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MULTIDIMENSIONAL LINEAR CONGRUENTIAL GRAPHS

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Abstract. – *Let d be an integer, F be a finite set of d-dimensional linear functions and $\vec{s} = (s_1, s_2, \ldots, s_d)$, be a d-dimensional vector of positive integers. We define* graph $G(F, \vec{s})$, *called a* linear congruential graph of dimension *d as a graph on the vertex set $V = Z_{s_1} \times Z_{s_2} \times \cdots \times Z_{s_d}$, in which any $\vec{x} \in V$ is adjacent to the vertices $f_i(\vec{x}) \bmod \vec{s}$, for any f_i in F.*

These graphs generalize several well known families of graphs, e.g. the de Bruijn graphs, chordal graphs, and linear congruential graphs. We show that, for a properly selected set of functions, multidimensional linear congruential graphs generate regular, highly connected graphs which are substantially larger than linear congruential graphs, or any other large family of graphs of the same degree and diameter. Some theoretical and empirical properties of these graphs are given and their structural properties are studied.

1. INTRODUCTION

In this paper we will consider undirected, simple graphs, *see* [3] for graph terminology not defined here. For a graph G with a vertex set V and an edge set E, the *diameter* is defined as the maximum distance $d(x, y)$ over all pairs of vertices x, y of V. The problem of constructing large graphs of a given degree d and diameter k, called (d, k) *graph problem*, and the related problem of constructing a graph of given size n and degree d of smallest possible diameter, proposed first in [7], has attracted attention of many researchers (*see* [4] for surveys), since this problem has practical applications in the design of interconnection network of processors in massively parallel computers.

An interconnection network can be represented as a graph in which the vertices correspond to the processors and the edges to the communication links. In an interconnection network which uses the store and forward mode of communication, the time needed for sending a message from one node to another is proportional to the distance of the nodes. Thus, to minimize

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the communication delays in a netword it is important to have a network of low diameter. To simplify the construction of a network, it is desirable to use an interconnection network in which each node is connected to the same number of nodes. Another important consideration for a network is the possibility to increased the size of network. In graph-theoretical terms we can state the desirable properties of a network as follows:

A network should correspond to a regular graph of low diameter, and for a fixed degree there should be a possibility to multiply the size of the graph without making substantial changes to the structure of the graph.

For a graph of degree $d > 2$ and diameter k there is the following upper bound on the size n of the graph, called the Moore bound:

$$n \leqq (d(d-1)^k - 2)/(d-2).$$

For $k > 2$, and $d > 2$, the Moore bound cannot be attained [4]. Hence the main interest has been in constructing networks of degree $d > 2$ and diameter $k > 2$ whose size approaches the Moore bound. *See* the tables in [4, 5] for the largest known graph sizes for small values of d and k. These largest known graphs are constructed by different methods for different degrees and diameters, these constructions are often applicable only for a small range of parameters [1] and they are not suitable for network design.

The best general constructions of large graphs of the given degree and diameter that satisfy the communication network requirements are de Bruijn graphs [15, 2] and their variations, such as Kautz graphs [9], generalized de Bruijn graphs [6] and Imase-Itoh graphs [8].

Recently, Opatrny, Sotteau, Srinivasan, and Thulasiraman proposed in [14] *DCC Linear Congruential Graphs* as a generalization of de Bruijn graphs. These graphs are much larger than de Bruijn graphs for the same degree and diameter. In this paper we present a generalization of linear congruential graphs, called *d-dimensional Linear Congruential Graphs.* Informally, in a d-dimensional linear congruential graph, the vertices are a subset of vectors of a d-dimensional space and the edge set is defined by a finite set of functions of type $f(\vec{x}) = \vec{x} A + \vec{b}$ where A is a $d \times d$ matrix of integers and $\vec{b}$ is a d-dimensional vector of integers. We will call these functions the *generators* of the graph.

In Section 2 of this paper we give a definition of d-dimensional linear congruential graphs.

