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UNIFORMLY GROWING BACKTRACK TREES (*)

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Abstract. – *We consider a general additive weight on a family of non-regularly distributed binary trees introduced by P. W. Purdom for the purpose of modelling backtrack trees. This weight depends on the structure of the subtrees and on weight functions defined on the number of leaves appearing in the tree and its subtrees. Choosing particular weight functions, the corresponding weight is a characteristic parameter of the tree.*
We shall derive a general approach to the computation of the average weight for arbitrary weight functions. This general result implies exact and asymptotic expressions for many types of average weights defined on the considered class of trees if the weight functions are arbitrary polynomials in the number of leaves.

Keywords: Trees, asymptotic analysis, additive weight.

Résumé. – *Nous considérons un poids additif général sur une famille d'arbres binaires irréguliers introduite par P. W. Purdom dans le but de modéliser des arbres à retours en arrière. Ce poids dépend de la structure des sous-arbres et de fonctions de poids définies sur le nombre de feuilles figurant dans l'arbre et dans ses sous-arbres. Pour un choix particulier de fonctions de poids, le poids correspondant est un paramètre caractéristique de l'arbre.*
Nous développons une approche générale pour le calcul du poids moyen pour des fonctions de poids arbitraires. Ce résultat général fournit des expressions exactes et asymptotiques pour de nombreux types de poids moyen définis sur la classe d'arbres considérée lorsque les fonctions de poids sont des polynômes quelconques en le nombre de feuilles.

Mots clés : Arbres, analyse asymptotique, poids additif.

1. INTRODUCTION AND BASIC DEFINITIONS

Let T be an *extended binary tree* ([8], p. 399) with the set of internal nodes, $I(T)$, the nonempty set of leaves, $L(T)$, and the root $r(T) \in I(T) \cup L(T)$. Throughout this paper we shall use the convention that the one-node tree has no internal nodes and exactly one leaf. For any two nodes $u, v \in I(T) \cup L(T)$, $d(u, v)$ stands for the distance from u to v defined as the length of the shortest path (=number of nodes in the path minus

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one) from u to v. A node $x \in I(T) \cup L(T)$ with $d(r(T), x) = l$ has the level l. The set of internal nodes and leaves appearing in T at level l is denoted by $I_l(T)$ and $L_l(T)$, respectively. The three T has the height h if the maximum level of a node is equal to h.

In [9], a family of trees has been introduced in order to estimate the number of nodes of a backtrack tree by doing partial backtrack search. This family of trees $\mathcal{F}_p(h)$, $p \in [0, 1]$, $h \in \mathbb{N}_0$, consists of all extended binary trees with height less than or equal to h, where each tree $T \in \mathcal{F}_p(h)$ is associated with a positive real number $\varphi_{p,h}(T)$ recursively defined by:

(*a*) If T is the one-node tree then $\varphi_{p,h}(T) := p\,\delta_{h,0} + 1 - p$, $h \geq 0$;

(*b*) If T has the left subtree $T_1 \in \mathcal{F}_p(h-1)$ and the right subtree $T_2 \in \mathcal{F}_p(h-1)$ then $\varphi_{p,h}(T) := p\,\varphi_{p,h-1}(T_1)\,\varphi_{p,h-1}(T_2)$, $h \geq 1$.

The number $\varphi_{p,h}(T)$ is called the *p-weight* of the tree T. The trees $T \in \mathcal{F}_p(h)$ with $h \leq 2$ are drawn in Figure 1. It is not hard to see that for each $(p, h) \in [0, 1] \times \mathbb{N}_0$, the p-weights $\varphi_{p,h}(T)$ define a probability distribution on the set $\mathcal{F}_p(h)$ (*see* [9]; Lemma 1). In this paper, we shall derive explicit and asymptotic expressions for the expected value of the following parameters defined on a tree $T \in \mathcal{F}_p(h)$:

Figure 1. – All trees $T \in \mathcal{F}_p(h)$ for $h \leq 2$. The root of a tree T is marked by its p-weight $\varphi_{p,h}(T)$. q stands for $(1-p)$.

- the degree $D(T)$ of the root;
- the number of internal [external] nodes $IN(T)$ [$LE(T)$];
- the left [right] branch length $LBL(T)$ [$RBL(T)$];
- the number of internal [external] nodes $IL(T)$ [$LL(T)$] appearing in the left subtree of the root;
- the number of internal [external] nodes $IR(T)$ [$LR(T)$] appearing in the right subtree of the root;
- the number of paths between internal [external; internal and external] nodes $IP(T)$ [$LP(T)$; $ILP(T)$];
- the number of root-free paths between internal [external; internal and external] nodes $IP_r(T)$ [$LP_r(T)$; $ILP_r(T)$];
- the internal [external] path length $IPL(T)$ [$EPL(T)$];
- the internal [external; internal-external] free path length $IFPL(T)$ [$EFPL(T)$; $IEFPL(T)$].

The formal definition of each of these quantities is presented in the second column of Table 1. In order to investigate these parameters, we use the general concept of "additive weights" introduced by the author in [4, 5, 6]. For a given tree $T \in \mathcal{F}_p(h)$, the *weight* $w_p(T)$ is recursively defined as follows:

Let $c_1, c_2 \in \mathbb{R}_+$ be two constants and let $g, \Phi_1, \Phi_2 : \mathbb{N}_0 \to \mathbb{R}$ be given mappings, the so-called *weight functions*.

(*a*) If $T \in \mathcal{F}_p(h)$ is the one-node tree then $w_p(T) := g(1)$;

(*b*) If $T \in \mathcal{F}_p(h)$ has the left subtree $T \in \mathcal{F}_p(h-1)$ and the right subtree $T_2 \in \mathcal{F}_p(h-1)$, $h \geq 1$, then

$$\begin{aligned} w_p(T) : &= c_1 w_p(T_1) + c_2 w_p(T_2) + g(|L(T)|) \\ &+ \Phi_1(|L(T_1)|) + \Phi_2(|L(T_2)|). \end{aligned}$$

All considered parameters can be characterized in a recursive way by choosing special values for c_1 and c_2, and special functions for g, Φ_1, Φ_2. The particular choice of these quantities for each parameter is given in Table 1.

First, we present a general approach to the computation of the average weight $\mathrm{w}_p(h)$ of a tree $T \in \mathcal{F}_p(h)$ for *arbitrary* weight functions g, Φ_1, Φ_2. It turns out that $\mathrm{w}_p(h)$ satisfies an inhomogeneous linear recurrence with constant coefficients (Theorem 1). If the weight functions are polynomials then the inhomogeneous part of that recurrence is simply a linear combination of the derivatives of a fixed function defined by another nonlinear recurrence (Theorem 2). Fortunately, it is possible to derive an asymptotic equivalent to these derivatives. This observation leads to an asymptotic equivalent to the average weight $\mathrm{w}_p(h)$ provided that the weight functions are polynomials (Theorem 3). In the main, we find that $\mathrm{w}_p(h)$ grows at most linearly in h if $p < 0.5$, and polynomially in h if $p = 0.5$; if $p > 0.5$ then $\mathrm{w}_p(h)$ has an exponential growth in h.

In the last section, we shall apply these rather general results to the weights defined in Table 1. We obtain exact and asymptotic expressions for these parameters together with some non-obvious relations (Tables 2, 3).

TABLE 1

The definition of distinguished parameters together with their representation as an additive weight. Here, T is a tree in $\mathcal{F}_p(h)$ with $r := r(T)$, $I := I(T)$, $L := L(T)$, the left subtree $T_1 \in \mathcal{F}_p(h-1)$ and the right subtree $T_2 \in \mathcal{F}_p(h-1)$, where $I_i := I(T_i)$, $L_i := L(T_i)$ and $r_i := r(T_i)$, $i \in \{1, 2\}$. The leftmost [rightmost] leaf of the subtree with the root v is denoted by a_v [b_v].

		characterized by the weight $w_p(T)$ with				
Parameter	Definition	c_1	c_2	$g(m)$	$\Phi_1(m)$	$\Phi_2(m)$
$D(T)$	$2(1-\delta_{1,\|I \cup L\|})$	0	0	$2(1-\delta_{1,m})$	0	0
$IN(T)$	$\|I\|$	1	1	$1-\delta_{m,1}$	0	0
$LE(T)$	$\|L\|$	1	1	$\delta_{m,1}$	0	0
$LBL(T)$	$d(r, a_r)$	1	0	0	1	0
$RBL(T)$	$d(r, b_r)$	0	1	0	0	1
$IL(T)$	$\|I_1\|$	0	0	0	$m-1$	0
$IR(T)$	$\|I_2\|$	0	0	0	0	$m-1$
$LL(T)$	$\|L_1\|$	0	0	0	m	0
$LR(T)$	$\|L_2\|$	0	0	0	0	m
$IP(T)$	$\binom{\|I\|}{2}$	0	0	$\binom{m-1}{2}$	0	0
$LP(T)$	$\binom{\|L\|}{2}$	0	0	$\binom{m}{2}$	0	0
$ILP(T)$	$\|I\|\,\|L\|$	0	0	$m(m-1)$	0	0
$IP_r(T)$	$\binom{\|I_1\|}{2}+\binom{\|I_2\|}{2}$	0	0	0	$\binom{m-1}{2}$	$\binom{m-1}{2}$
$LP_r(T)$	$\binom{\|L_1\|}{2}+\binom{\|L_2\|}{2}$	0	0	0	$\binom{m}{2}$	$\binom{m}{2}$
$ILP_r(T)$	$\|I_1\|\,\|L_1\|+\|I_2\|\,\|L_2\|$	0	0	0	$m(m-1)$	$m(m-1)$
$IPL(T)$	$\sum_{v \in I} d(v, r)$	1	1	0	$m-1$	$m-1$
$EPL(T)$	$\sum_{v \in L} d(v, r)$	1	1	0	m	m
$IFPL(T)$ [1]	$\frac{1}{2} \sum_{(u,v) \in I \times I} d(u, v)$	1	1	0	$(m-1)^2$	$(m-1)^2$
$EFPL(T)$ [2]	$\frac{1}{2} \sum_{(u,v) \in L \times L} d(u, v)$	1	1	0	m^2	m^2
$IEFPL(T)$ [3]	$\frac{1}{2} \sum_{(u,v) \in I \times L} d(u, v)$	1	1	0	$2m(m-1)$	$2m(m-1)$

[1] $IFPL(T)$ and $w_p(T)$ are interrelated by $w_p(T) = |I|\,IPL(T) - IFPL(T)$ (*see* [4, 5]).
[2] $EFPL(T)$ and $w_p(T)$ are interrelated by $w_p(T) = |L|\,EPL(T) - EFPL(T)$ (*see* [4, 5]).
[3] $IEFPL(T)$ and $w_p(T)$ are interrelated by $w_p(T) = |I|\,EPL(T) + |L|\,IPL(T) - IEFPL(T)$ (*see* [4, 5]).

