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AUTONOMOUS POSETS AND QUANTALES (*) (1)

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Abstract. – *In this paper we consider partially ordered algebraic structures arising in the semantics of formulas of a non commutative version of Girard linear logic. The non commutative version we treat is the one recently proposed by V. M. Abrusci. We introduce* autonomous quantales *and prove a completion theorem from autonomous posets to autonomous quantales and a representation theorem* "every autonomous quantale is isomorphic to a non commutative phase space quantale", *generalizing previous existing results valid in the commutative case.*

Résumé. – *Dans cet article nous considérons les structures algébriques partiellement ordonnées qui surviennent dans la sémantique des formules d'une version non commutative de la logique linéaire de Girard. La version non commutative que nous traitons est celle qui a récemment été proposée par V. M. Abrusci. Nous introduisons les* quantales autonomes *et nous prouvons un théorème de complétude* « tout quantale autonome est isomorphe à un quantale d'un espace de phase non commutatif » *généralisant ainsi des résultats antérieurs concernant le cas commutatif.*

1. INTRODUCTION

In this paper we consider partially ordered algebraic structures arising in the semantics of formulas of a non commutative version of the linear logic introduced in [17]: the non commutative version we consider is that proposed in [3].

Autonomous posets are closed posets, *i.e.* partially ordered monoids with two "linear implications" satisfying an "adjunction" property with respect to the monoid operation, such that every element is a fixed point of a "double negation" operator with respect to a so called *dualizing* element.

Autonomous quantale are quantales, *i.e.* complete lattices with a binary associative operation $\otimes$ such that $a \otimes -$ and $-\otimes a$ preserve arbitrary sups

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for any element *a* of the quantale, such that every element is a fixed point of a "double negation" operator with respect to a so called *dualizing* element.

The following two results are obtained:

1) a completion theorem from autonomous posets to autonomous quantales;

2) a representation theorem for autonomous quantales:

"every autonomous quantale is isomorphic to a non commutative phase space quantale".

The paper is organized as follows.

– in the first paragraph we consider autonomous posets,

– in the second paragraph we deal with autonomous quantales,

– in the third paragraph we study the so called "phase space" autonomous quantale and establish a completion and a representation theorem.

Related work in Logic and Computer Science can be presented as follows:

1) Algebraic structures.

The definitions and the results presented in this paper are a generalization in the non commutative case of Rosenthal's work [31] in the "cyclic" case [37].

Significant work on algebraic structures related to autonomous posets and quantales include references [8, 9, 13, 21, 22].

2) Categorical structures.

Autonomous posets are in close relationships with categorical structures studied in [6, 12, 15, 32, 33].

3) Semantics of formulas.

The Tarski-style semantics of formulas for propositional and first order "resource-sensitive" logics like linear logic are considered in [4, 5, 16, 19, 23, 24] and [27], representation theorems being related to completeness theorems.

4) Computer science applications.

Algebraic structures strongly related to autonomous posets and quantales arise in various areas of computer science:

– in the area of semantics and logics of programs: [18, 20, 35];

– in the area of semantics and logics of concurrent processes: [1, 2, 10, 11, 14, 25];

– in the area of logics for knowledge representation: [7, 28].

2. AUTONOMOUS POSETS

In this paragraph we consider two kinds of partially ordered algebraic structures. The closed posets are partially ordered (non commutative) monoids with two "linear implications". Then we treat closure operators satisfying a "compatibility condition" w.r.t. the monoid operation and called closed nuclei. More precisely we have two negation operators ${}^{\perp}(-)$ and $(-)^{\perp}$ and deal with a sufficient condition to obtain a closed nuclei $({}^{\perp}(-))^{\perp} = {}^{\perp}((-)^{\perp})$.

Finally we study the partially ordered algebraic structures called *autonomous posets* which are the non commutative version of *-autonomous posets. In such structures the two operators $({}^{\perp}(-))^{\perp}$ and ${}^{\perp}((-)^{\perp})$ are equal and every element is a fixed point of such operators. Moreover we define the non commutative version of the "par" connective of linear logic.

2.1. Closed posets

We introduce closed posets and closed maps and their unital version. Examples of such structures arise in various domains. Of particular interest in our context is the power set of a semigroup or a monoid.

