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EQUATIONS ON THE SEMIDIRECT PRODUCT OF A FINITE SEMILATTICE BY A $\mathcal{J}$-TRIVIAL MONOID OF HEIGHT k (*)

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Abstract. – *Let* $\mathbf{J}_k$ *denote the kth level of Simon's hierarchy of* $\mathcal{J}$*-trivial monoids. The 1st level* $\mathbf{J}_1$ *is the* $\mathbb{M}$*-variety of finite semilattices. In this paper, we give a complete sequence of equations for the product* $\mathbf{J}_1 \star \mathbf{J}_k$ *generated by all semidirect products of the form* $M \star N$ *with* $M \in \mathbf{J}_1$ *and* $N \in \mathbf{J}_k$. *Results of Almeida imply that this sequence of equations is complete for the product* $\mathbf{J}_1^{k+1}$ *or* $\mathbf{J}_1 \star \ldots \star \mathbf{J}_1$ $(k+1$ *times) generated by all semidirect products of* $k+1$ *finite semilattices and that* $\mathbf{J}_1 \star \mathbf{J}_k$ *is defined by a* finite *sequence of equations if and only if* $k = 1$. *The equality* $\mathbf{J}_1 \star \mathbf{J}_k = \mathbf{J}_1^{k+1}$ *implies that a conjecture of Pin concerning tree hierarchies of* $\mathbb{M}$*-varieties is false.*

Résumé. – *Soit* $\mathbf{J}_k$ *le niveau* k *de la hiérarchie de Simon des monoïdes* $\mathcal{J}$*-triviaux. Le premier niveau* $\mathbf{J}_1$ *est la* $\mathbb{M}$*-variété des monoïdes idempotents et commutatifs ou demi-treillis. Dans cet article, nous donnons une suite complète d'équations pour le produit* $\mathbf{J}_1 \star \mathbf{J}_k$ *engendré par les produits semidirects de la forme* $M \star N$ *avec* $M \in \mathbf{J}_1$ *et* $N \in \mathbf{J}_k$. *Des résultats d'Almeida entraînent que cette suite d'équations est aussi complète pour le produit* $\mathbf{J}_1^{k+1}$ *ou* $\mathbf{J}_1 \star \ldots \star \mathbf{J}_1$ $(k+1$ *fois) engendré par les produits semidirects de* $k+1$ *demi-treillis et que* $\mathbf{J}_1 \star \mathbf{J}_k$ *est défini par une suite* finie *d'équations si et seulement si* $k = 1$. *L'égalité* $\mathbf{J}_1 \star \mathbf{J}_k = \mathbf{J}_1^{k+1}$ *entraîne qu'une conjecture de Pin concernant des hiérarchies d'arbres de* $\mathbb{M}$*-variétés est fausse.*

1. INTRODUCTION

Let $\mathbf{J}_k$ denote the $\mathbb{M}$-variety of $\mathcal{J}$-trivial monoids of height k. The first level $\mathbf{J}_1$ is the $\mathbb{M}$-variety of finite semilattices. In this paper, we give an equational characterization of the product $\mathbf{J}_1 \star \mathbf{J}_k$ generated by all semidirect products of the form $M \star N$ with $M \in \mathbf{J}_1$ and $N \in \mathbf{J}_k$. A result of Almeida [3] gives an equational characterization of the product

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$\mathbf{J}_1 \star \ldots \star \mathbf{J}_1 (k+1$ times) or $\mathbf{J}_1^{k+1}$, which turns out to be our equational characterization of $\mathbf{J}_1 \star \mathbf{J}_k$. The equality $\mathbf{J}_1 \star \mathbf{J}_k = \mathbf{J}_1^{k+1}$ implies that a conjecture of Pin concerning tree hierarchies of $\mathbf{M}$-varieties is false. Almeida [3] implies that $\mathbf{J}_1 \star \mathbf{J}_k$ is defined by a *finite* sequence of equations if and only if $k = 1$. The methods used in this paper were developed by Almeida [1], [2].

1.1 Preliminaries

The reader is referred to the books of Eilenberg [15], Lallement [19] or Pin [20] for terminology not defined in this paper.

Let A be a finite set called an alphabet, whose elements are called letters. We will denote by A^* the *free monoid* over A. The elements of A^* are the finite sequences of letters called words. The empty word (denoted by 1) corresponds to the empty sequence.

Let L be a subset of A^* (or a *language* over A) and $\sim$ be an equivalence relation on A^*. We say that $\sim$ saturates L if L is a union of classes modulo $\sim$ or for every $u, v \in A^*$, $u \sim v$ and $u \in L$ imply $v \in L$.

The *syntactic congruence* of L is the congruence $\sim_L$ on A^* defined by $u \sim_L v$ if and only if for every $x, y \in A^*$, $xuy \in L$ if and only if $xvy \in L$. We can show that $\sim_L$ is the coarsest congruence saturating L. The *syntactic monoid* of L is the quotient monoid $M(L) = A^* / \sim_L$.

Let S and T be semigroups. We say that S is a *quotient* of T if there exists a surjective morphism $\varphi : T \rightarrow S$ and we say that S *divides* T $(S \prec T)$ if S is a quotient of a submonoid of T. The division relation is transitive. The syntactic monoid of a language. L is the smallest monoïd recognizing L, where smallest is taken in the sense of the division relation.