2. A GENERAL APPROACH TO THE COMPUTATION OF THE AVERAGE WEIGHT

In this section, we shall present a general approach to the computation of the average weight $\mathbf{w}_p(h)$ of a tree $T \in \mathcal{F}_p(h)$. The observations stated in the following lemma can be proved by induction on h using the recursive definition of the p-weight $\varphi_{p,h}(T)$ presented in the preceding section.

LEMMA 1: *Let $h \in \mathbb{N}_0$, $T \in \mathcal{F}_p(h)$ and $\varphi_{p,h}(T)$ be the p-weight of T.*

(a) We have: $\varphi_{p,h}(T) = p^{|I(T)|}(1-p)^{|L(T)|-|L_h(T)|}$

(b) The sequence $\varphi_{p,h}(T)$, $T \in \mathcal{F}_p(h)$, is a probability distribution on $\mathcal{F}_p(h)$. □

An inspection of the preceding lemma shows that we obtain big bushy trees with high probability if p is near 1 and little short trees with high probability if p is near 0; for p near 0.5, we obtain long skinny trees with high probability. Henceforth, we say that the tree $T \in \mathcal{F}_p(h)$ has the probability $\varphi_{p,h}(T)$ explicitly given in part (a) of Lemma 1.

Now, let $\mathcal{F}_p^{(h)}(m, k) := \{T \in \mathcal{F}_p(h) \,|\, (|L(T)|, |L_h(T)|) = (m, k)\}$ be the set of all trees $T \in \mathcal{F}_p(h)$ with m leaves and k leaves at level h. Note that each $T \in \mathcal{F}_p^{(h)}(m, k)$ has the probability $p^{m-1}(1-p)^{m-k}$. Introducing the generating function

$$\Pi_{p,h}(z, y) := \sum_{m \geq 1} \sum_{k \geq 0} t_p^{(h)}(m, k)\, z^m y^k \tag{1}$$

of the numbers $t_p^{(h)}(m, k) := |\mathcal{F}_p^{(h)}(m, k)|$, we find

$$\Pi_{p,0}(z, y) = zy, \qquad \Pi_{p,h}(z, y) = z + \Pi_{p,h-1}^2(z, y), \quad h \geq 1. \tag{2}$$

The initial condition of this recurrence follows from the relation $t_p^{(h)}(m, k) = \delta_{m,1}\delta_{k,1}$. The recurrence reflects the fact that a tree $T \in \mathcal{F}_p^{(h)}(m, k)$ is either the one-node tree [giving the contribution z to $\Pi_{p,h}(z, y)$] or a tree with a left subtree $T_1 \in \mathcal{F}_p^{(h-1)}(m_1, k_1)$ and a right subtree $T_2 \in \mathcal{F}_p^{(h-1)}(m_2, k_2)$, where $m_1 + m_2 = m$ and $k_1 + k_2 = k$ [giving the contribution $\Pi_{p,h-1}^2(z, y)$ to $\Pi_{p,h}(z, y)$]. Note that $p^{-1}\Pi_{p,h}(p(1-p), (1-p)^{-1}) = 1$ for all $h \in \mathbb{N}_0$ because $\varphi_{p,h}(T)$ is a probability distribution on $\mathcal{F}_p(h)$.

Next, let $\mathcal{F}_p^{(h)}(m, k, w) := \{T \in \mathcal{F}_p^{(h)}(m, k) \,|\, w_p(T) = w\}$ be the set of all trees $T \in \mathcal{F}_p(h)$ with m leaves, k leaves at level h and weight w.

The cardinality of $\mathcal{F}_p^{(h)}(m, k, w)$ is denoted by $t_p^{(h)}(m, k, w)$. Since each $T \in \mathcal{F}_p^{(h)}(m, k, w)$ occurs with probability $p^{m-1}(1-p)^{m-k}$, the average weight of a tree $\mathcal{F}_p(h)$ is given by

$$\mathbf{w}_p(h) = \sum_{T \in \mathcal{F}_p(h)} w(T)\,\varphi_{p,h}(T) = \sum_{m \geq 1} \sum_{k \geq 0} \sum_{w \geq 0} \sum_{T \in \mathcal{F}_p^{(h)}(m,k,w)} w\,\varphi_{p,h}(T)$$
$$= \sum_{m \geq 1} \sum_{k \geq 0} \sum_{w \geq 0} w\, t_p^{(h)}(m, k, w)\, p^{m-1}(1-p)^{m-k}.$$

Hence, introducing the generating function

$$W_{p,h}(z, y, x) := \sum_{m \geq 1} \sum_{k \geq 0} \sum_{w \geq 0} t_p^{(h)}(m, k, w)\, z^m\, y^k\, x^w \tag{3}$$

of the numbers $t_p^{(h)}(m, k, w)$, then we find for the average weight $\mathbf{w}_p(h)$

$$\mathbf{w}_p(h) = p^{-1} \frac{\partial}{\partial x} W_{p,h}(z, y, x) \Big|_{(z,y,x)=(p(1-p),(1-p)^{-1},1)}. \tag{4}$$

It remains to derive an expression for the function $W_{p,h}(z, y, x)$. This will be done in the proof of the following theorem.

THEOREM 1: *The average weight* $\mathbf{w}_p(h)$ *of a tree* $T \in \mathcal{F}_p(h)$ *is recursively given by*

$$\mathbf{w}_p(0) = g(1)$$
$$\mathbf{w}_p(h) = p(c_1 + c_2)\,\mathbf{w}_p(h-1) + \sum_{m \geq 1} \sum_{k \geq 0} [g(m)\, t_p^{(h)}(m, k)$$
$$+ p(\Phi_1(m) + \Phi_2(m))\, t_p^{(h-1)}(m, k)]\, p^{m-1}(1-p)^{m-k}.$$

Proof: Obviously, $W_{p,0}(z, y, x) = zyx^{g(1)}$. Thus, $\mathbf{w}_p(0) = g(1)$ by (4). Now, let $h \geq 1$. A tree $T \in \mathcal{F}_p^{(h)}(m, k, w)$ has m leaves, k leaves at level h and weight w if and only if the left [right] subtree $T_1 \in \mathcal{F}_p(h-1)$ [$T_2 \in \mathcal{F}_p(h-1)$] has m_1 [m_2] leaves, k_1 [k_2] leaves at level $h-1$ and weight w_1 [w_2], where $m_1 + m_2 = m$, $k_1 + k_2 = k$ and $w - g(m) = c_1 w_1 + c_2 w_2 + \Phi_1(m_1) + \Phi_2(m_2)$. Translating this fact into terms of the generating functions

$$E_{p,h}^{(i)}(z, y, x) := \sum_{m\geq 1}\sum_{k\geq 0}\sum_{w\geq 0} t_p^{(h)}(m, k, w)\, z^m\, y^k\, x^{c_i w+\Phi_i(m)},$$

$$i \in \{1, 2\},$$

and

$$R_{p,h}(z, y, x) := \sum_{m\geq 1}\sum_{k\geq 0}\sum_{w\geq 0} t_p^{(h)}(m, k, w)\, z^m\, y^k\, x^{w-g(m)},$$

we immediately find the relation

$$R_{p,h}(z, y, x) = z + E_{p,h-1}^{(1)}(z, y, x)\, E_{p,h-1}^{(2)}(z, y, x), \qquad h \geq 1. \quad (5)$$

Since $\sum_{w\geq 0} t_p^{(h)}(m, k, w) = t_p^{(h)}(m, k)$, we have $R_{p,h}(z, y, 1) = E_{p,h}^{(1)}(z, y, 1) = E_{p,h}^{(2)}(z, y, 1) = \Pi_{p,h}(z, y)$, where $\Pi_{p,h}(z, y)$ is defined in (1). Using these relations and taking the partial derivative on both sides of (5), we find by an elementary computation

$$\begin{aligned}&\frac{\partial}{\partial x} W_{p,h}(z, y, x)\Big|_{x=1} - \sum_{m\geq 1}\sum_{k\geq 0} g(m)\, t_p^{(h)}(m, k)\, z^m\, y^k\\ &= \Bigg[(c_1 + c_2)\frac{\partial}{\partial x} W_{p,h-1}(z, y, x)\Big|_{x=1}\\ &\quad + \sum_{m\geq 1}\sum_{k\geq 0}(\Phi_1(m) + \Phi_2(m))\, t_p^{(h-1)}(m, k)\, z^m\, y^k\Bigg]\Pi_{p,h-1}(z, y). \quad (6)\end{aligned}$$