In Section 3 and 4 we discuss two subfamilies of d-dimensional linear congruential graphs. For each subfamily sufficient conditions on generators to generate regular, maximally connected graphs are given. We also give tables of their diameters for various vertex set sizes and degrees. These graphs contain more vertices than DCC Linear Congruential Graphs or de Bruijn graphs of the same degree and diameter.

2. D-DIMENSIONAL LINEAR CONGRUENTIAL GRAPHS

We will use N to denote the set of nonnegative integers, and Z_p to denote the set $\{0, 1, \ldots, p-1\}$.

DEFINITION 2.1: For a positive integer d we defined a linear function of dimensional d as $f(\vec{x}) = \vec{x}A + \vec{b}$ where A is a $d \times d$ matrix of integers, $\vec{b}$ is a vector of integers of length d, and $\vec{x}$ is a vector of variables of length d.

DEFINITION 2.2: Let d be a positive integer. Let $\vec{s}$ be a constant vector of length d, $\vec{s} = (s_1, s_2, \ldots, s_d)$, $s_i \in N - \{0\}$ for $1 \leqq i \leqq d$, and F be a set of linear functions of dimension d, $F = \{ f_i(\vec{x}) \mid f_i(\vec{x}) = \vec{x}A_i + \vec{b_i}, 1 \leqq i \leqq k$ for some $k\}$. We define graph $G(F, \vec{s})$, called a *linear congruential graph of dimension* d as a graph on the vertex set $V = Z_{s_1} \times Z_{s_2} \times \cdots \times Z_{s_d}$, in which any $\vec{x} \in V$ is adjacent to the vertices $f_i(\vec{x}) \bmod \vec{s}$, for $1 \leqq i \leqq k$.

For a subset V_1 of V and a linear function g, we define a graph $G(F, \vec{s}, g, V_1)$ of dimension d as a graph on vertex set V, in which any $\vec{x} \in V$ is adjacent to the vertices $f_i(\vec{x}) \bmod \vec{s}$, $1 \leqq i \leqq k$ and any $\vec{x} \in V_1$ is also adjacent to the vertex $g(\vec{x}) \bmod \vec{s}$.

We use $G(F, \vec{s})$ to generate large regular graphs on even degrees, while $G(F, \vec{s}, g, V_1)$ is used to generate large regular graphs of odd degrees. The size of the above graphs is determined by $\vec{s}$ and is equal to $s_1 \star s_2 \star \cdots \star s_d$. *See* figure 1 for an example of a 2-dimensional linear congruential graph.

We call the linear functions in F and $F \cup g$, the *generators* of $G(F, \vec{s})$ and $G(F, \vec{s}, g, V_1)$, respectively. For any generator f, the graph $G(\{f\}, \vec{s})$ is called the *graph generated by* f *on* $\vec{s}$.

Clearly, Linear Congruential Graphs [13] and DCC Linear Congruential Graphs [14] are 1-dimensional linear congruential graphs. Since de Bruijn graphs are a special case of linear congruential graphs, de Bruijn graphs are also a special case of 1-dimensional linear congruential graphs.

DEFINITION 2.3: We say that a generator f is of cycle type k^i on $\vec{s}$ if the graph generated by f on $Z_{s_1} \times Z_{s_2} \times \cdots \times Z_{s_d}$ consists of k^i vertex-disjoint cycles of the same length.

When studying the linear congruential graphs, the best results with respect to the diameter of linear congruential graphs of size $n = k^i m$ and degree t generated by linear functions $f_1, f_2, \ldots$ were obtained in [14] when the generator f_i is of cycle type k^{i-1} on n for $1 \leqq i \leqq \lceil t/2 \rceil$ and all cycles are edge disjoint. Such a set of generators is called *Disjoint Consecutive Cycles (DCC for short) set of generators.* Any DCC set of generators generates regular, maximally connected graphs [14].