A *variety* V is a class of semigroups closed under division and products. By the well-known theorem of Birkhoff such a variety is defined by equations that must hold for all elements of semigroups in V. Thus equations give rise to varieties.

An $\mathbf{S}$-*variety* is a class of finite semigroups closed under division and finite products and an $\mathbf{M}$-*variety* is a class of finite monoids closed under division and finite products. Equivalently, a class $\mathbf{V}$ of finite monoids is an $\mathbf{M}$-variety if $\mathbf{V}$ satisfies the following two conditions:

- if $T \in \mathbf{V}$ and $S \prec T$, then $S \in \mathbf{V}$;
- if $S, T \in \mathbf{V}$, then $S \times T \in \mathbf{V}$.

Eilenberg has shown the existence of a bijection between the M-varieties and some classes of languages called the $\star$-varieties of languages.

A class $\mathcal{V}$ is a *$\star$-variety of languages* if

• for every alphabet A, $A^* \mathcal{V}$ is a set of recognizable languages over A closed under boolean operations;

• if $\varphi : A^* \to B^*$ is a free monoid morphism, then $L \in B^* \mathcal{V}$ implies $L\varphi^{-1} = \{u \in A^* \mid u\varphi \in L\}$ is in $A^* \mathcal{V}$;

• if $L \in A^* \mathcal{V}$ and $a \in A$, then $a^{-1} L = \{u \in A^* \mid au \in L\}$ and $L a^{-1} = \{u \in A^* \mid ua \in L\}$ are in $A^* \mathcal{V}$.

If $\mathbf{V}$ is an M-variety and A is an alphabet, we denote by $A^* \mathcal{V}$ the set of recognizable languages over A whose syntactic monoid is in $\mathbf{V}$. Equivalently, $A^* \mathcal{V}$ is the set of languages of A^* recognized by a monoid of $\mathbf{V}$. If $\mathcal{V}$ is a $\star$-variety of languages, we denote by $\mathbf{V}$ the M-variety generated by the monoids of the form $M(L)$ where $L \in A^* \mathcal{V}$ for some alphabet A.

A result of Simon enables us to describe the $\star$-variety of languages corresponding to the M-variety of $\mathcal{J}$-trivial monoids denoted by $\mathbf{J}$.

A word $a_1 \ldots a_i \in A^*$ is a subword of a word u of A^* if there exist words $u_0, u_1, \ldots, u_i \in A^*$ such that $u = u_0 a_1 u_1 \ldots a_i u_i$. For each integer $k \geq 0$, we define an equivalence relation $\sim_k$ on A^* by $u \sim_k v$ if and only if u and v have the same subwords of length less than or equal to k. We can verify that $\sim_k$ is a congruence on A^* with finite index. Note that $u \sim_1 v$ if and only if u and v have the same letters. The set of letters that occur in a work u will be denoted by $u\alpha$.

A language L over A is called *piecewise testable* if it is a union of classes modulo $\sim_k$ for some integer k, or equivalently if it is in the boolean algebra generated by all languages of the form $A^* a_1 A^* \ldots a_i A^*$ where $i \geq 0$ and $a_1, \ldots, a_i \in A$. Simon [24] has proved that a language is piecewise testable if and only if its syntactic monoid is $\mathcal{J}$-trivial. For every alphabet A, we will denote by $A^* \mathcal{J}_k$ the boolean algebra generated by all languages of the form $A^* a_1 A^* \ldots a_i A^*$,where $0 \leq i \leq k$ and $a_1, \ldots, a_i \in A$. One can show that $\mathcal{J}_k$ is a $\star$-variety of languages and we will denote by $\mathbf{J}_k$ the corresponding M-variety. The M-variety $\mathbf{J}$ is the union of the M-varieties $\mathbf{J}_k$.

1.2 Product of varieties of semigroups

Let S and T be semigroups. To simplify the notation we will represent S additively (without necessarily supposing that S is commutative) and T multiplicatively.

An *action* of T on S is a function

$$\begin{aligned} T \times S &\to S \\ (t,\, s) &\mapsto ts \end{aligned}$$

satisfying for every $t,\, t' \in T$ and $s,\, s' \in S$:

- $t\,(s + s') = ts + ts'$;
- $t\,(t'\,s) = (tt')\,s$.

Given an action of T on S, the *semidirect product* $S \star T$ is the semigroup defined on $S \times T$ by the multiplication

$$(s,\, t)\,(s',\, t') = (s + ts',\, tt').$$

The multiplication in $S \star T$ is associative. Thus $S \star T$ is a semigroup.

In this paper, we only consider semidirect products $S \star T$ given by actions of T on S that are described by monoid homomorphisms $\varphi : T^1 \to \mathrm{End}\, S$ from T^1 into the monoid of endomorphisms of S. In the terminology adopted by Eilenberg [15], this means that we only consider left unitary actions, that is actions of T on S that satisfy $1\, s = s$ for every $s \in S$. Here T^1 denotes the semigroup $T \cup \{1\}$ obtained from T by adjoining an identity if T does not have one, and $T^1 = T$ otherwise.