Multiplying this equation by p^{-1} and setting $(z, y) := (p(1-p), (1-p)^{-1})$, we obtain by (4)

$$\begin{aligned}&\mathbf{w}_p(h) - \sum_{m\geq 1}\sum_{k\geq 0} g(m)\, t_p^{(h)}(m, k)\, p^{m-1}(1-p)^{m-k}\\ &= \Bigg[(c_1 + c_2)\,\mathbf{w}_p(h-1) + \sum_{m\geq 1}\sum_{k\geq 0}(\Phi_1(m)\\ &\quad + \Phi_2(m))\, t_p^{(h-1)}(m, k)\, p^{m-1}(1-p)^{m-k}\Bigg]\Pi_{p,h-1}(p(1-p), (1-p)^{-1}).\end{aligned}$$

Since $\Pi_{p,h}(p(1-p),(1-p)^{-1}) = p$ for all $h \geq 0$ this equation is equivalent to the expression presented in our theorem. □

The general result presented in Theorem 1 shows that the average weight $\mathrm{w}_p(h)$ satisfies an inhomogeneous linear recurrence with constant coefficients. The inhomogeneous part of that recurrence depends on the weight functions $\Phi_i(m)$, $i \in \{1, 2\}$ and $g(m)$. If we restrict these functions to polynomials in the variable m then we can express the inhomogeneous part by the function $F_{p,h}(z) := p^{-1}\Pi_{p,h}(p(1-p)z,(1-p)^{-1})$. We obtain the following result.

THEOREM 2: *Assume that the weight functions $\Phi_i(m)$, $i \in \{1, 2\}$, and $g(m)$ have the representations*

$$g(m) := \sum_{0\leq\lambda\leq d} g_\lambda m^\lambda \quad \textit{and} \quad \Phi_i(m) =: \sum_{0\leq\lambda\leq d_i} f_\lambda^{(i)} m^\lambda, \qquad i \in \{1, 2\}.$$

Furthermore, let $\rho := \max\{d_1, d_2, d\}$. The average weight $\mathrm{w}_p(h)$ of a tree $T \in \mathcal{F}_p(h)$ is recursively given by

$$\begin{aligned}\mathrm{w}_p(0) &= g(1)\\ \mathrm{w}_p(h) &= p(c_1+c_2)\mathrm{w}_p(h-1) + \sum_{0\leq\lambda\leq\rho}\sum_{0\leq j\leq\lambda} \mathcal{S}_\lambda^{(j)}\\ &\quad\times [g_\lambda F_{p,h}^{(j)}(1) + p(f_\lambda^{(1)}+f_\lambda^{(2)})F_{p,h-1}^{(j)}(1)], \quad h\geq 1,\end{aligned}$$

where $\mathcal{S}_\lambda^{(j)}$ is a Stirling number of the second kind and $F_{p,h}^{(j)}$ denotes the j-th derivative with respect to z of the function $F_{p,h}(z)$ defined by the recurrence

$$F_{p,0}(z) = z, \qquad F_{p,h}(z) = (1-p)z + pF_{p,h-1}^2(z), \qquad h \geq 1.$$

Proof: Inserting the given representations of $g(m)$ and $\Phi_i(m)$, $i \in \{1, 2\}$, into the recurrence for $\mathrm{w}_p(h)$ presented in Theorem 1 and using the well-known identity $m^\lambda = \sum_{0\leq j\leq\lambda} j!\binom{m}{j}\mathcal{S}_\lambda^{(j)}$ as well as the definition of $\Pi_{p,h}(z, y)$ given in (1), we find for $h \geq 1$

$$\begin{aligned}\mathrm{w}_p(h) &= p(c_1+c_2)\mathrm{w}_p(h-1) + \sum_{0\leq\lambda\leq\rho}\sum_{0\leq j\leq\lambda}\mathcal{S}_\lambda^{(j)}\\ &\quad\left[g_\lambda \frac{d^j}{dz^j}\hat{F}_{p,h}(1) + p(f_\lambda^{(1)}+f_\lambda^{(2)})\frac{d^j}{dz^j}\hat{F}_{p,h-1}(1)\right],\end{aligned}$$

where $\hat{F}_{p,h}(z) := p^{-1}\,\Pi_{p,h}(p(1-p)z, (1-p)^{-1})$. Translating the recurrence for $\Pi_{p,h}(z, y)$ stated in (2) into terms of the function $\hat{F}_{p,h}(z)$, we immediately find that $\hat{F}_{p,h}(z)$ obeys the same recurrence as $F_{p,h}(z)$ introduced in our theorem. Thus, $\hat{F}_{p,h} \equiv F_{p,h}$. This completes the proof. □

Remark 1: Since $F_{p,h}(1) = 1$, we find by the recurrence for $F_{p,h}(z)$ presented in Theorem 2

$$F'_{p,0}(1) = 1, \qquad F'_{p,h}(1) = (1-p) + 2p\,F'_{p,h-1}(1)$$

and for $s \geq 2$

$$F^{(s)}_{p,0}(1) = 0, \quad F^{(s)}_{p,h}(1) = 2p \sum_{0 \leq j < s} \binom{s-1}{j} F^{(j)}_{p,h-1}(1)\, F^{(s-j)}_{p,h-1}(1). \quad (7)$$

The recurrence for $F'_{p,h}(1)$ can be solved by iteration. We find

$$F'_{p,h}(1) = \begin{cases} \dfrac{p}{2p-1}(2p)^h + \dfrac{1-p}{1-2p} & \text{if } p \in [0, 1]\backslash\left\{\dfrac{1}{2}\right\} \\ \dfrac{1}{2}h + 1 & \text{if } p = \dfrac{1}{2}. \end{cases} \quad (8)$$

Solving the recurrence (7) for $s := 2$, we find by (8) the exact expression

$$F''_{p,h}(1) = \begin{cases} \dfrac{p^2}{(2p-1)^3}(2p)^{2h} - \dfrac{2p(p-1)^2}{(2p-1)^3} \\ \quad + \dfrac{p[(p-1)(2h+1)-1]}{(2p-1)^2}(2p)^h \\ \qquad\qquad \text{if } p \in [0, 1]\backslash\left\{\dfrac{1}{2}\right\} \\ \dfrac{h}{24}(2h^2 + 9h + 13) \quad \text{if } p = \dfrac{1}{2}. \end{cases} \quad (9)$$

□

All the weights introduced in Table 1 [except $D(T)$, $IN(T)$ and $LE(T)$] have weight functions $g(m)$ and $\Phi_i(m)$, $i \in \{1, 2\}$, which are polynomials in m of degree less than or equal to two. Restricting Theorem 2 to this case, we obtain the following result by (8), (9) and the relation $F_{p,h}(1) = 1$.

COROLLARY 1: *Assume that the weight functions $\Phi_i(m)$, $i \in \{1, 2\}$, and $g(m)$ have the representations*

$$g(m) = g_0 + g_1 m + g_2 m^2 \quad \text{and} \quad \Phi_i(m) = f_0^{(i)} + f_1^{(i)} m + f_2^{(i)} m^2.$$

Moreover, let $\kappa_\lambda := f_\lambda^{(1)} + f_\lambda^{(2)}$ *and* $c := c_1 + c_2$. *The average weight* $\mathbf{w}_p(h)$ *of a tree* $T \in \mathcal{F}_p(h)$ *is recursively given by*

$$\mathbf{w}_p(0) = g(1)$$

$$\mathbf{w}_p(h) = \begin{cases} pc\,\mathbf{w}_p(h-1) + A_1 (2p)^{2h} + [A_2 + A_3 h](2p)^h + A_4 \\ \quad \text{if} \quad p \in [0,1] \backslash \left\{ \frac{1}{2} \right\} \\ \frac{1}{2} c\,\mathbf{w}_p(h-1) + B_1 h^3 + B_2 h^2 + B_3 h + B_4 \qquad \text{if} \quad p = \frac{1}{2} \end{cases}$$

where

$$\begin{aligned} A_1 &= \frac{p}{4(2p-1)^3} [4pg_2 + \kappa_2] \\ A_2 &= \frac{p}{2(2p-1)^2} [(2p-1)(2g_1 + \kappa_1) + (p-1)(6g_2 + \kappa_2)] \\ A_3 &= \frac{p(p-1)}{(2p-1)^2} [2g_2 + \kappa_2] \\ A_4 &= \frac{1}{(2p-1)^3} [(2p-1)^3 (g_0 + p\kappa_0) + (2p-1)^2 (p-1)(g_1 + p\kappa_1) \\ &\quad + (2p^2 - 2p + 1)(p-1)(g_2 + p\kappa_2)] \end{aligned}$$

$$\begin{aligned} B_1 &= \frac{1}{24} [2g_2 + \kappa_2] \\ B_2 &= \frac{1}{16} [6g_2 + \kappa_2] \\ B_3 &= \frac{1}{48} [24g_1 + 50g_2 + 12\kappa_1 + 13\kappa_2] \\ B_4 &= \frac{1}{8} [8(g_0 + g_1 + g_2) + 4\kappa_0 + 2\kappa_1 + \kappa_2] \quad \square \end{aligned}$$