We will now state the theorem which is used to obtain DCC sets of generators in the 1-dimensional case as it will be also used in the multidimensional case

THEOREM 2.1: [14, 10] *Let n be a positive integer such that $n = k^i m$ for some integers $k > 1$, $i \geqq 2$, and m. Let c_1 be an integer such that $gcd(c_1, n) = 1$, and let b_1 be a product of all prime factors of n; b_1 also has 4 as a factor if n is divisible by 4. Let $g_j(x) = (k^j b_1 + 1) x + k^j c_1$. For every j, $0 \leqq j \leqq i$ the function g_j is of cycle type k^j. Furthermore, the cycles generated by g_{j1}, g_{j2} are edge disjoint if $j_1 \neq j_2$.*

We say that integers a, b satisfy Theorem 2.1 with respect to n if the function $ax + b$ corresponds to one of the functions g_j from the theorem.

NOTATION 2.1: *We will assume in the sequel that in a generator $f_i(\vec{x}) = \vec{x} A_i + \vec{b}_i$,*

$$A_i = \begin{pmatrix} a_{i,11} & \cdots & a_{i,1d} \\ a_{i,21} & \cdots & a_{i,2d} \\ \cdots & & \\ a_{i,d1} & \cdots & a_{i,dd} \end{pmatrix}, \qquad \vec{b}_i = (b_{i,1}, \ldots, b_{i,d}).$$

In the multidimensional linear congruential graphs, we will distinguish the following two cases of generators.

DEFINITION 2.4: Let $f_i(\vec{x}) = \vec{x} A_i + \vec{b}_i$ be a d-dimensional linear congruential function. We say that f if of Type 1 if $a_{i,j} = 0$ for $i \neq j$, and we say that f is of Type 2 if there exists $i \neq j$ such that $a_{i,j} \neq 0$. We say that a d-dimensional congruential graph is of Type 1 if all generators are of type 1, and is of Type 2 otherwise.

Our intention is to generalize DCC linear congruential graphs to the d-dimensional case, since we expected that this might improve the number of nodes in a graph of a given degree and diameter. Thus one of the generators should generates a Hamiltonian cycle. It follows from the result in [10] that this is possible only when the value in $\vec{s}$ are relative primes. Thus we will restrict our attention only to this case.

3. D-DIMENSIONAL LINEAR CONGRUENTIAL GRAPHS OF TYPE 1

LEMMA 3.1: *A d-dimensional linear congruential graph $G(F, (s_1, \ldots, s_d))$ of type 1 can be obtained as a "product" $G_1 \circ G_2 \circ \cdots \circ G_d$ of d 1-dimensional graphs $G_1 = G(F_1, s_1)$, $G_2 = G(F_2, s_2), \ldots, G_d = G(F_d, s_d)$ where*

1. $F_j = \{a_{i,jj}\, x + b_{i,j} \mid 1 \leqq i \leqq k\}$, $1 \leqq j \leqq d$, and

2. the operation $\circ$ is defined as follows:

$G_1 \circ G_2 \circ \cdots \circ G_d$ is a graph with the vertex set $V(G_1) \times V(G_2) \times \cdots \times V(G_d)$ and the edge set $\{((u_1, u_2, \ldots, u_d), (v_1, v_2, \ldots, v_d)) | (u_1, v_1) \in E(G_1), (u_2, v_2) \in E(G_2) \ldots, (u_d, v_d) \in E(G_d)\}$

Proof: For any generator $f_i = \vec{x}_i A_i + b_i$ of type 1, all entries in A outside of the main diagonal are equal to 0. Thus the value of the jth component of $f_i(x)$ depends only on the value of the jth component of x, $a_{i,jj}$, and $b_{i,j}$. This implies that the values of f_i can be calculated using d ordinary linear functions, which allows the above decomposition of the graph into d 1-dimensional graphs. □

This decomposition allows us to find d-dimensional generators of specific cycle type by using the known results for DCC graphs.