If V and W are varieties of semigroups, the product $V \star W$ is the variety generated by all semigroups of the form $S \star T$ with $S \in V$ and $T \in W$. The product of two $\mathbf{S}$-varieties (or $\mathbf{M}$-varieties) is defined analogously. The operation $\star$ defined on varieties is associative.

There remain many problems to be solved on products of $\mathbf{S}$-varieties (or $\mathbf{M}$-varieties). The most important of these is the following. Given two decidable $\mathbf{S}$-varieties (or $\mathbf{M}$-varieties), is the product decidable? A particular case of this problem is well known in the theory of semigroups. Karnofsky and Rhodes [18] have established the decidability of the $\mathbf{M}$-varieties $\mathbf{A} \star \mathbf{G}$ and $\mathbf{G} \star \mathbf{A}$. Here, $\mathbf{A}$ denotes the $\mathbf{M}$-variety of aperiodic monoids and $\mathbf{G}$ the $\mathbf{M}$-variety of groups.

This paper deals in particular with products of the form $\mathbf{J}_1^k$. It is known that $\bigcup_{k \geq 0} \mathbf{J}_1^k$ is the $\mathbf{M}$-variety $\mathbf{R}$ of all finite $\mathcal{R}$-trivial monoids (Stiffler [25]) and that $\mathbf{J}_1^k$ is decidable (Pin [21]).

1.3 Equations on products of varieties of semigroups

Let A^+ be the free semigroup over a denumerable alphabet A and let u, $v \in A^+$. We say that a semigroup S satisfies the equation $u = v$ or the equation $u = v$ holds in S (and we write $S \models u = v$) if for every morphism $\varphi : A^+ \to S$, $u\varphi = v\varphi$. This means that, it we substitute elements of S for the letters in u and v, we reach equalities in S. For example, S is idempotent if it satisfies the equation $x = x^2$ and S is commutative if it satisfies the equation $xy = yx$. For a sequence $\mathcal{E}$ of equations and an equation $u = v$, $\mathcal{E} \vdash u = v$ (and we say $u = v$ is deducible from $\mathcal{E}$) means that for every semigroup S, if $S \models \mathcal{E}$, then $S \models u = v$.

Let $\mathbf{V}(u, v)$ be the class of finite semigroups S satisfying the equation $u = v$. It is easy to show that $\mathbf{V}(u, v)$ is an $\mathbf{S}$-variety.

Let $(u_i, v_i)_{i>0}$ be a sequence of pairs of words of A^+. Consider the following $\mathbf{S}$-varieties:

$$\mathbf{W} = \bigcap_{i>0} \mathbf{V}(u_i, v_i)$$

$$\mathbf{W}' = \bigcup_{I>0} \bigcap_{i \geq I} \mathbf{V}(u_i, v_i).$$

We say that $\mathbf{W}$ is *defined* by the equations $u_i = v_i\,(i > 0)$. This corresponds to the fact that a finite semigroup is in $\mathbf{W}$ if and only if it satisfies the equations $u_i = v_i$ for every $i > 0$. We say that $\mathbf{W}'$ is *ultimately defined* by the equations $u_i = v_i\,(i > 0)$. This corresponds to the fact that a finite semigroup is in $\mathbf{W}'$ if and only if it satisfies the equations $u_i = v_i$ for every i sufficiently large.

The arguments above apply equally well to $\mathbf{M}$-varieties. We only need to replace A^+ by A^* throughout.

Eilenberg and Schützenberger [16] have proved the following result. Every nonempty $\mathbf{M}$-variety is ultimately defined by a sequence of equations, or every $\mathbf{S}$-variety containing the trivial semigroup is ultimately defined by a sequence of equations. If $\mathbf{V}$ is the $\mathbf{S}$-variety ultimately defined by the equations $u_i = v_i$, $i > 0$, then the same equations ultimately define the $\mathbf{M}$-variety consisting of all the monoids in $\mathbf{V}$. Also every $\mathbf{M}$-variety generated by a single monoid is defined by a (finite or infinite) sequence of equations.

Equational characterizations of all the $\mathbf{M}$-varieties $\mathbf{J}_k$ are known [23], [5], [6], [10], [11]. In particular,

• the $\mathbf{M}$-variety $\mathbf{J}_1$ is defined by the equations $x = x^2$ and $xy = yx$, so $\mathbf{J}_1$ is the $\mathbf{M}$-variety of idempotent and commutative monoids;

• the $\mathbf{M}$-variety $\mathbf{J}_2$ is defined by the equations $xyzx = xyxzx$ and $(xy)^2 = (yx)^2$;

• the $\mathbf{M}$-variety $\mathbf{J}_3$ is defined by the equations $xzyxvxwy = xzxyxvxwy$, $ywxvxyzx = ywxvxyxzx$ and $(xy)^3 = (yx)^3$.

DEFINITION 1.1: *Let $k \geq 1$ and let $A = \{x_1, x_2, \dots\}$ be a denumerable alphabet of variables including $x\,(x = x_1)$.*

$\mathcal{E}_k$ is the sequence of all equations (over A) of the form

$$u_i \dots u_1\, v_1 \dots v_j = u_i \dots u_1\, x v_1 \dots v_j$$

where

$$\{x\} \subseteq u_1\,\alpha \subseteq \dots \subseteq u_i\,\alpha$$

$$\{x\} \subseteq v_1\,\alpha \subseteq \dots \subseteq v_j\,\alpha$$

and where $i + j = k$.