Next, we shall derive an asymptotic equivalent to $\mathbf{w}_p(h)$ for large h. In the corresponding computations, we shall often use the following identities (convention $0^0 := 1$)

$$\sum_{0 \le i \le h} (h-i)^m x^i = \begin{cases} \dfrac{h^m}{1-x} + \mathcal{O}(h^{m-1}) & \text{if} \quad x < 1 \\ \dfrac{h^{m+1}}{m+1} + \mathcal{O}(h^m) & \text{if} \quad x = 1, \\ \dfrac{A_m(x^{-1})}{(1-x^{-1})^{m+1}} x^h + \mathcal{O}(h^m) & \text{if} \quad x > 1 \end{cases} \tag{10}$$

where $x \in \mathbb{R}_+$ and $m \in \mathbb{N}_0$ fixed. Here, $A_m(x)$ denotes the m-th Eulerian polynomial ([2], p. 244; [3], p. 214). For $x < 1$, the identity follows by applying the binomial theorem to $(h-i)^m$. For $x = 1$, the identity is a consequence of the relation ([2], p. 155; [3], p. 215)

$$\sum_{0 \le k \le h} k^m = \frac{1}{m+1} [B_{m+1}(h+1) - B_{m+1}],$$

where $B_m(x) = x^m + \mathcal{O}(x^{m-1})$ is the m-th Bernoulli polynomial ([2], p. 48; [3], p. 215) and $B_m = B_m(0)$ is the m-th Bernoulli number. For $x > 1$, the identity follows by reversing the sum and by applying the general relation ([2], p. 245; [3], p. 214)

$$\sum_{\lambda \ge 0} \lambda^m x^\lambda = \frac{A_m(x)}{(1-x)^{m+1}}, \qquad |x| < 1.$$

Now, we are ready to prove the following technical result.

LEMMA 2: *Let $F_{p,h}(z)$ be the function introduced in Theorem 2.*
(a) Let $p < 0.5$. We have for fixed $s \ge 1$:

$$F_{p,h}^{(s)}(1) = \alpha_s + \mathcal{O}(h^{s-1}(2p)^h),$$

where α_s satisfies the recurrence

$$\alpha_1 = \frac{1-p}{1-2p}, \qquad \alpha_s = \frac{2p}{1-2p} \sum_{1 \le j < s} \binom{s-1}{j} \alpha_j \alpha_{s-j}, \qquad s \ge 2.$$

(b) Let $p = 0.5$. We have for fixed $s \ge 1$:

$$F_{p,h}^{(s)}(1) = \beta_s h^{2s-1} + \mathcal{O}(h^{2s-2}),$$

where β_s satisfies the recurrence

$$\beta_1 = \frac{1}{2}, \qquad \beta_s = \frac{1}{2s-1} \sum_{1 \le j < s} \binom{s-1}{j} \beta_s \beta_{s-j}, \qquad s \ge 2.$$

(c) Let $p > 0.5$. We have for fixed $s \ge 1$:

$$F_{p,h}^{(s)}(1) = \gamma_s (2p)^{hs} + \mathcal{O}(h^{1-\delta_{1,s}} (2p)^{h(s-1)}),$$

where γ_s satisfies the recurrence

$$\gamma_1 = \frac{p}{2p-1}, \qquad \gamma_s = \frac{1}{(2p)^{s-1} - 1} \sum_{1 \le j < s} \binom{s-1}{j} \gamma_j \gamma_{s-j}, \qquad s \ge 2.$$

Proof: An inspection of formula (8) shows that our statements are valid for $s = 1$. The general recurrence for $F_{p,h}^{(s)}(1)$ presented in (7) implies for $s \geq 2$

$$F_{p,h}^{(s)}(1) = \sum_{0 \leq \lambda < h} (2p)^{\lambda+1} \times \sum_{1 \leq j < s} \binom{s-1}{j} F_{p,h-\lambda-1}^{(j)}(1) \, F_{p,h-\lambda-1}^{(s-j)}(1), \quad s \geq 2. \tag{11}$$

Now, the results presented in (*a*), (*b*) and (*c*) can easily be verified by induction. The induction step in each case is as follows:

(*a*) We find by the recurrence for α_s and by (10) with $x = 1$

$$\begin{aligned} F_{p,h}^{(s)}(1) &= \sum_{0 \leq \lambda < h} (2p)^{\lambda+1} \\ &\quad \times \sum_{1 \leq j < s} \binom{s-1}{j} [\alpha_j + \mathcal{O}((h-\lambda-1)^{j-1} (2p)^{h-\lambda-1})] \\ &\quad \times [\alpha_{s-j} + \mathcal{O}((h-\lambda-1)^{s-j-1} (2p)^{h-\lambda-1})] \\ &= \sum_{0 \leq \lambda < h} (2p)^{\lambda+1} \left[\frac{1-2p}{2p} \alpha_s + \mathcal{O}((h-\lambda-1)^{s-2} (2p)^{h-\lambda-1})\right] \\ &= \alpha_s [1-(2p)^h] + \mathcal{O}(h^{s-1} (2p)^h). \end{aligned}$$

(*b*) Using the recurrence for β_s, we obtain by (10) with $x = 1$

$$\begin{aligned} F_{p,h}^{(s)}(1) &= \sum_{0 \leq \lambda < h} 1 \\ &\quad \times \sum_{1 \leq j < s} \binom{s-1}{j} [\beta_j (h-\lambda-1)^{2j-1} + \mathcal{O}((h-\lambda-1)^{2j-2})] \\ &\quad \times [\beta_{s-j} (h-\lambda-1)^{2s-2j-1} + \mathcal{O}((h-\lambda-1)^{2s-2j-2})] \\ &= \sum_{0 \leq \lambda < h} [(2s-1) \beta_s (h-\lambda-1)^{2s-2} + \mathcal{O}((h-\lambda-1)^{2s-3})] \\ &= \beta_s h^{2s-1} + \mathcal{O}(h^{2s-2}). \end{aligned}$$

(*c*) Making use of the recurrence for γ_s, we find by (10) with $x = 1$ and $x < 1$

$$\begin{aligned} F_{p,h}^{(s)}(1) &= \sum_{0 \le \lambda < h} (2p)^{\lambda+1} \sum_{1 \le j < s} \binom{s-1}{j} [\gamma_j (2p)^{j(h-\lambda-1)} \\ &\quad + \mathcal{O}((h-\lambda-1)^{1-\delta_{1,j}} (2p)^{(j-1)(h-\lambda-1)})] \\ &\quad \times [\gamma_{s-j} (2p)^{(s-j)(h-\lambda-1)} \\ &\quad + \mathcal{O}((h-\lambda-1)^{1-\delta_{1,s-j}} (2p)^{(s-j-1)(h-\lambda-1)})] \\ &= [(2p)^{s-1} - 1] \gamma_s (2p)^{sh-(s-1)} \sum_{0 \le \lambda < h} (2p)^{-(s-1)\lambda} \\ &\quad + \mathcal{O}\left(2^{h(s-1)} \sum_{1 \le j < s} \binom{s-1}{j} (\gamma_j + \gamma_{s-j}) h^{1-\delta_{1,j}+\delta_{2,s}}\right) \\ &= \gamma_s (2p)^{sh} [1 - (2p)^{-(s-1)h}] + \mathcal{O}(h^{1-\delta_{1,s}} 2^{h(s-1)}). \quad \square \end{aligned}$$

Remark 2:

(i) The recurrence for α_s presented in part (a) of the preceding Lemma 2 can be solved in the following way: let $\alpha_0 := 0$ and let $A(z) := \sum_{s \ge 0} \frac{\alpha_s}{s!} z^s$ be the exponential generating function of the numbers α_s. Translating the recurrence into terms of $A(z)$, we immediately find the differential equation

$$A(z) A'(z) = \frac{1-2p}{2p} \left[A'(z) - \frac{1-p}{1-2p}\right], \qquad p \neq 0.$$

Integrating this equation on both sides and using the initial condition $A(0) = 0$, we further obtain

$$\frac{1}{2} A^2(z) = \frac{1-2p}{2p} A(z) - \frac{1-p}{2p} z.$$

Solving this quadratic equation, we find

$$A(z) = \frac{1-2p}{2p} \left[1 - \left(1 - 4 \frac{p(1-p)}{(1-2p)^2} z\right)^{1/2}\right],$$

because $A(0) = 0$. Now, using the known evaluation ([3], p. 206)

$$1 - (1-4y)^{1/2} = 2 \sum_{k \ge 0} \frac{1}{k+1} \binom{2k}{k} y^{k+1}$$

with $y := \dfrac{p(1-p)}{(1-2p)^2}\, z$, we finally obtain [also for $p = 0$]

$$\alpha_s = \frac{(2s-2)!p^{s-1}(1-p)^s}{(s-1)!\,(1-2p)^{2s-1}}, \qquad s \geq 1. \tag{12}$$

(ii) In the same way, the recurrence for β_s given in part (b) of the preceding Lemma 2 can be solved. Let $\beta_0 := 0$ and let $A(z) := \sum_{s \geq 0} \frac{\beta_s}{s!} z^s$ be the exponential generating function of the numbers β_s. Here, the recurrence implies the differential equation

$$A(z)\,A'(z) = 2\,[z\,A'(z)]' - A'(z) - \frac{1}{2}.$$

Integrating this equation on both sides, we obtain by means of the initial condition $A(0) = 0$

$$\frac{1}{2} A^2(z) = 2\,z\,A'(z) - A(z) - \frac{1}{2}\,z.$$

Now, the substitution $A(z) := z^{1/2} A_1(z)$ yields $A_1^2(z) = 4\,z^{1/2} A_1'(z) - 1$. This Riccati differential equation can be solved by separating the variables. We obtain $A_1(z) = \tan\left(\frac{1}{2} z^{1/2}\right)$. Thus, $A(z) = z^{1/2} \tan\left(\frac{1}{2} z^{1/2}\right)$. Using the known evaluation ([1], p. 75)

$$\tan(z) = \sum_{k \geq 1} \frac{(-1)^{k-1}\, 2^{2k}\,(2^{2k}-1)\,B_{2k}}{(2k)!}\, z^{2k-1},$$

where B_m is the m-th Bernoulli number, we finally obtain

$$\beta_s = \frac{2\,s!\,(2^{2s}-1)}{(2s)!}\,(-1)^{s-1}\,B_{2s}, \qquad s \geq 1. \tag{13}$$

(iii) Unfortunately, the procedure used in (ii) and (iii) does not lead to an explicit expression for γ_s defined by the recurrence stated in Lemma 2(c). Introducing the exponential generating function $A(z) := \sum_{s \geq 0} \frac{\gamma_s}{s!} z^s$, we obtain the functional equation $A^2(z) = p^{-1} A(2pz) - 2\,A(z)$. To date, the author is unable to solve this equation. ◇

Now, we are ready to prove the following rather general result.