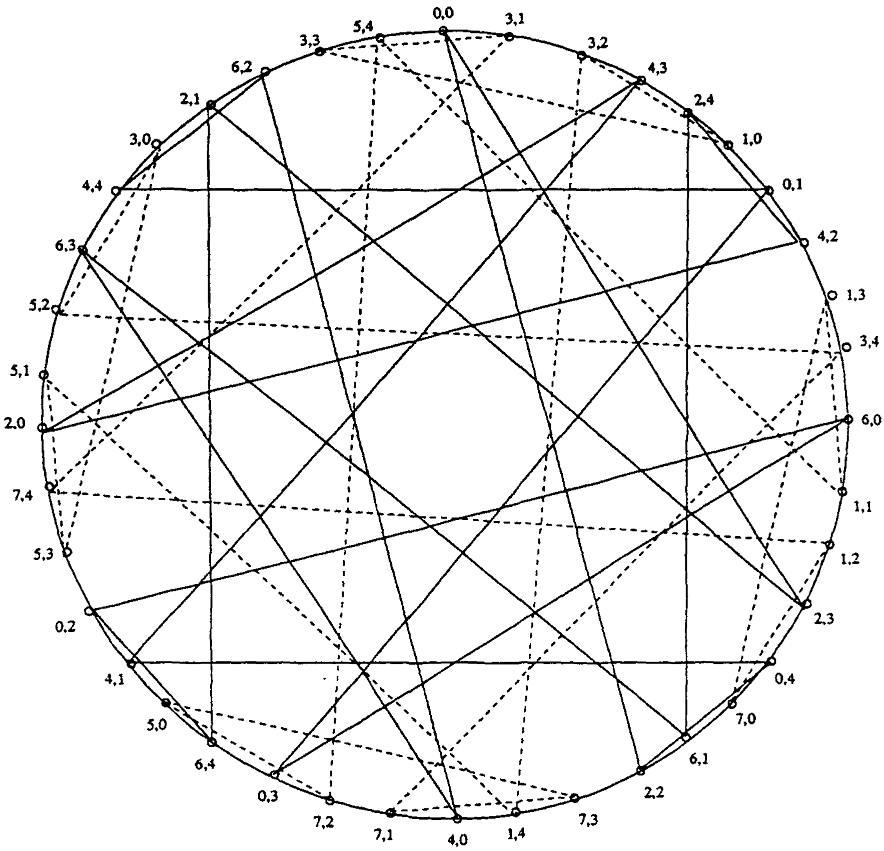
LEMMA 3.2: *Let $f_i = \vec{x} A_i + \vec{b}_i$ be a type 1 generator. If for $1 \leqq j \leqq d$, $f_{i,j}(x) = a_{i,jj}\, x + b_{i,j}$ is of cycle type c_j on s_j and s_j, s_k are relative primes for $1 \leqq j < k \leqq d$, then f_i is of cycle type $\prod_{j=1}^{d} c_j$ on $(s_1, \ldots, s_d)$.*

Proof: It follows immediately from Lemma 3.1.

THEOREM 3.1: *Let r be an integer, s be a d dimensional vector of sizes such that $s_1 = k^l m$ and $s_1, s_2, \ldots, s_d$ be pairwise relatively prime numbers. Let $a_{i,11}$ and $b_{i,1}$ be integers so that $a_{i,11}\, x + b_{i,1}$ is of cycle type k^{i-1} on $k^l m$ according to Theorem 2.1, and let $a_{i,jj}\, x + b_{i,j}$ generate a Hamiltonian cycle on s_j for $2 \leqq j \leqq d$ and $1 \leqq i \leqq r$. Then the graph $G(\{f_1, f_2, \ldots, f_r\}, \vec{s})$ is a graph such that*

1. f_i, is of cycle type $k^{i-1}$for $1 \leqq i \leqq r$, and all the cycles are edge-disjoint, i. e. *the cycle structure is identical to the cycle structure of DCC graphs,*

2. the graph is regular of degree $2r$,



$$f_1 = \vec{x}\begin{pmatrix} 5 & 0 \\ 1 & 1 \end{pmatrix} + (3,\, 1) \qquad f_2 = \vec{x}\begin{pmatrix} 9 & 0 \\ 2 & 1 \end{pmatrix} + (2,\, 3)$$

Figure 1.