THEOREM 1.1 [10]: *Let $k \geq 1$. The $\mathbf{M}$-variety $\mathbf{J}_k$ is defined by $\mathcal{E}_k$.*

These results lead to the following question. Can the $\mathbf{M}$-varieties $\mathbf{J}_k$ be defined by a *finite* sequence of equations? This question has been answered in [11]. The $\mathbf{M}$-varieties $\mathbf{J}_k$ can be defined by a *finite* sequence of equations if and only if $k = 1$, 2 or 3.

Equations are known for the product of the $\mathbf{S}$-variety of semilattices, groups, and $\mathcal{R}$-trivial semigroups by the $\mathbf{S}$-variety of locally trivial semigroups [15]. These results have important applications to language theory [14], [15].

Pin [22] has shown that the $\mathbf{M}$-variety $\mathbf{J}_1 \star \mathbf{J}_1$ is defined by the equations $xux = xux^2$ and $xuyvxy = xuyvyx$. A result of Irastorza [17] shows that the $\mathbf{M}$-varieties $\mathbf{J}_1 \star (Z_k)$ are not defined by finite sequences of equations. Here, (Z_k) denotes the $\mathbf{M}$-variety generated by the cyclic group Z_k of order k which is defined by the equations $x^k = 1$ and $xy = yx$. Almeida [3] has shown that $\mathbf{J}_1^k$ is defined by a finite sequence of equations if and only if $k = 1$ or 2. Ash [4] has shown that $\mathbf{J}_1 \star \mathbf{G} = \mathbf{Inv}$ is defined by the equation $x^\omega y^\omega = y^\omega x^\omega$. The $\mathbf{M}$-variety of groups $\mathbf{G}$ is defined by the equation $x^\omega = 1$, and $\mathbf{Inv}$ denotes the $\mathbf{M}$-variety generated by the inverse semigroups.

2. ON A COMPLETE SEQUENCE OF EQUATIONS FOR $\mathbf{J}_1 \star \mathbf{J}_k$

In this section, in order to simplify the notation, we will denote also by $\mathbf{J}_k$ the S-variety generated by $\mathbf{J}_k$. It will be convenient to denote by $\mathbf{J}_0$ the S-variety defined by the equation $x = y$. In this section, we work essentially with semigroups.

Our results follow from an approach to the semidirect product that was introduced in Almeida [1].

The free object on the set X in the variety generated by an S-variety (or M-variety) $\mathbf{V}$ will be denoted by $F_X \mathbf{V}$. We will also write $F_i \mathbf{V}$ as an abbreviation for $F_{\{x_1, \ldots, x_i\}} \mathbf{V}$. For every $i \geq 1$ and $k \geq 1$, the free object $F_i(\mathbf{J}_k)$ can be viewed as a set of representatives of classes modulo $\sim_k$ of words over $\{x_1, \ldots, x_i\}$. This set is finite. For $i \geq 1$ and $k \geq 1$, let $p_{i,k} : \{x_1, \ldots, x_i\}^+ \to F_i(\mathbf{J}_1 \star \mathbf{J}_k)$ be the canonical projection that maps the letter x_j onto the generator x_j of $F_i(\mathbf{J}_1 \star \mathbf{J}_k)$, and let $q_{i,k} : \{x_1, \ldots, x_i\}^+ \to F_i(\mathbf{J}_k)$ be the canonical projection that maps the letter x_j onto the generator x_j of $F_i(\mathbf{J}_k)$. If $u \in \{x_1, \ldots, x_i\}^+$, then $uq_{i,k}$ can be viewed as a representtive of the class modulo $\sim_k$ of u.

DEFINITION 2.1: *Let $k \geq 1$ and $u \in \{x_1, \ldots, x_i\}^+$.*
$u\,\alpha_{i,k}$ is the set of all pairs of the form

$$(u'\,q_{i,k},\, x) \in (F_i(\mathbf{J}_k))^1 \times \{x_1, \ldots, x_i\}$$

where $u = u'\,xu''$ for some $u', u'' \in \{x_1, \ldots, x_i\}^*$.

In the case of $k = 0$, $(F_i(\mathbf{J}_0))^1 = \{1\}$ and so $u\,\alpha_{i,0} = \{1\} \times u\,\alpha$.

The following lemmas will help us give an equational characterization of $\mathbf{J}_1 \star \mathbf{J}_k$. Lemma 2.1 provides an algorithm to decide when an equation holds in $\mathbf{J}_1 \star \mathbf{J}_k$.

LEMMA 2.1: *Let $k \geq 0$ and $u, v \in \{x_1, \ldots, x_i\}^+$. Then*

$$\mathbf{J}_1 \star \mathbf{J}_k \models u = v$$

if and only if $u\,\alpha_{i,k} = v\,\alpha_{i,k}$.

