examples; we are just intended in partiality here. We have seen that (Fr, E) is a fibration over Ré. Each fiber of Ré over S is a poset and we can think $(\text{Ré}, p)$ as a generalized poset with joins and finite meets. If we denote it, for a moment by Ω, we can see (Fr, E) as an Ω-set as it was defined by Fournan and Scott (4). But for this, we must define an equality $[| = |]$ or a restriction $\lceil$ satisfying some properties. We have in fact this:

PROPOSITION 1: *There exists a cartesian functor* $\lceil$ *from* $(\text{Fr}, q) \times (\text{Ré}, p)$ *to* (Fr, q) *such that on objects* $\lceil ((f, I), (A, I)) = (g, I)$ *where g is defined by: for* $x \in I$, $g(x) \downarrow$ *iff* $f(x) \downarrow$ *and* $x \in A$, *and in that case one has* $g(x) = f(x)$. *So g is the restriction of f to A.*

Proof: is staightforward and is not of interest here.

If we want generalize this situation, we fix a category S, a fibration (B, p) over S with some properties and we study fibration (F, q) over S equipped with cartesian functors $E : (F, q) \to (B, p)$ and $\lceil : (F, q) \times (B, p) \to (F, q)$ satisfying some equalities.

In fact, we may made this program in a simpler and more general manner.

Take again the main example. (Fr, E) is a fibration over Ré: let $(u) : (B, J) \to (A, I)$ be a morphism of Ré and $(f, I) \in$ Fr in the fibre over (A, I). A cartesian morphism over (u) is $\langle u \rangle : (g, J) \to (f, I)$ where g is defined by: for $x \in J$, $g(x) \downarrow$ iff $x \in B$ and in this case $g(x) = f(u(x))$. In particular for $(\mathrm{Id}_I) : (B, I) \to (A, I)$ and $(f, I) \in$ Fr, a cartesian morphism over (Id_I) is exactly $\langle \mathrm{Id}_I \rangle : (g, I) \to (f, I)$ where g is the restriction of f to B.

We then have a notion of restriction by using only the fact that (Fr, E) is a fibration over Ré. Morever the usual equations which must satisfy E and $\lceil$ for Ω-set are automaticly true here. We show this fact in a general manner. In (6), Hyland-Johnston-Pitts define the $\mathcal{P}$-valued set by using equality $[| = |]$ and so a generalization of Ω-valued set. Here we present a generalization of presheaf over a locale by considering presheaf over a fibration.

Let S be a category and $(\mathbb{H}, p)$ a fixed fibration over S.

DEFINITION: *A presheaf over $\mathbb{H}$ is a fibration (F, q) over S such that there exists a cartesian functor E from (F, q) to $(\mathbb{H}, p)$ such that (F, E) be a fibration over $\mathbb{H}$. We will denote (F, E) a presheaf over $\mathbb{H}$; it is understand that $q = p.E$.*

Let (F, E) be a presheaf over $\mathbb{H}$, $X \in F$ and $f : B \to E(X)$ a morphism of $\mathbb{H}$. We call restriction of X to f a cartesian morphism over f; we denote it by $X \lceil f$.

PROPOSITION 2: 1°) $X \lceil \mathrm{Id}_{E(X)} \stackrel{\sim}{=} X$

2°) $E(X \lceil f) = f$

3°) *Denote by Y the domain of $X \lceil f$. Then $X \lceil f.g \stackrel{\sim}{=} Y \lceil g$, where g is a morphism of $\mathbb{H}$ with codomain $E(Y)$.*

PROPOSITION 3: *If (F', E') is an another presheaf over $\mathbb{H}$ and if G is a cartesian functor from (F, E) to (F', E') then $G(X \lceil f) \stackrel{\sim}{=} G(X) \lceil f$.*

If we suppose now that the fibration $(\mathbb{H}, p)$ has finite products we have the particular definition: let $X \in F$ and $B \in \mathbb{H}$ such that $q(X) = p(B)$. We denote by $X \lceil B$ the domain of the cartesian morphism

over $B \wedge E(X) \to E(X)$; we remark that B and $E(X)$ are in the same fibre and so we can take her product. We have the following picture:

If we denote by π_2 the secund projection from $B \wedge E(X)$ to $E(X)$, then $X\lceil B = X\lceil \pi_2$.

PROPOSITION 4: 1°) $X\lceil E(X) \cong X$

2°) $E(X\lceil B) = E(X) \wedge B$

3°) $X\lceil B\lceil C \cong X\lceil B \wedge C$

We remark that these equations are exactly those used for a presheaf on a locale (4).

PROPOSITION 5: *If G is a cartesian functor from (F, E) to an another presheaf (F', E') then $G(X\lceil B) \cong G(X)\lceil B$.*

Let (F, E) be a presheaf over $\mathbb{H}$, X, $Y \in F$ such that $q(X) = q(Y)$. Put $A = E(X)$ and $B = E(Y)$. We say that X and Y are compatible if $X\lceil B \cong Y\lceil A$.

Thus we can see that this notion of presheaf over a fibration is a good notion for the study of partiality. It's developped in a forthcoming paper where we define the sheaf'property. If we have an object ω of S. Thus we call a recursive fibration a presheaf (F, E) over $\mathbb{H}$ such that (F, q) is strongly ω-enumered.

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