3. the graph is maximally connected,

4. furthermore, the graph $G\left(\{f_1, f_2, \ldots, f_r\}, (k^p m, s_2, \ldots, s_d)\right)$ *has the 3 above properties for any* $p > i$.

Proof: It follows directly from the properties of DCC graphs [14].

We could not obtain any good bound on the diameter of d-dimensional linear congruential graphs. We have therefore calculated the diameters of Type 1 graphs by computer. So far, we mostly investigated 2-dimensional linear congruential graphs.

The best values for the diameter of Type 1 graphs are summarized in Tables 1 and 2.

Table 1 a

Degree	Diameter	Size	Generators	
4	8	$32 * 9 * 5 = 1440$	5 0 0 0 2 0 0 0 2	1 1 1
4	8	$32 * 9 * 5 = 1440$	9 0 0 0 4 0 0 0 1	2 1 1
4	9	$32 * 81 = 2592$	5 0 0 2	1 2
4	9	$32 * 81 = 2592$	9 0 0 4	4 1
6	5	$8 * 81 = 648$	5 0 0 1	2 3
6	5	$8 * 81 = 648$	9 0 0 4	3 2
6	5	$8 * 81 = 648$	17 0 0 10	1 3
6	6	$16 * 81 = 1296$	5 0 0 1	2 3
6	6	$16 * 81 = 1296$	9 0 0 4	3 2
6	6	$16 * 81 = 1296$	17 0 0 10	1 3

NOTE 3.1: *In Tables 1b, 2, and 3, each row is labeled by a degree and each column by the size of the graph. Each entry specifies the matrix and the constant vector of generators, and the upper right corner gives the diameter of the graph.*

Type 1 graphs gave us many improvements on the size of a graph of a given degree and diameter in comparisons to DCC graphs and de Bruijn graphs [14].

4. D-DIMENSIONAL LINEAR CONGRUENTIAL GRAPHS OF TYPE 2

For type 2 multidimensional graphs, we do not have a simple decomposition of multidimensional linear congruential graphs into

Table 1 b

Deg \ Size	9216	18432	36864	73728	147456
	1024 * 9	2048 * 9	4096 * 9	8192 * 9	16384 * 9
3				5 0 **19** 0 1 3 4 9 0 0 2 2 2	5 0 **20** 0 1 3 4 9 0 0 2 2 1
4	5 0 **10** 0 1 3 5 9 0 0 4 2 2	5 0 **11** 0 1 3 5 9 0 0 4 2 2	5 0 **12** 0 1 2 1 9 0 0 5 1 1		5 0 **13** 0 1 3 4 9 0 0 2 2 1
5	5 0 **8** 0 1 6 1 9 0 0 1 5 2 17 0 0 1 6 4	5 0 **9** 0 1 7 1 9 0 0 1 7 2 17 0 0 1 4 3		5 0 **10** 0 1 7 1 9 0 0 1 7 2 17 0 0 1 4 3	5 0 **11** 0 1 7 1 9 0 0 1 7 2 17 0 0 1 4 3
6	5 0 **7** 0 1 7 1 9 0 0 1 7 2 17 0 0 1 4 3		5 0 **8** 0 1 9 1 9 0 0 1 8 2 17 0 0 1 9 3		5 0 **9** 0 1 7 1 9 0 0 1 7 2 17 0 0 1 4 3

1-dimensional linear congruential graphs, and thus we cannot use the known results from the 1-dimensional linear congruential graphs in a straigh-forward manner. At the beginning of our investigations we had to obtain an insight on type 2 generators using computers. To simplify the problem, we considered only the case of two dimensions and $s_1 = 2^i$ since this case gave us many different graphs whose size is within reasonable computational limits. For that reason we have so far results on the cycle structure of type 2 generators when we have 2 dimensions, and $s_1 = 2^i$ for some positive integer i, which will be assume in this section.