Proof: For $k = 0$, we have that $\mathbf{J}_1 \models u = v$ if and only if $u\,\alpha = v\,\alpha$. Since $F_i(\mathbf{J}_k)$ is finite for every $i \geq 1$ and $k \geq 1$, a representation of free objects for a semidirect product of S-varieties obtained in [1] implies that $F_i(\mathbf{J}_1 \star \mathbf{J}_k)$ is also finite for every $i \geq 1$ and $k \geq 1$. Moreover, there is an embedding of $F_i(\mathbf{J}_1 \star \mathbf{J}_k)$ into $F_Y(\mathbf{J}_1) \star F_i(\mathbf{J}_k)$ that maps x_j into $((1, x_j), x_j)$. Here $Y = (F_i(\mathbf{J}_k))^1 \times \{x_1, \ldots, x_i\}$ and the action in the semidirect product of the free objects is given by $x_j(s, x_{j'}) = (x_j s, x_{j'})$ for $s \in (F_i(\mathbf{J}_k))^1$. The word $x_{j_1} \ldots x_{j_r}$ is mapped into

$$((1, x_{j_1}) + (x_{j_1}, x_{j_2}) + \ldots + (x_{j_1} \ldots x_{j_{r-1}}, x_{j_r}), x_{j_1} \ldots x_{j_r})$$

Suppose that $\mathbf{J}_1 \star \mathbf{J}_k \models u = v$, or that $up_{i,k} = vp_{i,k}$. This is equivalent to the two conditions $u\alpha_{i,k} = v\alpha_{i,k}$ and $\mathbf{J}_k \models u = v$. Observe that $\mathbf{J}_k \models u = v$ if and only if $uq_{i,k} = vq_{i,k}$. The result follows since $u\alpha_{i,k} = v\alpha_{i,k}$ implies $uq_{i,k} = vq_{i,k}$. □

Let $k \geq 1$. Let $u, v \in \{x_1, \ldots, x_i\}^+$ be such that $u\alpha_{i,k} = v\alpha_{i,k}$. Let $x \in u\alpha$ and consider the first occurrence of x in u.

Case 1. If x is the last letter occurring for the first time in u, then there is a factorization $u = u_1 x u_2$ with $u_1, u_2 \in \{x_1, \ldots, x_i\}^*$, $x \notin u_i \alpha$ and $u_2 \alpha \subseteq (u_1 x)\alpha$. In such a case, since $u\alpha_{i,k} = v\alpha_{i,k}$, there is also a factorization $v = v_1 x v_2$ with $v_1, v_2 \in \{x_1, \ldots, x_i\}^*$ and $x \notin v_1 \alpha$.

Case 2. If x is not the last letter occurring for the first time in u, then there is a factorization $u = u_1 x u_2 y u_3$ with $u_1, u_2, u_3 \in \{x_1, \ldots, x_i\}^*$, $x \notin u_1 \alpha$, $u_2 \alpha, \subseteq (u_1 x)\alpha$ and $y \notin (u_1 x u_2)\alpha$. In such a case, since $u\alpha_{i,k} = v\alpha_{i,k}$, there is also a factorization $v = v_1 x v_2 y v_3$ with $v_1, v_2, v_3 \in \{x_1, \ldots, x_i\}^*$, $x \notin v_1 \alpha$ and $y \notin (v_1 x v_2)\alpha$.

LEMMA 2.2: *In Case 1 and Case 2,* $u_2 \alpha_{i,k-1} = v_2 \alpha_{i,k-1}$.

Proof: Let $u_2 = u_2' z u_2''$ with $z \in \{x_1, \ldots, x_i\}$. Consider the pair $(u_2' q_{i,k-1}, z)$ in $u_2 \alpha_{i,k-1}$. The pair $((u_1 x u_2') q_{i,k}, z)$ is in $u\alpha_{i,k}$. Since $u\alpha_{i,k} = v\alpha_{i,k}$, there is a factorization $v = v' z v''$ with $(u_1 x u_2') q_{i,k} = v' q_{i,k}$. It follows that the $\sim_k$-class of $u_1 x u_2'$ is equal to the $\sim_k$-class of v' and hence $x \in v'\alpha$ and, in Case 2, $y \notin v'\alpha$. Therefore, the chosen occurrence of z in $v = v' z v''$ must be in v_2. There is then a factorization $v_2 = v_2' z v_2''$ such that $v' = v_1 x v_2'$. Hence $(u_2' q_{i,k-1}, z) = (v_2' q_{i,k-1}, z)$ and the pair $(u_2' q_{i,k-1}, z)$ is in $v_2 \alpha_{i,k-1}$. Then inclusion $u_2 \alpha_{i,k-1} \subseteq v_2 \alpha_{i,k-1}$ follows. The reverse inclusion is similar. □

DEFINITION 2.2: *Let $k \geq 1$ and let $A = \{x_1, x_2, x_3, \ldots\}$ be a denumerable alphabet of variables including x and y ($u = x_1$ and $y = x_2$).*

C_k is the sequence of all equations (over A) of the form

$$u_k \ldots u_1 x = u_k \ldots u_1 x^2$$

where

$$\{x\} \subseteq u_1\, \alpha \subseteq \ldots \subseteq u_k\, \alpha$$

$\mathcal{D}_k$ *is the sequence of all equations (over* A*) of the form*

$$u_k \ldots u_1\, xy = u_k \ldots u_1\, yx$$

where

$$\{x,\, y\} \subseteq u_1\, \alpha \subseteq \ldots \subseteq u_k\, \alpha.$$

We define $\mathcal{C}_0$ as the sequence consisting of the equation $x = x^2$ and $\mathcal{D}_0$ the sequence consisting of $xy = yx$.