THEOREM 3: *Assume that the weight functions* $\Phi_i(m)$, $i \in \{1, 2\}$, *and* $g(m)$ *have the representations*

$$g(m) := \sum_{0 \le \lambda \le d} g_\lambda m^\lambda \quad \text{and} \quad \Phi_i(m) := \sum_{0 \le \lambda \le d_i} f_\lambda^{(i)} m^\lambda, \qquad i \in \{1, 2\}.$$

Furthermore, let $\rho = \max\{d_1, d_2, d\}$, $c := c_1 + c_2$ *and* $f_\lambda := p[f_\lambda^{(1)} + f_\lambda^{(2)}]$. *The average weight* $\mathbf{w}_p(h)$ *has the following asymptotic equivalents for large* h*:*

(a) If $\rho = 0$ *then:*

$$\mathbf{w}_p(h) = \begin{cases} \dfrac{g_0 + f_0}{1 - pc} + \mathcal{O}([pc]^h) & if \quad pc < 1 \\ (g_0 + f_0)\, h + \mathcal{O}(1) & if \quad pc = 1 \\ \dfrac{g_0\, pc + f_0}{pc - 1}\, [pc]^h + \mathcal{O}(1) & if \quad pc > 1 \end{cases}$$

(b) If $\rho \ge 1$ *then:*

(b1) If $p < 0.5$ *then:*

$$\mathbf{w}_p(h) = \begin{cases} \dfrac{C_\rho(p)}{p(1 - pc)} + \mathcal{O}(\psi_\rho(h)) & if \quad pc < 1 \\ p^{-1}\, C_\rho(p)\, h + \mathcal{O}(1) & if \quad pc = 1 \\ \mathcal{O}([pc]^h) & if \quad pc > 1, \end{cases}$$

where $\psi_\rho(h) = (2p)^h h^{\rho - 1 + \delta_{2,c}}$ *if* $c \le 2$, *and* $\psi_\rho(h) = [pc]^h$ *if* $c > 2$. *Here,* $C_\rho(p)$ *is the constant*

$$C_\rho(p) = p(g_0 + f_0) + (1 - 2p) \sum_{1 \le \lambda \le \rho} (g_\lambda + f_\lambda)\, Q_\lambda\left(\frac{p(1-p)}{(1-2p)^2}\right),$$

where $Q_\lambda(x)$ *is the polynomial* $Q_\lambda(x) = \sum_{1 \le j \le \lambda} \mathcal{S}_\lambda^{(j)} \dfrac{(2j-2)!}{(j-1)!}\, x^j$.

(b2) If $p = 0.5$ *then:*

$$\mathbf{w}_p(h) = \begin{cases} \dfrac{2\, C_\rho}{2 - c}\, h^{2\rho - 1} + \mathcal{O}(h^{2\rho - 2}) & if \quad c < 2 \\ \dfrac{C_\rho}{2\rho}\, h^{2\rho} + \mathcal{O}(h^{2\rho - 1}) & if \quad c = 2 \\ \mathcal{O}(c^h\, 2^{-h}) & if \quad c > 2 \end{cases}$$

Here, C_ρ *is the constant* $C_\rho = 2\,(g_\rho + f_\rho)\,\frac{\rho!\,(2^{2\rho}-1)}{(2\,\rho)!}\,(-1)^{\rho-1}\,B_{2\,\rho}$*, where* $B_{2\,\rho}$ *is a Bernoulli number.*

(b3) If $p > 0.5$ *then:*

$$\mathbf{w}_p\,(h) = \begin{cases} \frac{C_\rho}{(2\,p)^\rho - pc}\,(2\,p)^{\rho\,h} + \mathcal{O}\,(\psi_\rho\,(h)) \\ if \quad pc \le 1 \vee (pc > 1 \wedge \rho > \tau) \\ C_\rho\,(2\,p)^{\rho\,(h-1)}\,h + \mathcal{O}\,([pc]^h) & if \quad pc > 1 \wedge \rho = \tau \\ \mathcal{O}\,([pc]^h) & if \quad pc > 1 \wedge \rho < \tau \end{cases}$$

where $\tau = \ln(pc)/\ln(2\,p)$ *and* $\psi_\rho\,(h) = (2\,p)^{h\,(\rho-1)}\,h^{1-\delta_{\rho,\,1}\,(1-\delta_{pc,\,1})}$ *if* $pc \le 1$, $\psi_\rho\,(h) = (2\,p)^{h\,(\rho-1)}\,h^{1+\delta_{\rho,\,\tau+1}}$ *if* $pc > 1 \wedge \rho \ge \tau + 1$*, and* $\psi_\rho\,(h) = [pc]^h$ *if* $pc > 1$ *and* $\tau < \rho < \tau + 1$*. Here,* C_ρ *is the constant* $C_\rho = \gamma_s\,[(2\,p)^\rho\,g_\rho + f_\rho]$*, where* γ_s *is given by the recurrence established in Lemma 2(c).*

Proof: Iterating the recurrence for $\mathbf{w}_p\,(h)$ presented in Theorem 2, we obtain

$$\mathbf{w}_p\,(h) = (pc)^h\,g\,(1) + \sum_{0 \le \mu < h} (pc)^\mu \sum_{0 \le \lambda \le \rho} 1 \times \sum_{0 \le j \le \lambda} \mathcal{S}_\lambda^{(j)}\,[g_\lambda\,F_{p,\,h-\mu}^{(j)}\,(1) + f_\lambda\,F_{p,\,h-\mu-1}^{(j)}\,(1)],$$

or equivalently

$$\mathbf{w}_p\,(h) = (pc)^h\,g\,(1) + (g_0 + f_0) \sum_{0 \le \mu < h} (pc)^\mu + \sum_{1 \le \lambda \le \rho}\ \sum_{1 \le j \le \lambda} \mathcal{S}_\lambda^{(j)}\,G_{\lambda,\,j}\,(h), \tag{14}$$

where

$$G_{\lambda,\,j}\,(h) = \sum_{0 \le \mu < h} (pc)^\mu\,[g_\lambda\,F_{p,\,h-\mu}^{(j)}\,(1) + f_\lambda\,F_{p,\,h-\mu-1}^{(j)}\,(1)]. \tag{15}$$

For $\rho = 0$, we have $g\,(1) = g_0$ and our result established in part (a) follows by using the closed formed expression for the geometric series.

Now, let $\rho \geq 1$.

(i) The case $p < 0.5$.

Inserting the expression for $F_{p,h}^{(s)}(1)$ presented in Lemma 2(a) into (15), we immediately obtain

$$G_{\lambda,j}(h) = \alpha_j(g_\lambda + f_\lambda) \sum_{0 \leq \mu < h} (pc)^\mu + \mathcal{O}\left((2p)^h \sum_{0 \leq \mu < h} (h-\mu)^{j-1} c^\mu 2^{-\mu}\right).$$

By (10), the $\mathcal{O}$-term is equal to $\mathcal{O}(\psi_j(h))$, where $\psi_j(h)$ is the function introduced in part (b1) of our lemma. Hence, by (14)

$$\mathbf{w}_p(h) = (pc)^h g(1) + \left[g_0 + f_0 + \sum_{1 \leq \lambda \leq \rho} \sum_{1 \leq j \leq \lambda} \mathcal{S}_\lambda^{(j)} \alpha_j (g_\lambda + f_\lambda)\right] \times \sum_{0 \leq \mu < h} (pc)^\mu + \mathcal{O}(\psi_\rho(h)).$$

Now, our result established in part (b1) follows by the definition of α_j presented in (12) and the closed formed expression for the geometric series.

(ii) The case $p = 0.5$.

Using Lemma 2(b), we find by (15) for $c \neq 0$

$$G_{\lambda,j}(h) = \beta_j [g_\lambda + 2c^{-1} f_\lambda] \sum_{0 \leq \mu < h} [c^\mu 2^{-\mu} (h-\mu)^{2j-1} - 2c^{-1} f_\lambda \beta_j h^{2j-1} + \mathcal{O}\left(\sum_{0 \leq \mu < h} c^\mu 2^{-\mu} (h-\mu)^{2j-2}\right)$$

and further by (10)

$$G_{\lambda,j}(h) = \begin{cases} \dfrac{2}{2-c}(g_\lambda + f_\lambda)\beta_j h^{2j-1} + \mathcal{O}(h^{2j-2}) & \text{if } c < 2 \\ \dfrac{1}{2j}(g_\lambda + f_\lambda)\beta_j h^{2j} + \mathcal{O}(h^{2j-1}) & \text{if } c = 2 \\ \mathcal{O}(c^h 2^{-h}) & \text{if } c > 2 \end{cases}$$

For $c = 0$, this result can be verified directly using (14), (15) and Lemma 2(b). Inserting the derived expression for $G_{\lambda,j}(h)$ into (14), the result stated in part (b2) of our lemma follows with (13).