THEOREM 4.1: *Let* $f(\vec{x}) = \vec{x}A + \vec{b}$ *be a 2-dimensional generator, and* $\vec{s} = (2^i, 4k+1)$. *If* $a_{11} - 1$ *contains* 4 *as a factor,* $a_{12} = 0$, $a_{22} = 1$, b_1 *is odd, and* b_2, $4k+1$ *are relative primes, then* $f(x)$ *generates a Hamiltonian cycle on* $(2^i, 4k+1)$ *for any* $i \geqq 0$.

Proof: We will use induction on i. If $i = 0$ then we have again a 1-dimensional case and the theorem is true by Theorem 2.1.

Assume that the theorem is true for some $i = n$. Let $\vec{s}_1 = (2^i, 4k+1)$, $\vec{s}_2 = (2^{i+1}, 4k+1)$. Let $G_1 = G(\{f\}, \vec{s}_1)$, $G_2 = G(\{f\}, \vec{s}_2)$. For any vertex $\vec{x} = (x_1, x_2)$ of G_1 there are two possibilities for the value $f(\vec{x}) \bmod \vec{s}_2$:

Table 2

Deg \ Size	576	1152	2304	4608	9216	18432	36864	73728	147456
	64 * 9	128 * 9	256 * 9	512 * 9	1024 * 9	2048 * 9	4096 * 9	8192 * 9	16384 * 9
7		5 0 0 1 5 1 9 0 0 1 6 2 17 0 0 1 2 3 33 0 0 1 2 8 **[5]**		5 0 0 1 5 1 9 0 0 1 6 2 17 0 0 1 2 3 33 0 0 1 2 8 **[6]**		5 0 0 1 5 1 9 0 0 1 6 2 17 0 0 1 2 3 33 0 0 1 2 8 **[7]**			
8	5 0 0 1 5 1 9 0 0 1 6 2 17 0 0 1 2 3 33 0 0 1 2 8 **[4]**		5 0 0 1 5 1 9 0 0 1 6 2 17 0 0 1 2 3 33 0 0 1 3 7 **[5]**		5 0 0 1 5 1 9 0 0 1 6 2 17 0 0 1 2 3 33 0 0 1 3 7 **[6]**				5 0 0 1 15 1 9 0 0 1 14 2 17 0 0 1 16 3 33 0 0 1 4 7 **[8]**
9			5 0 0 1 2 1 9 0 0 1 5 2 17 0 0 1 1 3 33 0 0 1 1 7 65 0 0 1 3 4 **[5]**			5 0 0 1 2 1 9 0 0 1 5 2 17 0 0 1 1 3 33 0 0 1 1 7 65 0 0 1 3 4 **[6]**			
10		5 0 0 1 2 1 9 0 0 1 5 2 17 0 0 1 1 3 33 0 0 1 1 7 65 0 0 1 3 4 **[4]**		5 0 0 1 2 1 9 0 0 1 5 2 17 0 0 1 3 3 33 0 0 1 1 7 65 0 0 1 1 1 **[5]**			5 0 0 1 2 1 9 0 0 1 5 2 17 0 0 1 3 3 33 0 0 1 1 7 65 0 0 1 1 1 **[6]**		5 0 0 1 2 1 9 0 0 1 5 2 17 0 0 1 3 3 33 0 0 1 1 7 65 0 0 1 1 1 **[7]**

1. $f(\vec{x}) \bmod \vec{s}_2 = f(\vec{x}) \bmod \vec{s}_1$.

2. $f(\vec{x}) \bmod \vec{s}_2 = f(\vec{x}) \bmod \vec{s}_1 + 2^i$. In this case $f(\vec{x} + (2^i, 0)) \bmod \vec{s}_2 = f(\vec{x}) \bmod \vec{s}_1$, and we will say that there is an *edge change at the pair* $\vec{x}$, $\vec{x} + (2^i, x_2)$.