Let J_k denote the variety of all semigroups that satisfy all the equations in $\mathcal{E}_k$. The variety J_k is locally finite, or every finitely generated semigroup in J_k is finite. For a class $\mathcal{C}$ of semigroups, we denote by $\mathcal{C}^F$ the class of all finite semigroups of $\mathcal{C}$. The equality $\mathbf{J}_k = (J_k)^F$ holds. By [1], if $k \geq 1$, then the equality $(J_1 \star J_k)^F = \mathbf{J}_1 \star \mathbf{J}_k$ holds and $J_1 \star J_k$ is locally finite. Hence $J_1 \star J_k$ is generated by $\mathbf{J}_1 \star \mathbf{J}_k$ and so $F_i\,(\mathbf{J}_1 \star \mathbf{J}_k)$ is the free object on $\{x_1,\, \ldots,\, x_i\}$ in the variety $J_1 \star J_k$.

THEOREM 2.1: *Let* $k \geq 0$. *The variety* $J_1 \star J_k$ *is defined by* $\mathcal{C}_k \cup \mathcal{D}_k$.

Proof: We first want to show that $J_1 \star J_k \models \mathcal{C}_k \cup \mathcal{D}_k$. Let $u,\, v \in \{x_1,\, \ldots,\, x_i\}^+$ be such that $u = v$ is an equation in $\mathcal{D}_k$ (the case of equations in $\mathcal{C}_k$ is similar). By Lemma 2.1, it suffices to show that $u\, \alpha_{i,\,k} = v\, \alpha_{i,\,k}$. Let $u = u_k \ldots u_1\, xy$ and $v = u_k \ldots u_1\, yx$ be such that $\{x,\, y\} \subseteq u_1\, \alpha \subseteq \ldots \subseteq u_k\, \alpha$. Note that

$$((u_k \ldots u_1)\, q_{i,\,k},\, x) = ((u_k \ldots u_1\, y)\, q_{i,\,k},\, x)$$

since the words $u_k \ldots u_1$ and $u_k \ldots u_1\, y$ are $\sim_k$-equivalent. Note also that

$$((u_k \ldots u_1\, x)\, q_{i,\,k},\, y) = ((u_k \ldots u_1)\, q_{i,\,k},\, y)$$

The equality $u\, \alpha_{i,\,k} = v\, \alpha_{i,\,k}$ follows.

Conversely, we want to show that if $u,\, v \in \{x_1,\, \ldots,\, x_i\}^+$ are such that $u\, \alpha_{i,\,k} = v\, \alpha_{i,\,k}$, then $\mathcal{C}_k \cup \mathcal{D}_k \vdash u = v$. So, assume that $u\, \alpha_{i,\,k} = v\, \alpha_{i,\,k}$. Let $x \in u\, \alpha$ and consider the first occurrence of x in u and v. As in Lemma 2.2, we denote by u_1 (respectively v_1) the longest prefix of u (respectively v) in which the letter x does not occur, and we denote by u_2 (reespectively v_2) the longest segment of u (respectively v) following the first occurrence of x in u (respectively v) that does not involve any new letters. By Lemma 2.2, the equality $u_2\, \alpha_{i,\,k-1} = v_2\, \alpha_{i,\,k-1}$ holds. By the inductive hypothesis on

k, we conclude that the equation $u_2 = v_2$ is deducible from $\mathcal{C}_{k-1} \cup \mathcal{D}_{k-1}$. By a result of [3] (Proposition 2.3), since $\mathcal{C}_{k-1} \cup \mathcal{D}_{k-1} \vdash u_2 = v_2$ and $u_2\,\alpha \subseteq (u_1\,x)\,\alpha$, then $\mathcal{C}_k \cup \mathcal{D}_k \vdash u_1\,xu_2 = u_1\,xv_2$.

Let $z \in \{x_1, \ldots, x_i\}$. Let u'(respectively v') be the longest prefix of u (respectively v) before the first occurrence of z. We show that the equation $u' = v'$ is deducible from $\mathcal{C}_k \cup \mathcal{D}_k$. If z is the first letter in u (and so also the first letter in v), then the equation $u' = v'$ becomes $1 = 1$. We assume that it is true for the first occurrence of $z = x$ (as in Lemma 2.2), or $\mathcal{C}_k \cup \mathcal{D}_k \vdash u_1 = v_1$. Here $u_1\,xu_2 = u_1\,xv_2 = v_1\,xv_2$ is deducible from $\mathcal{C}_k \cup \mathcal{D}_k$. If x is the last letter occurring for the first time in u (as in Case 1 of Lemma 2.2), we obtain that the equation $u = v$ is deducible from $\mathcal{C}_k \cup \mathcal{D}_k$. Otherwise, the induction step allows us to proceed until the first occurrence of another letter, say $z = y$ (as in Case 2 of Lemma 2.2). After every letter of u has been found, we obtain the deducibility of the equation $u = v$ from $\mathcal{C}_k \cup \mathcal{D}_k$. $\square$

Since $\mathbf{J}_1 \star \mathbf{J}_k = (J_1 \star J_k)^F$, any sequence of equations for $J_1 \star J_k$ is also a sequence of equations for $\mathbf{J}_1 \star \mathbf{J}_k$.