2.1.1. DEFINITION: (i) A *Closed Poset* is a structure $(P, \leqq, \otimes, -\!\!\bullet_l, -\!\!\bullet_r)$ such that:

- $(P, \leqq)$ is a partially ordered set,
- $(P, \otimes)$ is a semigroup,
- for all $a, b, c \in P$: $a \otimes c \leqq b$ iff $c \leqq a -\!\!\bullet_r b$

$$c \otimes a \leqq b \text{ iff } c \leqq a -\!\!\bullet_l b$$

A *Unitial closed poset* $(P, \leqq, \otimes, 1)$ is a closed poset such that $(P, \otimes, 1)$ is a monoid with two-sided identity 1.

(ii) Let $(P, \leqq_P, \otimes_P)$ and $(Q, \leqq_Q, \otimes_Q)$ two closed posets. A *Closed Map* $f: P \to Q$ is an order preserving function from P to Q such that for all $a, b \in P$:

$$f(a) \otimes_Q f(b) \leqq f(a \otimes_P b).$$

Let $(P, \leqq_P, \otimes_P, 1_P)$ and $(Q, \leqq_Q, \otimes_Q, 1_Q)$ be two unital closed posets. A *Unital closed map* $f: P \to Q$ is a closed map from P to Q such that $1_Q \leqq f(1_P)$.

2.1.2. *Remark:* The partially ordered algebraic structure of autonomous posets can be generalized in a categorical setting in at least two ways:

(i) as a monoidal biclosed category,

(ii) as an enriched category: a quantaloid.

2.1.3. *Examples:* 1) Every partially ordered group $(G, \cdot, (\ )^{-1})$ is a unital closed posed with:

$$a \otimes b = a.b, \quad a \multimap_r b = a^{-1}.b, \quad a \multimap_l b = b.a^{-1}.$$

2) For every semigroup $(M, -)$ the power set of M forms a closed posed $(\mathscr{P}(M), \subseteq, \otimes)$ with:

- inclusion $\subseteq$ as partial ordering;
- $A \otimes B = \{a.b \mid a \in A, b \in B\}$;
- $A \multimap_r B = \{x \mid \text{for every } a \in A\colon a.x \in B\}$;
- $A \multimap_l B = \{x \mid \text{for every } a \in A\colon x.a \in B\}$.

3) Let R be a (non commutative) ring. The left ideals of R form a closed poset $(\mathrm{Lidl}(R), \subseteq, \otimes)$ with

- $A \otimes B = \left\{\sum_{i}^{n} a_i b_i \mid a_i \in A, b_i \in B\right\}$;
- $A \multimap_l B = \{x \in R \mid \text{for every } a \in A\colon x.a \in B\}$;
- $A \multimap_r B = \sum \{x \in \mathrm{Lidl}(R) \mid A x \in B\}$.

Similarly the right ideals of R form a closed poset $(\mathrm{Ridl}(R), \subseteq, \otimes)$ with partial ordering as above, $A \otimes B$ as above and

- $A \multimap_r B = \{x \in R \mid \text{for every } a \in A\colon a.x \in B\}$;
- $A \multimap_l B = \sum \{x \in \mathrm{Ridl}(R) \mid x A \in B\}$.

Basic facts used in the following are presented.

2.1.4. Facts: Let P be a closed poset, for every $a, b, c \in P$:

1) $c \leq a \multimap_r b$ iff $a \leq c \multimap_l b$;

2) $a \otimes (a \multimap_r b) \leq b$ and $(a \multimap_l b) \otimes a \leq b$;

3) $(a \otimes b) \multimap_r c = b \multimap_r (a \multimap_r c)$ and $(a \otimes b) \multimap_l c = a \multimap_l (b \multimap_l c)$;

4) $a \multimap_r (b \multimap_l c) = b \multimap_l (a \multimap_r c)$;

5) $a \leq b$ implies $c \multimap_r a \leq c \multimap_r b$ and $c \multimap_l a \leq c \multimap_l b$

Sketch of proof:

1) $c \leqq a \multimap_r b$ iff $a \otimes c \leqq b$ iff $a \leqq c \multimap_l b$.

For points 2), 3) and 4) *see* Rosenthal's book (proposition 2.1.1).

5) From $c \otimes (c \multimap_r a) \leqq a$ and $a \leqq b$ we have $c \otimes (c \multimap_r a) \leqq b$ with is equivalent to $(c \multimap_r a) \leqq (c \multimap_r b)$. From $(c \multimap_l a) \otimes c \leqq a$ and $a \leqq b$ we have $(c \multimap_l a) \otimes c \leqq b$ which is equivalent to $(c \multimap_l a) \leqq (c \multimap_l b)$

2.2. Closure operators and closed nuclei: Double negation operator

Let us recall the definition of closure operators. In particular we are interested in the closure operators called "double negation": we have two distinct ones since we are in a non commutative context.