Thus, G_2 can be constructed from two copies of G_1 by modifying the edges for all the pairs having the edge change, similarly as it can be done for linear congruential graphs [13]. Since f generates a Hamiltonian cycle on $\vec{s}_1$ by the inductive hypothesis, then if the number of pairs of vertices having an edge change is odd, then f will also generate a Hamiltonian cycle on $\vec{s}_2$. For every j, $0 \leqq j \leqq 4k$ consider the linear function $f^j(x)$ obtained from $f(x, y)$ by fixing the value of $y = j$.

If a_{21} is even then f^j generates a Hamiltonian cycle on 2^{i+1} for any j, $0 \leqq j \leqq 4k$ and each function contributes an odd number of edge changes, and thus the total number of edge changes is odd.

If a_{21} is odd then f^j generates a Hamiltonian cycle on 2^{i+1} for any $j = 0, 2, \ldots$, and it generates two disjoint cycles otherwise, *see* Theorem 2.1. Thus, $2k + 1$ of the functions f^j contribute odd number of edge changes, and $2k$ of them contribute an even number of edge changes, which gives an odd number of edge changes in total.

Thus, in either case, since the total of pairs having edge changes is odd, f generates a Hamiltonian cycle on s_2. □

THEOREM 4.2: *Let* $G(\{f_1, f_2\}, (2^i, s_2))$ *be a two-dimensional linear congruential graph,* $f_1(\vec{x}) = \vec{x} A_1 + b_1$, $f_2(\vec{x}) = \vec{x} A_2 + \vec{b}_2$. *Generators* $f_1(\vec{x})$, $f_2(\vec{x})$ *will create edge-disjoint cycles on* $(2^i, s_2)$ *if* $a_{1,11} x + b_{1,1}$, $a_{2,11} x + b_{2,1}$ *satisfy Theorem 2.1,* $b_{1,1}$, $b_{2,1}$ *are of different parities, and* $a_{1,21}$, $a_{2,21}$ *are of the same parity.*

Proof: It is easy to show that in this case the values of the first dimension of $f_1(\vec{x})$ and $f_2(\vec{x})$ will be of different parities and, therefore, the cycles generated by the two generators are disjoint. □

When analysing 2-dimensional linear congruential graphs of type 2, we usually selected one generator according to Theorem 4.1 and the remaining generators were selected using Theorem 4.2. This ensures that the graphs are connected.

Notice that in the above theorems, the constants in the generators which generate Hamiltonian cycles or edge-disjoint cycles are independant of the value of i in the size vector $\vec{s} = (2^i, s_2)$. Thus, type 2 graphs obtained with generators satisfying the above theorems are extensible by any factor of 2.

Table 3

Size / Deg	640	1280	2560	5120	10240	20480	40960
	128 * 5	256 * 5	512 * 5	1024 * 5	2048 * 5	4096 * 5	8192 * 5
3	5 0 [10] 6 1 13 1	5 0 [12] 0 1 6 8	5 0 [13] 0 1 6 8	5 0 [14] 10 1 5 1		5 0 [17] 4 1 1 1	5 0 [18] 0 1 7 1
	9 0 6 1 14 1	9 0 0 1 3 4	9 0 0 1 3 4	9 0 10 1 6 1		9 0 0 1 4 1	9 0 0 1 12 1
4	5 0 [7] 1 1 1 1	5 0 [8] 1 1 1 1	5 0 [9] 0 1 1 1			5 0 [11] 1 1 1 1	5 0 [12] 1 1 1 1
	9 0 5 1 8 1	9 0 5 1 4 1	9 0 0 1 4 1			9 0 3 1 4 1	9 0 3 1 4 1
5	5 0 [6] 4 1 5 1		5 0 [7] 0 1 3 1				
	9 0 4 1 2 1		9 0 0 1 4 2				
	17 0 0 1 4 1		17 0 0 1 1 1				
6	5 0 [5] 4 1 5 1	5 0 [6] 0 1 3 1	—				
	9 0 4 1 2 1	9 0 0 1 2 1					
	17 0 0 1 4 1	17 0 0 1 1 1					

Table 3 summarizes the results that were obtained using type 2 generators. Similarly as in the case of type 1 graphs, we do not have good estimate on the diameter of type 2 graphs and, therefore, the values in Table 3 were derived by a computer analysis. Type 2 graphs improve further the size of graphs for many different values of degrees and diameters.