COROLLARY 2.1: *Let $k \geq 0$. The* S*-variety $\mathbf{J}_1 \star \mathbf{J}_k$ is defined by $\mathcal{C}_k \cup \mathcal{D}_k$.*

Note that if two words u and v form an equation $u = v$ for $\mathbf{J}_1 \star \mathbf{J}_k$, then $u \sim_{k+1} v$. Equations for other S-varieties generalizing the S-varieties $\mathbf{J}_k$ have been built from properties of congruences generalizing the congruences $\sim_k$ (*see* [7], [8], [9], [12]).

Pin has given the equational characterization of $\mathbf{J}_1 \star \mathbf{J}_1$ of Theorem 2.2 and Almeida the characterization of $\mathbf{J}_1^k$ of Theorem 2.3.

THEOREM 2.2. (Pin [22]): *The* S*-variety $\mathbf{J}_1 \star \mathbf{J}_1$ is defined by $\mathcal{C}_1 \cup \mathcal{D}_1$ or equivalently by the two equations $xux = xux^2$ and $xuyvxy = xuyvyx$.*

THEOREM 2.3 (Almeida [3]): *Let $k \geq 0$. The* S*-variety $\mathbf{J}_1^{k+1}$ is defined by $\mathcal{C}_k \cup \mathcal{D}_k$.*

From the preceding results, we deduce the following corollary.

COROLLARY 2.2: *Let $k \geq 0$. The* S*-varieties $\mathbf{J}_1 \star \mathbf{J}_k$ and $\mathbf{J}_1^{k+1}$ are equal and hence the* S*-variety $\mathbf{J}_1 \star \mathbf{J}_k$ is decidable.*

A result of Almeida [3] implies the following.

COROLLARY 2.3: *The* S*-variety $\mathbf{J}_1 \star \mathbf{J}_k$ is defined by a finite sequence of equations if and only if $k = 1$.*

As mentioned at the beginning of this section, we have worked essentially with semigroups in section 2. As explained in [3], since the $\mathbf{S}$-variety generated by the $\mathbf{M}$-variety $\mathbf{J}_k$ is monoidal, results such as Theorems 2.2 and 2.3, and Corollaries 2.1, 2.2 and 2.3 can be translated to results on the $\mathbf{M}$-varieties $\mathbf{J}_1 \star \mathbf{J}_k$ and $\mathbf{J}_1^{k+1}$.

3. ON A CONJECTURE OF PIN

Theorem 3.1 gives a new proof that a conjecture of Pin concerning tree-hierarchies of $\mathbf{M}$-varieties is false (another proof was given in [13] using different techniques). Let $M_1, \ldots, M_k$ be finite monoids. The Schützenberger product of $M_1, \ldots, M_k$, denoted by $\Diamond_k (M_1, \ldots, M_k)$, is the submonoid of upper triangular $k \times k$ matrices with the usual multiplication of matrices, of the form $x = (x_{ij})$, $1 \leq i, j \leq k$, in which the (i, j)-entry is a subset of $M_1 \times \ldots \times M_k$ and all of whose diagonal entries are singletons, that is

1. $x_{ij} = \varnothing$ if $i > j$;
2. $x_{ii} = \{(1, \ldots, 1, m_i, 1, \ldots, 1)\}$ for some $m_i \in M_i$ (here, m_i is the ith component in the k-tuple);
3. $$x_{ij} \subseteq \{(m_1, \ldots, m_k) \in M_1 \times \ldots \times M_k \mid m_1 = \ldots = m_{i-1} = 1 = m_{j+1} = \ldots = m_k\}$$

(here, 1 is the identity of $M_1, \ldots, M_k$).

Condition (2) allows to identify x_{ii} with an element of M_i and Condition (3) x_{ij} with a subset of $M_i \times \ldots \times M_j$. If

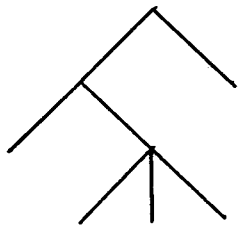
$$\bar{m} = (m_i, \ldots, m_j) \in M_i \times \ldots \times M_j$$

and

$$\bar{m}' = (m'_{i'}, \ldots, m'_{j'}) \in M_{i'} \times \ldots \times M_{j'},$$

then $\bar{m}\bar{m}' = (m_i, \ldots, m_{j-1}, m_j m'_{i'}, m'_{i'+1}, \ldots, m'_{j'})$ if $j = i'$, and is undefined otherwise. This multiplication is extended to sets in the usual fashion; addition is given by set union.