2.2.1. DEFINITION: A *Closure operator* on a closed poset $(P, \leqq, \otimes)$ is a closure operator $j: P \rightarrow P$ on the partially ordered set $(P, \leqq)$ *i.e.*

- for every $a \in P: a \leqq j(a)$;
- for every $a, b \in P: a \leqq b$ implies $j(a) \leqq j(b)$;
- for every $a \in P: j(j(a)) \leqq j(a)$.

Let us consider now the two operators corresponding to double negations.

Notation: Let P be a closed poset, and $a, c \in P$, let us denote $a \multimap_l c$ by ${}^c a$ and $a \multimap_r c$ by a^c.

The basic properties of the double negations are given in the following:

2.2.2. LEMMA: *Let P be a closed poset and $c \in P$.*

1) *The maps* $({}^c(-))^c: P \rightarrow P$ *and* ${}^c((-)^c): P \rightarrow P$ *are closure operators.*

2) *For all* $a, b \in P$:

(i) $a \leqq b$ *implies* ${}^c b \leqq {}^c a$ *and* $b^c \leqq a^c$;

(ii) $a \multimap_l b \leqq {}^c b \multimap_r {}^c a$ *and* $a \multimap_r b \leqq b^c \multimap_l a^c$,

(iii) $a \multimap_l b \leqq ({}^c a)^c \multimap_l ({}^c b)^c$ and $a \multimap_r b \leqq {}^c(a^c) \multimap_r {}^c(b^c)$.

Proof: 1) *See* Rosenthal Lemma 3.3.1.

2) (i) From $a \leqq b$ and $b \leqq {}^c(b^c)$ we have $a \leqq {}^c(b^c)$, hence by fact 2.1.4. 1), $b^c \leqq a^c$. Similarly $a \leqq b$ implies ${}^c b \leqq {}^c a$.

(ii) From 1): $a \leqq (a^b) \multimap_l b$, by adjunction: $a \otimes (a^b) \leqq b$. So by 2) (i): $b \multimap_r c \leqq (a \otimes a^b) \multimap_r c$, by adjunction $(a \otimes a^b) \otimes (b \multimap_r c) \leqq c$, by associativity $a \otimes (a^b \otimes b^c) \leqq c$, by adjunction $a^b \otimes b^c \leqq a \multimap_r c$, by adjunction $a^b \leqq b^c \multimap_l (a \multimap_r c)$.

(iii) From 2) (i), 2) (ii) and transitivity.

Closed nuclei are now introduced: they are closure operators "compatible" with the monoid structure of a closed poset. Our aim is to study under which conditions the two double negation closure operators coincide and are closed nuclei.

2.2.3. Definition: A map $j: P \to P$ is a *closed nucleus* iff j is a closure operator and j is a closed map *i.e.* for every a, $b \in P$: $j(a) \otimes j(b) \leqq j(a \otimes b)$.

In a first step let us consider some properties of these operators, related to the closed nucleus condition, which hold in general in every closed poset.

2.2.4. Lemma: *Let P be a closed poset and $c \in P$*

1) (i) *${}^c(a^c) = a$ iff there exists $b \in P$ such that $a = {}^c b$;*

(ii) *$({}^c a)^c = a$ iff there exists $b \in P$ such that $a = b^c$.*

2) (i) *${}^c(a \otimes b) \leqq {}^c(a \otimes ({}^c b)^c)$ and $(a \otimes b)^c \leqq ({}^c(a^c) \otimes b)^c$;*

(ii) *${}^c(({}^c a)^c \otimes b) \leqq {}^c(a \otimes b)$ and ${}^c({}^c(a^c) \otimes b) \leqq {}^c(a \otimes b)$, $(a \otimes {}^c(b^c))^c \leqq (a \otimes b)^c$ and $(a \otimes ({}^c b)^c)^c \leqq (a \otimes b)^c$.*

Proof: 1) *See* Rosenthal 1990, Proposition 2.1.1. (10) and (11).

2) (i) Let us prove ${}^c(a \otimes b) \leqq {}^c(a \otimes ({}^c b)^c)$. Since $b \multimap_l c \leqq ({}^c b)^c \multimap_l c$, by fact 1.1.4. 5) we obtain: $a \multimap_l (b \multimap_l c) \leqq a \multimap_l ((({}^c b)^c \multimap_l c)$, hence by fact 2.1.4. 3): $(a \otimes b) \multimap_l c \leqq (a \otimes ({}^c b)^c) \multimap_l c$.