5. CONCLUSIONS

The d-dimensional linear congruential graphs presented in this paper form a very interesting family graphs. Similarly as DCC linear congruential graphs, they are defined for both, odd and even degrees, and can be defined for many different graph sizes. They also satisfy the extensibility requirements in network design. As seen from the tables, they are much larger, for the same diameter and degree, than any other general construction. The improvement in the graph size when compared to the family of linear congruential graphs in more than 15%, and the size of de Bruijn graphs is a fraction of the size of multidimensional linear congruential graphs of the same degree and diameter.

Further studies of higher dimensional linear congruential graphs could give additional increase in the size of graphs than can be obtained for the same degree and diameter.

The problem of obtaining a good upper bound on the diameter of d-dimensional linear congruential graphs is a very interesting open problem that requires further studies.

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REFERENCES

1. J. C. Bermond, C. Delorme and J. J. Quisquater, Strategies for interconnection networks: some methods from graph theory, *Journal of Parallel and Distributed Computing*, 1986, *3*, pp. 433-449.
2. J. C. Bermond, C. Peyrat, de Bruijn and Kautz networks: a competitor for the hypercube? *Hypercube and Distributed Computers*, 1989, pp. 279-294.
3. J. A. Bondy and U. S. R. Murty, *Graphs Theory with Applications*, North Holland, 1976.
4. F. R. K. Chung, *Diameters of graphs: old problems and new results*, Proceedings of the 18th South-Eastern Conference on Combinatorics, Graph Theory, and Computing, Congressus Numerantium, 1987, pp. 295-317.
5. C. Delorme, *A Table of Large Graphs of Small Degrees and Diameters*, personal communication, 1990.
6. D. Z. Du and F. K. Hwang, Generalized de Bruijn Digraphs, *Networks*, 1988, *18*, pp. 28-38.
7. B. Elpas, Topological Constrains on Interconnection Limited Logic, *Switching Circuits Theory and Logical Design*, 1964, *5*, pp. 133-147.
8. M. Imase and M. Itoh, Design to minimize diameter on building block network, *IEEE Trans. on Computers*, 1981, *C-30*, pp. 439-442.
9. W. H. Kautz, Bounds on directed (d, k) graphs, Theory of Cellular Logic Networks and Machines, *SRI Project 7258*, 1968, pp. 20-28.
10. D. E. Knuth, *The art of computer programming, Seminumerical Algorithms*, Addison-Wesley, *II*, 1972.
11. C. C. Koung, *Multi-dimensional Linear Congruential Network Models*, Master's Thesis, Dept. of Comp. Sci. Concordia University, Montreal, 1993.
12. W. Leland and M. Solomon, Dense trivalent graphs for processor interconnection, *IEEE Trans. on Computers*, 1982, *31*, No. 3, pp. 219-222.

13. J. Opatrny and D. Sotteau, *Linear Congruential Graphs, Graph Theory, Combinatorics, Algorithms, and Applications*, SIAM proceedings series, 1991, pp. 404-426.
14. J. Opatrny, D. Sotteau, N. Srinivasan and K. Thulasiraman, DCC Linear Congruential Graphs, a New Network Model, *IEEE Trans. Comput.*, to appear.
15. M. R. Samantham and D. K. Pradham, The de Bruijn Multiprocessor Network: A Versatile Parallel Processing and Sorting Network for VLSI, *IEEE Trans. Comput.*, 1989, *38*, No. 4, pp. 567-581.