(iii) The case $p > 0.5$.

Inserting the expression for $F_{p,h}^{(s)}(1)$ given in Lemma 2(c) into (15), we obtain for $c \neq 0$

$$G_{\lambda,j}(h) = (2p)^{j(h-1)} \gamma_j [(2p)^j g_\lambda + f_\lambda] \sum_{0 \le \mu < h} (pc)^\mu (2p)^{-\mu j} + \mathcal{O}\left((2p)^{h(j-1)} \sum_{0 \le \mu < h} (pc)^\mu (2p)^{-\mu(j-1)} (h-\mu)^{1-\delta_{j,1}}\right).$$

Discussing the particular choices for j [*i.e.* $pc < (2p)^k$, $pc = (2p)^k$, $pc > (2p)^k$ for $k \in \{j, j-1\}$], we find with (10) by a lengthy computation

$$G_{\lambda,j}(h) = \begin{cases} \dfrac{\gamma_j [(2p)^j g_\lambda + f_\lambda]}{(2p)^j - pc} (2p)^{jh} + \mathcal{O}(\psi_j(h)) \\ \text{if} \quad pc \le 1 \vee (pc > 1 \wedge j > \tau) \\ \gamma_j [(2p)^j g_\lambda + f_\lambda] (2p)^{j(h-1)} h + \mathcal{O}([pc]^h) \\ \text{if} \quad pc > 1 \wedge j = \tau \\ \mathcal{O}([pc]^h) \qquad \text{if} \quad pc > 1 \wedge j < \tau \end{cases}$$

where the number τ and the function $\psi_j(h)$ are introduced in part (b3) of our lemma. This equation is also valid for $c = 0$. Now, our result established in part (b3) follows immediately by (14). □

3. THE GENERALIZED WEIGHT

In particular applications (*cf.* Section 4) the average "generalized weight"

$$\begin{aligned} \mathrm{w}_p^{[1]}(h) &= \sum_{T \in \mathcal{F}_p(h)} |L(T)|\, w(T)\, \varphi_{p,h}(T) \\ &= \sum_{m \ge 1} \sum_{k \ge 0} \sum_{w \ge 0} m\, w\, t_p^{(h)}(m, k, w)\, p^{m-1} (1-p)^{m-k} \end{aligned}$$

is of interest. Using the definition of the function $W_{p,h}(z, y, x)$ introduced in (3), we obtain

$$\mathrm{w}_p^{[1]}(h) = (1-p) \frac{\partial}{\partial x} \frac{\partial}{\partial z} W_{p,h}(z, y, x) \Bigg|_{(z,y,x) = (p(1-p), (1-p)^{-1}, 1)}. \tag{16}$$

Taking the partial derivative with respect to z on both sides of equation (6), we find by means of (4), (16) and the relation $\Pi_{p,h}(p(1-p), (1-p)^{-1}) = p$

$$\begin{aligned}\mathbf{w}_p^{[1]}(h) &= p(c_1+c_2)\,\mathbf{w}_p^{[1]}(h-1) + \sum_{m\geq 1}\sum_{k\geq 0} m\,[g(m)\,t_p^{(h)}(m,k) \\ &\quad + p(\Phi_1(m)+\Phi_2(m))\,t_p^{(h-1)}(m,k)]\,p^{m-1}(1-p)^{m-k} \\ &\quad + \Big[p(c_1+c_2)\,\mathbf{w}_p(h-1) \\ &\quad + \sum_{m\geq 1}\sum_{k\geq 0}(\Phi_1(m)+\Phi_2(m))\,t_p^{(h-1)} \\ &\quad \times (m,k)\,p^m(1-p)^{m-k}\Big]\,M_{p,h-1},\end{aligned}$$

where $M_{p,h} := (1-p)\dfrac{\partial}{\partial z}\Pi_{p,h}(z,y)\Big|_{(z,y)=(p(1-p),(1-p)^{-1})}$.

Now, the recurrence (2) implies that

$$M_{p,h} = \begin{cases} 1+p\,\dfrac{(2p)^h-1}{2p-1} & \text{if } p\in[0,1]\backslash\left\{\dfrac{1}{2}\right\} \\ 1+\dfrac{1}{2}h & \text{if } p=\dfrac{1}{2}\end{cases}. \tag{17}$$

Using this relation and Theorem 1, the above recurrence for $\mathbf{w}_p^{[1]}(h)$ can be transformed into

$$\begin{aligned}\mathbf{w}_p^{[1]}(0) &= g(1) \\ \mathbf{w}_p^{[1]}(h) &= p(c_1+c_2)\,\mathbf{w}_p^{[1]}(h-1) + M_{p,h-1}\,\mathbf{w}_p(h) \\ &\quad + \sum_{m\geq 1}\sum_{k\geq 0}[(m-M_{p,h-1})\,g(m)\,t_p^{(h)}(m,k) \\ &\quad + pm(\Phi_1(m)+\Phi_2(m))\,t_p^{(h-1)}(m,k)]\,p^{m-1}(1-p)^{m-k}.\end{aligned} \tag{18}$$

Restricting the preceding considerations to the case that the weight functions $\Phi_i(m)$, $i\in\{1,2\}$, and $g(m)$ are polynomials in m of degree less than or equal to one then we find by (18) the following result.

COROLLARY 2: *Assume that the weight functions $\Phi_i(m)$, $i\in\{1,2\}$ and $g(m)$ have the representations*

$$g(m) = g_0 + g_1 m \qquad \textit{and} \qquad \Phi_i(m) = f_0^{(i)} + f_1^{(i)} m.$$

Moreover, let $\kappa_\lambda := f_\lambda^{(1)} + f_\lambda^{(2)}$ *and* $c : c_1 + c_2$. *The average generalized weight* $\mathbf{w}_p^{[1]}(h)$ *of a tree* $T \in \mathcal{F}_p(h)$ *is recursively given by*

$$\mathbf{w}_p^{[1]}(0) = g(1)$$

$$\mathbf{w}_p^{[1]}(h) = \begin{cases} pc\,\mathbf{w}_p^{[1]}(h-1) + C_1\,\mathbf{w}_p(h) + C_2\,(2p)^{2h} \\ \quad + (C_3 + C_4\,h)(2p)^h + C_5 \\ \quad if \quad p \in [0,\, 1]\backslash\left\{\frac{1}{2}\right\} \\ \frac{1}{2}\,c\,\mathbf{w}_p^{[1]}(h-1) + D_1\,\mathbf{w}_p(h) + D_2\,h^3 + D_3\,h^2 + D_4\,h + D_5 \\ \quad if \quad p = \frac{1}{2} \end{cases}$$

where

$$C_1 = \frac{1}{2(2p-1)}\left[2(p-1) + (2p)^h\right]$$
$$C_2 = \frac{p}{4(2p-1)^3}\left[2g_1 + \kappa_1\right]$$
$$C_3 = \frac{1}{2(2p-1)^2}\left[(2p-1)^2 g_0 + (p-1)(4p-1)g_1 + p(2p-1)\kappa_0 + p(p-1)\kappa_1\right]$$
$$C_4 = \frac{p(p-1)}{(2p-1)^2}\left[2g_1 + \kappa_1\right]$$
$$C_5 = \frac{p(p-1)}{(2p-1)^3}\left[g_1 + (2p-1)^2\kappa_0 + (2p^2 - 2p + 1)\kappa_1)\right]$$

$$D_1 = \frac{1}{2}(h+1)$$
$$D_2 = \frac{1}{24}\left[2g_1 + \kappa_1\right]$$
$$D_3 = \frac{1}{16}\left[2g_1 + \kappa_1\right]$$
$$D_4 = \frac{1}{48}\left[14g_1 + 12\kappa_0 + 13\kappa_1\right]$$
$$D_5 = \frac{1}{8}\left[4(g_0 + g_1) + 2\kappa_0 + \kappa_1\right] \quad \square$$

4. APPLICATIONS

In this section, we shall apply the general results presented in the previous sections to the particular weights introduced in Table 1. Using the general relations established in Theorem 1, Corollary 1 and Corollary 2, the computation of exact expressions for the average weights does not present any difficulty. Let us consider two examples.

(1) The average number of internal nodes $\mathbf{IN}(h)$ of a tree $T \in \mathcal{F}_p(h)$ is characterized by the parameters

$$(c_1, c_2, g(m), \Phi_1(m), \Phi_2(m)) = (1, 1, 1 - \delta_{m,1}, 0, 0).$$

Interesting these quantities into the recurrence established in Theorem 1, we find

$$\begin{aligned}
\mathbf{IN}(0) &= 0\\
\mathbf{IN}(h) &= 2p\,\mathbf{IN}(h-1) + \sum_{m\geq 2}\sum_{k\geq 0} t_p^{(h)}(m, k)\,p^{m-1}(1-p)^{m-k}\\
&= 2p\,\mathbf{IN}(h-1) + [p^{-1}\,\Pi_{p,h}(p(1-p), (1-p)^{-1})\\
&\quad - \sum_{k\geq 0} t_p^{(h)}(1, k)(1-p)^{1-k}]\\
&= 2p\,\mathbf{IN}(h-1) + p, \qquad h \geq 1,
\end{aligned}$$

where $\Pi_{p,h}$ is the function given in (1). Here, we have used the relation

$$p^{-1}\,\Pi_{p,h}(p(1-p), (1-p)^{-1}) = 1$$

and

$$t_p^{(h)}(1, k) = \delta_{h,0}\,\delta_{k,1} + (1 - \delta_{h,0})\,\delta_{k,0}.$$

Solving the above recurrence for $\mathbf{IN}(h)$, we obtain the result for $\mathbf{IN}(h)$ presented in Table 2.