To prove $(a \otimes b)^c \leqq ({}^c(a^c) \otimes b)^c$ we proceed as follows:

Since $a \multimap_r c \leqq {}^c(a^c) \multimap_r c$, by fact 2.1.4. 5):

$$b \multimap_r (a \multimap_r c) \leqq b \multimap_r ({}^c(a^c) \multimap_r c),$$

hence

$$(a \otimes b) \multimap_r c \leqq ({}^c(a^c) \otimes b) \multimap_r c.$$

2) (ii) Let us prove ${}^c(({}^c a)^c \otimes b) \leqq {}^c(a \otimes b)$.

Since $a \leqq ({}^c a)^c$ then

$$({}^c a)^c \multimap_l (b \multimap_l c) \leqq a \multimap_l (b \multimap_l c),$$

hence $(({}^c a)^c \otimes b) \multimap_l c \leqq (a \otimes b) \multimap_l c$. Similarly for ${}^c({}^c(a^c) \otimes b) \leqq {}^c(a \otimes b)$.

Finally since $b \leqq ({}^c b)^c$ then

$$({}^c b)^c \multimap_r (a \multimap_r c) \leqq b \multimap_r (a \multimap_r c)$$

hence $(a \otimes ({}^c b)^c) \multimap_r c \leqq (a \otimes b) \multimap_r c$. Similarly for $(a \otimes {}^c(b^c))^c \leqq (a \otimes b)^c$.

In a second step we consider some properties of these operators, related to the closed nucleus condition, which hold in closed posets containing a so called "central" element: a condition formulated by Abrusci for the "power set of monoid" example for the semantics of formulas of his non commutative version of linear logic. Here we formulate it in the general framework of closed posets: compare it with point 1) of the above lemma. The relationship between the two formulations is shown in paragraph 4.1.

2.2.5. LEMMA: *Let P be a closed poset.*

If there exists an element, called central, $c \in P$ such that, for all $a \in P$:

- *if there exists* $b \in P$ *such that* $a = {}^c b$ *then* $({}^c a)^c \leqq a$,
- *if there exists* $b \in P$ *such that* $a = b^c$ *then* ${}^c(a^c) \leqq a$,

then, for every $a, b \in P$:

(i) ${}^c(a \otimes b) \leqq {}^c(({}^c a)^c \otimes b)$ *and* $(a \otimes b)^c \leqq (a \otimes {}^c(b^c))^c$

(ii) ${}^c({}^c(a^c) \otimes b) \leqq {}^c(({}^c a)^c \otimes b)$ *and* $(a \otimes ({}^c b)^c)^c \leqq (a \otimes {}^c(b^c))^c$.

Proof: (i) $a \otimes b \multimap_l c = a \multimap_l (b \multimap_l c) \leqq ({}^c a)^c \multimap_l ({}^c(b \multimap_l c))^c$

Since c is central: $({}^c(b \multimap_l c))^c = b \multimap_l c$, thus:

$$a \otimes b \multimap_l c \leqq ({}^c a)^c \multimap_l (b \multimap_l c) = {}^c(a^c) \otimes b \multimap_l c.$$

$$a \otimes b \multimap_r c = b \multimap_r (a \multimap_r c) \leqq {}^c(b^c) \multimap_r {}^c((a \multimap_r c)^c).$$

Since c is central, ${}^c((a \multimap_r c)^c) \multimap_r a \multimap_r c$ hence:

$$a \otimes b \multimap_r c \leqq {}^c(b^c) \multimap_r (a \multimap_r c) = (a \otimes {}^c(b)^c) \multimap_r c.$$

(ii) from 2.2.4. 2) (ii) second inequality and point (i) first inequality above, from 2.2.4. 2) (ii) fourth inequality and point (i) second inequality above.

Finally we obtain the required result concerning double negation operators which are closed nuclei.

2.2.6. PROPOSITION: *Let P be a unital closed poset. If c is a central element of P then*

1) ${}^c((-)^c) = ({}^c(-))^c : P \to P$ *denoted* j_c.