We will denote by $\mathcal{T}$ the set of trees on the alphabet $\{a, \bar{a}\}$. Formally, $\mathcal{T}$ is the set of words in $\{a, \bar{a}\}^*$ congruent to 1 in the congruence generated by the relation $a\bar{a} = 1$. Intuitively, the words in $\mathcal{T}$ are obtained as follows: we draw a tree and starting from the root we code a for going down and $\bar{a}$ for going up. For example,



is coded by $aa\bar{a}aa\bar{a}a\bar{a}a\bar{a}\bar{a}\bar{a}a\bar{a}$. The number of leaves of a word t in $\{a, \bar{a}\}^*$, denoted by $l(t)$ is by definition the number of occurrences of the factor $a\bar{a}$ in t. Each tree t factors uniquely into $t = at_1\, \bar{a}at_2\, \bar{a} \ldots at_k\, \bar{a}$ where $k \geq 0$ and where the t_i's are trees. Let t be a tree and let $t = t_1\, at_2\, \bar{a}t_3$ be a factorization of t. We say that the occurrences of a and $\bar{a}$ defined by this factorization are related if t_2 is a tree. Let t and t' be two trees. We say that t is *extracted* from t' if t is obtained from t' by removing in t' a certain number of related occurrences of a and $\bar{a}$. We now give Pin's tree hierarchy construction using Schützenberger's product.

To each tree t and to each sequence $\mathbf{V}_1, \ldots, \mathbf{V}_{l(t)}$ of $\mathbf{M}$-varieties is associated an $\mathbf{M}$-variety $\Diamond_t(\mathbf{V}_1, \ldots, \mathbf{V}_{l(t)})$ defined recursively by:

1. $\Diamond_1(\mathbf{V}) = \mathbf{V}$ for every $\mathbf{M}$-variety $\mathbf{V}$;

2. if $t = at_1\, \bar{a}at_2\, \bar{a} \ldots at_k\, \bar{a}$ with $k \geq 0$ and $t_1, \ldots, t_k \in \mathcal{T}$, $\Diamond_t(\mathbf{V}_1, \ldots, \mathbf{V}_{l(t)})$ is the $\mathbf{M}$-variety of monoids that divide some $\Diamond_k(M_1, \ldots, M_k)$ with $M_1 \in \Diamond_{t_1}(\mathbf{V}_1, \ldots, \mathbf{V}_{l(t_1)}), \ldots, M_k \in \Diamond_{t_k}(\mathbf{V}_{l(t_1)+\ldots+l(t_{k-1})+1}, \ldots, \mathbf{V}_{l(t_1)+\ldots+l(t_k)})$.

When $\mathbf{V}_1 = \ldots = \mathbf{V}_{l(t)} = \mathbf{V}$, we denote simply by $\Diamond_t(\mathbf{V})$ the $\mathbf{M}$-variety $\Diamond_t(\mathbf{V}_1, \ldots, \mathbf{V}_{l(t)})$. More generally, if T is a language contained in $\mathcal{T}$, we denote by $\Diamond_T(\mathbf{V})$ the smallest $\mathbf{M}$-variety containing the $\mathbf{M}$-varieties $\Diamond_t(\mathbf{V})$ with $t \in T$.

Let $\mathbf{I}$ denote the trivial $\mathbf{M}$-variety. In [21], the following equalities are shown: $\Diamond_{(a\bar{a})^{k+1}}(\mathbf{I}) = \mathbf{J}_k$ and $\Diamond_{(a\bar{a})^*}(\mathbf{I}) = \mathbf{J}$. Also, it is shown there that if $\mathbf{V}$ is an arbitrary $\mathbf{M}$-variety, then $\Diamond_{(a\bar{a})^2}(\mathbf{V}, \mathbf{I}) = \mathbf{J}_1 \star \mathbf{V}$.

Among the many problems concerning these tree hierarchies, is the comparison between the $\mathbf{M}$-varieties inside a hierarchy. More precisely, the problem consists in comparing the different $\mathbf{M}$-varieties $\Diamond_t(\mathbf{V})$ (or even $\Diamond_T(\mathbf{V})$). A partial result and a conjecture on this problem was given in Pin [21]. It was shown that for every $\mathbf{M}$-variety $\mathbf{V}$, if t is extracted from t', then $\Diamond_t(\mathbf{V}) \subseteq \Diamond_{t'}(\mathbf{V})$, and it was conjectured that if $t, t' \in T'$, $\Diamond_t(\mathbf{I}) \subseteq \Diamond_{t'}(\mathbf{I})$ if and only if t is extracted from t'. Here, T' denotes the set of trees in which each node is of arity different from 1.

THEOREM 3.1: *The above conjecture is false.*

Proof: To see this, let $k > 1$ and let $t = a^{k+1}(\bar{a}a\bar{a})^{k+1}$ and $t' = a(a\bar{a})^{k+1}\bar{a}a\bar{a}$. The equalities $\Diamond_t(\mathbf{I}) = \mathbf{J}_1^{k+1}$ and $\Diamond_{t'}(\mathbf{I}) = \Diamond_{(a\bar{a})^2}(\mathbf{J}_k, \mathbf{I}) = \mathbf{J}_1 \star \mathbf{J}_k$ hold. But $\mathbf{J}_1 \star \mathbf{J}_k = \mathbf{J}_1^{k+1}$ by Corollary 2.2 (M-variety version), and it is easy to verify that the tree t is not extracted from the tree t'. □

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