(2) The average internal free path length $\mathbf{IFPL}(h)$ of a tree $T \in \mathcal{F}_p(h)$ is given by [*cf.* Table 1]

$$\mathbf{IFPL}(h) = \mathbf{IPL}^{[1]}(h) - \mathbf{IPL}(h) - \mathbf{w}_p(h), \tag{19}$$

where $\mathbf{IPL}^{[1]}(h)$ is the average "generalized internal path length" [*cf.* Section 3] and $\mathbf{w}_p(h)$ is the average weight [*cf.* Table 1] which is characterized by the parameters $(c_1, c_2, g(m), \Phi_1(m), \Phi_2(m)) = (1, 1, 0, (m-1)^2, (m-1)^2)$. For $p \neq 0.5$, the result established in Corollary 1 yields

$$\begin{aligned}\mathbf{w}_p(0) &= 0\\ \mathbf{w}_p(h) &= 2p\,\mathbf{w}_p(h-1) + \frac{p}{2(2p-1)^3}(2p)^{2h}\\ &\quad+\frac{p}{(2p-1)^2}[1-3p+2h(p-1)](2p)^h\\ &\quad+\frac{2p^2(2p^2-1)}{(2p-1)^3}, \qquad h \geq 1.\end{aligned}$$

This recurrence can be solved by iteration. We obtain

$$\begin{aligned}\mathbf{w}_p(h) = \frac{p}{(2p-1)^4}&[p(2p)^{2h}\\ &+(h(2p-1)^2(hp-2p-h)-3p+4p^3)(2p)^h\\ &+2p(1-2p^2)].\end{aligned} \tag{20}$$

An inspection of Table 1 shows that the average internal path length $\mathbf{IPL}(h)$ is characterized by the parameters

$$(c_1, c_2, g(m), \Phi_1(m), \Phi_2(m)) = (1, 1, 0, m-1, m-1).$$

Here, Corollary 1 leads to the recurrence

$$\begin{aligned}&\mathbf{IPL}(0) = 0\\ &\mathbf{IPL}(h) = 2p\,\mathbf{IPL}(h-1) + \frac{p}{2p-1}(2p)^h - \frac{2p^2}{2p-1}, \qquad h \geq 1,\end{aligned}$$

which has the solution

$$\mathbf{IPL}(h) = \frac{p}{(2p-1)^2}[(2hp-2p-h)(2p)^h + 2p]. \tag{21}$$

Now, the result presented in Corollary 2 tells us

$$\begin{aligned}&\mathbf{IPL}^{[1]}(0) = 0\\ &\mathbf{IPL}^{[1]}(h) = 2p\,\mathbf{IPL}^{[1]}(h-1) + \frac{1}{2(2p-1)}[(2p)^h + 2(p-1)]\,\mathbf{IPL}(h)\\ &\qquad+\frac{1}{4(2p-1)^3}(2p)^{2h+1} + \frac{p}{(2p-1)^2}[2h(p-1)-p](2p)^h\\ &\qquad-\frac{4p^2(p-1)^2}{(2p-1)^3}, \qquad h \geq 1.\end{aligned}$$

Inserting the explicit expression for $\mathbf{IPL}(h)$ given in (21) into this equation, we obtain a recurrence for $\mathbf{IPL}^{[1]}(h)$ which can also be solved by iteration. We find the explicit expression

$$\begin{aligned}\mathbf{IPL}^{[1]}(h) = \frac{p}{2(2p-1)^4} &[(2hp-2p-h)(2p)^{2h+1} \\ &+ \{3h^2(p-1)(2p-1)^2 - h(p-1)(2p-1)(2p+3) \\ &- 4p(2p^2-6p+3)\}(2p)^h + 4p(p-1)(2p-3)]. \quad (22)\end{aligned}$$

TABLE 2

The exact values of the average weights introduced in Table 1.

Parameter	exact expression	
	$p \neq 0.5$	$p = 0.5$
$\mathbf{D}(h)$	$2p$	1
$\mathbf{IN}(h)$	$\frac{p}{2p-1}[(2p)^h - 1]$	$\frac{h}{2}$
$\mathbf{LE}(h)$	$\frac{1}{2p-1}[p(2p)^h + p - 1]$	$\frac{h+2}{2}$
$\mathbf{LBL}(h) = \mathbf{RBL}(h)$	$\frac{p}{p-1}[p^h - 1]$	$1 - 2^{-h}$
$\mathbf{IL}(h) = \mathbf{IR}(h)$	$\frac{p}{2(2p-1)}[(2p)^h - 2p]$	$\frac{h-1}{4}$
$\mathbf{LL}(h) = \mathbf{LR}(h)$	$\frac{p}{2(2p-1)}[(2p)^h + 2(p-1)]$	$\frac{h+1}{4}$
$\mathbf{IP}(h)$	$\frac{p}{2(2p-1)^3}[p(2p)^{2h} + (2p-1) \times (2hp-3p-2h)(2p)^h + 2p(3p-2)]$	$\frac{h(h-1)(2h+11)}{48}$
$\mathbf{ILP}(h) = 2\,\mathbf{LP}(h)$	$\frac{p}{(2p-1)^3}[p(2p)^{2h} + (2p-1) \times (2hp+p-2h-2)(2p)^h - 2(p-1)^2]$	$\frac{h(2h^2+9h+13)}{24}$
$\mathbf{IP}_r(h)$	$\frac{p}{4(2p-1)^3}[(2p)^{2h} + 2(2p-1) \times (2hp-5p-2h+2)(2p)^h + 8p^2(3p-2)]$	$\frac{(h-1)(h-2)(2h+9)}{48}$

TABLE 2 (*continued*)

Parameter	exact expression	
	$p \neq 0.5$	$p = 0.5$
$\mathbf{ILP}_r(h) = 2\,\mathbf{LP}_r(h)$	$\frac{p}{2(2p-1)^3}[(2p)^{2h} + 2(2p-1) \times (2hp - p - 2h)(2p)^h - 8p(p-1)^2]$	$\frac{(h-1)(2h^2+5h+6)}{24}$
$\mathbf{IPL}(h)$	$\frac{p}{(2p-1)^2}[(2hp - 2p - h)(2p)^h + 2p]$	$\frac{h(h-1)}{4}$
$\mathbf{EPL}(h)$	$\frac{p}{(2p-1)^2}[(2hp + 2p - h - 2)(2p)^h - 2(p-1)]$	$\frac{h(h+3)}{4}$
$\mathbf{IFPL}(h)$	$\frac{p}{2(2p-1)^4}[(2hp - 2p - h - 1)(2p)^{2h+1} + \{h^2(p-1)(2p-1)^2 - h(2p-1) \times (2p^2 - 3p - 1) + 2p(4p-1)\}(2p)^h - 4p(p-1)]$	$\frac{h(h-1)(h+1)(h+2)}{32}$
$\mathbf{EFPL}(h)$	$\frac{p}{2(2p-1)^4}[(2hp + 2p - h - 3)(2p)^{2h+1} + \{h^2(p-1)(2p-1)^2 + h(p-1) \times (2p-1)(6p-7) - 2(4p^2 - 7p + 2)\}(2p)^h - 4(p-1)^2]$	$\frac{h(h+1)(h+2)(3h+13)}{96}$
$\mathbf{IEFPL}(h)$	$\frac{p}{(2p-1)^4}[(2hp - h - 2)(2p)^{2h+1} + \{h^2(p-1)(2p-1)^2 + h(2p-1) \times (2p^2 - 5p + 4) + 2(3p-1)\}(2p)^h - 2(p-1)]$	$\frac{h(h+1)(h+2)(3h+5)}{48}$

Finally, inserting (20), (21) and (22) into (19), we obtain the corresponding result for $\mathbf{IFPL}(h)$ presented in Table 2. For $p = 0.5$, the explicit expression for $\mathbf{IFPL}(h)$ can be computed in the same way. Alternatively, we can apply L'Hospital's rule to the expression for $\mathbf{IFPL}(h)$ with $p \neq 0.5$.

The asymptotic results displayed in Table 3 can be verified by using the explicit expressions presented in Table 2 or by a direct application of Theorem 3. For example, consider the average external path length $\mathbf{EPL}(h)$

TABLE 3

The asymptotic equivalents of the average weights introduced in Table 1.