2) j_c *is a closed nucleus.*

Proof: 1) From 2.2.5. (ii) first inequality by putting $b = 1$, we have: ${}^c({}^c(a^c)) \leqq {}^c(({}^c a)^c)$. Hence by 2.1.4 (1): $({}^c a)^c \leqq ({}^c({}^c(a^c)))^c$. Since c is central: $({}^c({}^c(a^c)))^c \leqq {}^c(a^c)$. Thus by transitivity: $({}^c a)^c \leqq {}^c(a^c)$. From 2.2.5 (ii) second inequality by putting $a = 1$, we have $(({}^c b)^c)^c \leqq ({}^c(b^c))^c$. Hence: ${}^c(b^c) \leqq {}^c((({}^c b)^c)^c)$. Since c is central ${}^c((({}^c b)^c)^c) \leqq ({}^c b)^c$. Thus ${}^c(b^c) \leqq ({}^c b)^c$.

2) Let us prove $({}^c a)^c \otimes ({}^c b)^c \leqq ({}^c(a \otimes b))^c$. From 2.2.4. 2) (i) first inequality and by 2.2.5. (i) first inequality (since c is central) we obtain:

$$ {}^c(a \otimes b) \leqq {}^c(a \otimes ({}^c b)^c) \leqq {}^c(({}^c a)^c \otimes ({}^c b)^c). $$

Now from ${}^c(a \otimes b) \leqq {}^c(({}^c a)^c \otimes ({}^c b)^c)$ we have $({}^c(({}^c a)^c \otimes ({}^c b)^c))^c \leqq ({}^c(a \otimes b))^c$. Moreover since $({}^c(-))^c$ is a closure operator: $({}^c a)^c \otimes ({}^c b)^c \leqq ({}^c(({}^c a)^c \otimes ({}^c b)^c))^c$.

2.3. Autonomous posets

We now consider autonomous posets, the non commutative version of *-autonomous posets.

2.3.1. Definition: 1) An element d of closed poset P is a *dualizing element* iff for every $a \in P$ ${}^d(a^d) = ({}^d a)^d = a$.

2) A closed poset P is an *autonomous poset* iff it contains a dualizing element d. (a^d will be denoted by $a^\perp$ and ${}^d a$ by ${}^\perp a$.)

2.3.2. Facts: Let $(P, \leqq, \otimes, d)$ be an autonomous poset.

For every $a, b \in P$

1) $a \leqq b$ iff ${}^\perp b \leqq {}^\perp a$ and $a \leqq b$ iff $b^\perp \leqq a^\perp$;

2) $a \multimap_l b = {}^\perp b \multimap_r {}^\perp a$ and $a \multimap_r b = b^\perp \multimap_l a^\perp$;

3) $a \multimap_l b = {}^\perp(a \otimes b^\perp)$ and $a \multimap_r b = ({}^\perp b \otimes a)^\perp$;

$$ {}^\perp(b \multimap_r a^\perp) = a \otimes b = (a \multimap_l {}^\perp b)^\perp; $$

4) An autonomous poset is a unital closed poset with unit $1 = {}^\perp d = d^\perp$.

Proof: 1) An alternative proof is the following one:

$b^\perp \leqq a^\perp$ iff $b \multimap_r d \leqq b \multimap_r d$ iff $a \otimes (b \multimap_r d) \leqq d$ iff $a \leqq (b \multimap_r d) \multimap_l d = b$, since $b \in P$ and P is an autonomous poset.

2) $a \multimap_r b \leqq b^\perp \multimap_l a^\perp$, from lemma 2.2.2 (2) (ii)

$$ b^\perp \multimap_l a^\perp \leqq (a^\perp \multimap_l d) \multimap_r (b^\perp \multimap_l d) = a \multimap_r b $$

3) Let us prove $a \multimap_l b = {}^\perp(a \otimes b^\perp)$ in two steps:

Step 1:

$$ {}^\perp(a \otimes b^\perp) \leqq a \multimap_l b $$

$$ \begin{aligned} {}^\perp(a \otimes b^\perp) \leqq {}^\perp(a \otimes b^\perp) \quad & \text{iff } (a \otimes b^\perp) \leqq ({}^\perp(a \otimes b^\perp)) \multimap_r d \\ & \text{iff } {}^\perp(a \otimes b^\perp) \otimes (a \otimes b^\perp) \leqq d \quad \text{iff } ({}^\perp(a \otimes b^\perp) \otimes a) \otimes b^\perp \leqq d \\ & \text{iff } {}^\perp(a \otimes b^\perp) \otimes a \leqq (b^\perp \multimap_l d) = b \quad \text{iff } {}^\perp(a \otimes b^\perp) \leqq a \multimap_l b \end{aligned} $$

Step 2:

$$a \multimap_l b \leqq {}^{\perp}(a \otimes b^{\perp})$$

$a \multimap_l b \leqq a \multimap_l b$ iff $(a \multimap_l b) \otimes a \leqq b = b^{\perp} \multimap_l d$

iff $((a \multimap_l b) \otimes a) \otimes b^{\perp} \leqq d$ iff $(a \multimap_l b) \otimes (a \otimes b^{\perp}) \leqq d$

iff $a \otimes b^{\perp} \leqq (a \multimap_l b) \multimap_r d$ iff $a \multimap_l b \leqq {}^{\perp}(a \otimes b^{\perp})$.

4) $$a \otimes d^{\perp} = (a \multimap_l {}^{\perp}(d^{\perp}))^{\perp} = (a \multimap_l d)^{\perp} = ({}^{\perp}a)^{\perp} = a$$

$${}^{\perp}d \otimes a = {}^{\perp}(a \multimap_r ({}^{\perp}d)^{\perp}) = {}^{\perp}(a \multimap_r d) = {}^{\perp}(a^{\perp}) = a$$

We now consider the non commutative version of the "par" connective of linear logic and obtain the adequate generalizations of the equations holding in the commutative case.

2.3.3. Definition: Let $(P, \leqq, \otimes, d)$ be an autonomous poset.
For every $a, b \in P : a \operatorname{par} b = ({}^{\perp}b \otimes {}^{\perp}a)^{\perp} = {}^{\perp}(b^{\perp} \otimes a^{\perp})$.

2.3.4. Facts:

1) $d \operatorname{par} a = a \operatorname{par} d = a$;

2) ${}^{\perp}(a \operatorname{par} b) = {}^{\perp}b \otimes {}^{\perp}a$ and $(a \operatorname{par} b)^{\perp} = b^{\perp} \otimes a^{\perp}$;

3) ${}^{\perp}a \multimap_r b = a \operatorname{par} b = b^{\perp} \multimap_l a$ and $a \multimap_l b = b \operatorname{par} {}^{\perp}a$ and $a \multimap_r b = a^{\perp} \operatorname{par} b$.

Sketch of proof:

1) $d \operatorname{par} a = {}^{\perp}(d^{\perp} \otimes {}^{\perp}a) = {}^{\perp}(1 \otimes {}^{\perp}a) = {}^{\perp}(a^{\perp}) = a$,
$a \operatorname{par} d = {}^{\perp}(a^{\perp} \otimes d^{\perp}) = {}^{\perp}(a^{\perp} \otimes 1) = {}^{\perp}(a^{\perp}) = a$.

2) From the definitions.

3) From Facts 2.3.2. 3) and 4).

3. AUTONOMOUS QUANTALES

In this paragraph we first study (non commutative) quantales: complete lattices with a monoid structure and satisfying two "infinite distributive laws". So we have a "left" negation and a "right" "negation".

Then quantic nuclei are considered following the treatment given for closed nuclei. The same sufficient condition is used for having the two "double negation" equal and forming a quantic nucleus.

Finally autonomous quantales are studied, which are the non commutative version of the so called Girard quantales. Moreover we obtain properties of infs and sups of autonomous quantales with respect to the two negations.

3.1. Quantales

Let us recall the definition of a quantale, some examples and properties. More details can be found in Rosenthal's book.

3.1.1. Definition: (1.1) A *quantale* $(Q, \leqq, \otimes)$ is a complete lattice $(Q, \leqq)$ such that $(Q, \otimes)$ is a semigroup and for any $a \in Q$, $\{b_\alpha\} \subset Q$.

$$(\sup_\alpha b_\alpha) \otimes a = \sup_\alpha (b_\alpha \otimes a) \qquad \text{and} \qquad a \otimes (\sup_\alpha b_\alpha) = \sup_\alpha (a \otimes b_\alpha)$$

(1.2) Let Q be a quantale:

(i) An element $1 \in Q$ is a *left* (*right*) *unit* iff $1 \otimes a = a \, (a \otimes 1 = a)$ for any $a \in Q$; an element $1 \in Q$ is a *unit* iff is both a right and left unit.

(ii) A quantale Q is *right* (*left*) *unital* iff it has a right (left) unit 1; a quantale Q is *unital* if and only if it has a *unit* 1.

(2.1) Let P, Q be quantales, $f: P \to Q$ is a *quantale homomorfism* if and only if it preserves sups and the operation $\otimes$.

(2.2) Let P, Q be unital quantales with units 1_P, 1_Q respectively. A quantale homomorphism $f: P \to Q$ is a *unital quantale homomorfism* iff $f(1_P) = 1_Q$.