Parameter	asymptotic expression for $h \to \infty$		
	$p < 0.5$	$p = 0.5$	$p > 0.5$
$\mathbf{D}(h)$	$2p$	1	$2p$
$\mathbf{IN}(h)$	$\frac{p}{1-2p}$	$\frac{h}{2}$	$\frac{p}{2p-1}(2p)^h$
$\mathbf{LE}(h)$	$\frac{1-p}{1-2p}$	$\frac{h}{2}$	$\frac{p}{2p-1}(2p)^h$
$\mathbf{LBL}(h) = \mathbf{RBL}(h)$	$\frac{p}{p-1}$	1	$\begin{cases} \frac{p^{h+1}}{1-p} & if\ p \neq 1 \\ h & if\ p = 1 \end{cases}$
$\mathbf{IL}(h) = \mathbf{IR}(h)$	$\frac{p^2}{1-2p}$	$\frac{h}{4}$	$\frac{p}{2(2p-1)}(2p)^h$
$\mathbf{LL}(h) = \mathbf{LR}(h)$	$\frac{p(1-p)}{1-2p}$	$\frac{h}{4}$	$\frac{p}{2(2p-1)}(2p)^h$
$\mathbf{IP}(h)$	$\frac{p^2(2-3p)}{(1-2p)^3}$	$\frac{h^3}{24}$	$\frac{p^2}{2(2p-1)^3}(2p)^{2h}$
$\mathbf{ILP}(h) = 2\,\mathbf{LP}(h)$	$\frac{2p(1-p)^2}{(1-2p)^3}$	$\frac{h^3}{12}$	$\frac{p^2}{(2p-1)^3}(2p)^{2h}$
$\mathbf{IP}_r(h)$	$\frac{2p^3(2-3p)}{(1-2p)^3}$	$\frac{h^3}{24}$	$\frac{p}{4(2p-1)^3}(2p)^{2h}$
$\mathbf{ILP}_r(h) = 2\,\mathbf{LP}_r(h)$	$\frac{4p^2(1-p)^2}{(1-2p)^3}$	$\frac{h^3}{12}$	$\frac{p}{2(2p-1)^3}(2p)^{2h}$
$\mathbf{IPL}(h)$	$\frac{2p^2}{(1-2p)^2}$	$\frac{h^2}{4}$	$\frac{ph}{2p-1}(2p)^h$
$\mathbf{EPL}(h)$	$\frac{2p(1-p)}{(1-2p)^2}$	$\frac{h^2}{4}$	$\frac{ph}{2p-1}(2p)^h$
$\mathbf{IFPL}(h)$	$\frac{2p^2(1-p)}{(1-2p)^4}$	$\frac{h^4}{32}$	$\frac{p^2h}{(2p-1)^3}(2p)^{2h}$
$\mathbf{EFPL}(h)$	$\frac{2p(1-p)^2}{(1-2p)^4}$	$\frac{h^4}{32}$	$\frac{p^2h}{(2p-1)^3}(2p)^{2h}$
$\mathbf{IEFPL}(h)$	$\frac{2p(1-p)}{(1-2p)^4}$	$\frac{h^4}{16}$	$\frac{2p^2h}{(2p-1)^3}(2p)^{2h}$

which is characterized by the parameters

$$(c_1, c_2, g(m), \Phi_1(m), \Phi_2(m)) = (1, 1, 0, m, m).$$

Applying Theorem 3, we find $\rho = 1$, $c = 2$ and $f_\lambda = 2p\,\delta_{\lambda,1}$. Hence, part (b1) of that theorem tells us for $p < 0.5$

$$\mathbf{EPL}(h) = \frac{C_1(p)}{p(1-2p)} + \mathcal{O}((2p)^h\, h)$$

with

$$C_1(p) = (1-2p)2p\,Q_1\left(\frac{p(1-p)}{(1-2p)^2}\right) = \frac{2p^2(1-p)}{1-2p}.$$

For $p = 0.5$, part (b2) of Theorem 3 yields

$$\mathbf{EPL}(h) = \frac{C_1}{2}\, h^2 + \mathcal{O}(h),$$

where $C_1 = 2\,\frac{1!\,(2^2-1)}{2!}\,(-1)^0\, B_2 = \frac{1}{2}$. Finally, we obtain by part (b3) for $p > 0.5$ [note that $\tau = \rho = 1$]

$$\mathbf{EPL}(h) = C_1(2p)^{h-1}\, h + \mathcal{O}((2p)^h),$$

where $C_1 = \gamma_1\, 2p = \frac{2p^2}{2p-1}$. Note that all $\mathcal{O}$-terms are exact [*see* Table 2].

Remark 3: The average number of nodes appearing in a tree $T \in \mathcal{F}_p(h)$ is given by

$$\mathbf{IN}(h) + \mathbf{LE}(h) = \frac{(2p)^{h+1}-1}{2p-1}\left(= h+1 \quad \text{for } p = \frac{1}{2}\right).$$

This result has already been proved in [9]. The remaining results are new.

An inspection of Table 3 shows that the given weights satisfy some interesting relations. For example, the following facts are valid for large h:

• The average number $\mathbf{IP}_r(h)$ [resp. $\mathbf{LP}_r(h)$; $\mathbf{ILP}_r(h)$] of root-free paths between internal [resp. external; internal and external] nodes is asymptotically proportional to the average number $\mathbf{IP}(h)$ [resp. $\mathbf{LP}(h)$; $\mathbf{ILP}(h)$] of paths between internal [resp. external; internal and external] nodes; the proportional factor is $2p$ if $p \leq 0.5$ and $(2p)^{-1}$ if $p > 0.5$;

• The average internal [resp. internal free] path length $\mathbf{IPL}(h)$ [resp. $\mathbf{IFPL}(h)$] is asymptotically proportional to the average external [resp.

external free] path length $\mathbf{EPL}(h)$ [resp. $\mathbf{EFPL}(h)$]; the proportional factor is asymptotically $p(1-p)^{-1}$ if $p \leq 0.5$ and 1 if $p > 0.5$. The same relation holds for the average number of internal nodes $\mathbf{IN}(h)$ and the average number of external nodes $\mathbf{LE}(h)$;

• The product of the average internal free path length $\mathbf{IFPL}(h)$ and the average external free path length $\mathbf{EFPL}(h)$ is asymptotically proportional to the quadrat $\mathbf{IEFPL}^2(h)$ of the average internal-external free path length; the proportional factor is equal to $p(1-p)$ if $p \leq 0.5$ and $\frac{1}{4}$ if $p > 0.5$;

• The product of the average number $\mathbf{IP}(h)$ of paths between internal nodes and the average number $\mathbf{LP}(h)$ of paths between external nodes is asymptotically proportional to the quadrat $\mathbf{ILP}^2(h)$ of the average number of paths between internal and external nodes; the proportional factor is equal to $\frac{1}{4}p(2-3p)(1-p)^2$ if $p \leq 0.5$ and $\frac{1}{4}$ if $p > 0.5$. The same fact holds for the corresponding average numbers of root-free paths. ◇

5. FINAL REMARKS

In this paper, we have presented a general approach to the computation of the expected additive weight, $\mathbf{w}_p(h)$ of a special class of binary backtrack trees. There are three technical questions raised by the given approach:

(1) What is the exact solution of the recurrence for γ_s established in Lemma 2(c) or of the functional equation presented in Remark 2(iii)?

(2) Can the polynomials $Q_\lambda(x)$ appearing in Theorem 3(b1) be expressed by familiar polynomials?

It is not hard to show that

$$Q_1(x) = x \quad \text{and} \quad Q_\lambda(x) = x(1+4x)\,Q'_{\lambda-1}(x) - 2x\,Q_{\lambda-1}(x), \quad \lambda \geq 2.$$

Introducing the substitution $Q_\lambda(x) := (1+4x)^{1/2}\,h_\lambda(x)$, this recurrence implies $h_1(x) = x(1+4x)^{-1/2}$ and $h_\lambda(x) = x(1+4x)\,h'_{\lambda-1}(x)$ for $\lambda \geq 2$.

(3) What is the asymptotic behaviour of the average "generalized weight" introduced in Section 3?

Finally note that the defined class of backtrack trees can be generalized by introducing a new p_h for each h. Thus, the definition of the p-weights of a tree alters to

(*a*) If T is the one-node tree then $\varphi_{\vec{p},h}(T) := p_h\,\delta_{h,0} + 1 - p_h$, $h \geq 0$;

(*b*) If T has the left [right] subtree T_1 [T_2] then

$$\varphi_{\vec{p}, h}(T) := p_h \, \varphi_{\vec{p}, h-1}(T_1) \, \varphi_{\vec{p}, h-1}(T_2), \qquad h \geq 1.$$

It is not hard to see that the sequence $\varphi_{\vec{p}, h}(T)$ is again a probability distribution on the set $\mathcal{F}_{\vec{p}}(h)$ consisting of the backtrack trees with height less than or equal to h. The probability $\varphi_{\vec{p}, h}(T)$ of such a tree $T \in \mathcal{F}_{\vec{p}}(h)$ is now

$$\varphi_h(T) = \prod_{1 \leq j \leq h} p_j^{|I_{h-j}(T)|} (1 - p_j)^{|L_{h-j}(T)|}.$$

Generalizing the presented methods, the average behaviour of the expected values of the parameters defined in Table 1 can also be computed for the family $\mathcal{F}_{\vec{p}}(h)$. For example, we find $\mathbf{D}(h) = 2 p_h$ and

$$\mathbf{LE}(h) = \mathbf{IN}(h) + 1 = 1 + \sum_{0 \leq j < h} 2^j \prod_{0 \leq \lambda \leq j} p_{h-\lambda}$$

$$\mathbf{LBL}(h) = \mathbf{RBL}(h) = \sum_{0 \leq j < h} \prod_{0 \leq \lambda \leq j} p_{h-\lambda}$$

$$\mathbf{EPL}(h) = \sum_{0 \leq j < h} (j + 2) \, 2^j \sum_{0 \leq \lambda \leq j} p_{h-\lambda}$$

$$\mathbf{IPL}(h) = \sum_{0 \leq j < h} j \, 2^j \prod_{0 \leq \lambda \leq j} p_{h-\lambda}$$

It would be interesting to derive the asymptotic average behaviour of the introduced additive weight for this family $\mathcal{F}_{\vec{p}}(h)$ in general.

Another generalization of the presented concept is its extension to general rooted trees. Such an extension has been discussed in [10]; there, the concepts developed in this paper have been generalized to a subclass of simply generated trees with a given finite set of allowed node degrees.

Finally, notice that particular non-additive weights have been discussed in [7] and [11]. There, the average behaviour of the "stackfunction" and of the "registerfunction" has been considered for the family of trees $\mathcal{F}_p(h)$.

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