(3) **Quant** (**Unquant**) denotes the category respectively of (unital) quantales and (unital) homomorfisms.

3.1.2. *Examples*: (1) Complete partially ordered groups.

(2) A frame (*e.g.* [21]) is a commutative idempotent quantale with $\otimes$ as conjunction.

(3) The power set of a semigroup (monoid) as in 2.1.3. 2) is a (unital) quantale.

(4) The left ideals of a ring as in 2.1.3. 3) form a quantale. The right ideals of a ring as in 2.1.3. 3) form a quantale.

3.1.3. Fact: For any $a \in Q$, a quantale is a closed poset in which

$$a \multimap_r \_ : Q \to Q \qquad \text{and} \qquad \multimap_l \_ : Q \to Q$$

are as follows for any $b \in Q$:

$$a \multimap_r b = \sup \{x \in Q \mid a \otimes x \leqq b\} \quad \text{and} \quad a \multimap_l b = \sup \{x \in Q \mid x \otimes a \leqq b\}$$

3.1.4. PROPOSITION (*e.g.* Rosenthal 1990 and Abramsky-Vickers 1990): *There exists a functor* $\mathscr{P}$: **Mon** → **Unquant** *which is left adjoint to the forgetful functor* $\mathscr{U}$: **Unquant** → **Mon** *with:*

1) *for every monoid* M, $\mathscr{P}(M)$ *is the unital quantale considered in* 3.1.2. 3),

2) *if* $f: M \to N$ *is a monoid homomorphism then* $\mathscr{P}(f): \mathscr{P}(M) \to \mathscr{P}(N)$ *defined by* $\mathscr{P}(f)(A) = \{f(A) \mid a \in A\}$ *is a unital quantale homomorphism.*

3.2. Quantic nuclei

A very useful notion in quantales theory is the quantic quotient: the quantales representation theorem says that every quantale is isomorphic to a certain quantic quotient.

3.2.1. DEFINITION: A mapping $j: Q \to Q$ is a *quantic nucleus* on a quantale Q iff j is a closure operator and a closed map of quantale.

3.2.2. PROPOSITION (Niefield-Rosenthal 1988): *Let* $(Q, \leqq, \otimes)$ *be a quantale,* $j: Q \to Q$ *a quantic nucleus:*

Then $(Q_j = \{a \in Q \mid j(a) = a\}, \leqq, \otimes_j)$ *is a quantale with* $a \otimes_j b = j(a \otimes b)$, *called the* quantic quotient *of* Q *w.r.t.* j, *and* $j: Q \to Q_j$ *is a quantale homomorfism.*

Each surjective mapping in **Quant** *corresponds to a quantic nucleus.*

Why quantic quotients are so important in quantales theory? The answer is that each quantale is isomorphic to a quantic quotient.

3.2.3. PROPOSITION (Representation theorem for quantales). (Rosenthal 1990): *Let* Q *be a quantale, then there exist a semigroup* S *and a quantic nucleus* $j: \mathscr{P}(S) \to \mathscr{P}(S)$ *such that* $Q \cong \mathscr{P}(S)_j$.

3.2.4. PROPOSITION: *Let* Q *be a quantale and* $c \in Q$ *a central element, then:*

1) $({}^c(-))^c = {}^c((-)^c) : Q \to Q$ *is a quantic nucleus (denoted* j_c),

2) *The quantic quotient of* Q w.r.t. j_c *is a quantale.*

Proof: From 2.2.6 and 3.2.2.

3.3. Autonomous quantales

To introduce negation operators in a quantale we need the existence of a fixed element called *dualizing*. If this element is cyclic we have just one negation operation, linear negation, and we have just one negation operation, linear negation, and we have a cyclic autonomous quantale. If we do not make this hypothesis of cyclicity of commutativity or the monoid operation there exist two negations, right and lef negation, and we obtain an *autonomous quantale*.

3.3.1. DEFINITION: (1) An element d of a quantale Q is a *dualizing element* if and only if

$$(a \multimap_l d) \multimap_r d = (a \multimap_r d) \multimap_l d = a \quad \text{for any } a \in Q.$$

(2) An *autonomous quantale* is a quantale containing a dualizing element d. If we write for any $a \in Q$, ${}^{\perp}a = a \multimap_l d$ and $a^{\perp} = a \multimap_r d$ we get respectively the *right and left negation;* if d is a dualizing element in Q, then for any $a \in Q$, $({}^{\perp}a)^{\perp} = {}^{\perp}(a^{\perp}) = a$.

3.3.2. *Remark:* An element s of a quantale Q is *cyclic* iff ${}^{s}a = a^{s}$ for any $a \in Q$.

A quantale Q is a *cyclic autonomous quantale* if and only if it has a cyclic dualizing element c. $(-)^{\perp}$ is the *linear negation.*

3.3.3. FACT: Let Q be a quantale and $c \in Q$ a central element, then the quantic quotient of Q w.r.t. $({}^{c}(-))^{c} = {}^{c}((-)^{c}) : Q \to Q$ (denoted j_c) is an autonomous quantale with dualizing element c.

Proof: easy.

Let us consider now the "additive connectives" $\cap$ and $\cup$ and their relationships with the two negations and the "par" connective.

3.3.4. PROPOSITION: *Let Q be an autonomous quantale and let us denote by $\cup$ and $\cap$ the* inf *and* sup *respectively of two elements of Q.*

For every $a, b \in Q$:

1) ${}^{\perp}(a \cup b) = {}^{\perp}a \cap {}^{\perp}b$ *and* $(a \cup b)^{\perp} = a^{\perp} \cap b^{\perp}$.

2) ${}^{\perp}(a \cap b) = {}^{\perp}a \cup {}^{\perp}b$ *and* $(a \cap b)^{\perp} = a^{\perp} \cup b^{\perp}$.

3) $a \,\mathrm{par}\, (b \cap c) = (a \,\mathrm{par}\, b) \cap (a \,\mathrm{par}\, c)$.

Proof: 1) Right de Morgan w.r.t. $\cup$. First we prove: $a^{\perp} \cap b^{\perp} \geqq (a \cup b)^{\perp}$. From $a \leqq a \cup b$ and $b \leqq a \cup b$ we obtain: $(a \cup b)^{\perp} \leqq a^{\perp}$ and $(a \cup b)^{\perp} \leqq b^{\perp}$. So from the definition of inf we have the inequality. Then we prove: $a^{\perp} \cup b^{\perp} \leqq (a \cup b)^{\perp}$. From $a^{\perp} \cap b^{\perp} \leqq a^{\perp}$ and $a^{\perp} \cap b^{\perp} \leqq b^{\perp}$ we obtain: $a \leqq {}^{\perp}(a^{\perp} \cap b^{\perp})$ and $b \leqq {}^{\perp}(a^{\perp} \cap b^{\perp})$. So from the definition of sup we have: $a \cup b \leqq {}^{\perp}(a^{\perp} \cap b^{\perp})$, thus $({}^{\perp}(a^{\perp} \cap b^{\perp}))^{\perp} \leqq (a \cup b)^{\perp}$, hence $a^{\perp} \cap b^{\perp} \leqq (a \cup b)^{\perp}$. Left de Morgan w.r.t. $\cup$. is obtained in a similar way.

2)
$$(a \cap b)^{\perp} = ({}^{\perp}(a^{\perp}) \cap {}^{\perp}(b^{\perp})) = ({}^{\perp}(a^{\perp} \cup b^{\perp}))^{\perp} = a^{\perp} \cup b^{\perp}$$
$${}^{\perp}(a \cap b) = {}^{\perp}(({}^{\perp}a)^{\perp} \cap ({}^{\perp}b^{\perp})) = {}^{\perp}(({}^{\perp}a \cup {}^{\perp}b)^{\perp}) = {}^{\perp}a \cup {}^{\perp}b$$

3) $a \operatorname{par}(b \cap c) = a \operatorname{par}(({}^{\perp}b)^{\perp} \cap ({}^{\perp}c)^{\perp}) = a \operatorname{par}({}^{\perp}b \cup {}^{\perp}c)^{\perp}$
$= ({}^{\perp}(({}^{\perp}b \cup {}^{\perp}c)^{\perp}) \otimes {}^{\perp}a)^{\perp} = (({}^{\perp}b \cup {}^{\perp}c) \otimes {}^{\perp}a)^{\perp}$
$= (({}^{\perp}b \otimes {}^{\perp}a) \cup ({}^{\perp}c \otimes {}^{\perp}a))^{\perp} = ({}^{\perp}b \otimes {}^{\perp}a)^{\perp} \cap ({}^{\perp}c \otimes {}^{\perp}a)^{\perp} = (a \operatorname{par} b) \cap (a \operatorname{par} c)$.

4. THE PHASE SPACE: COMPLETION AND REPRESENTATION THEOREMS

We prove a completion theorem of autonomous posets in autonomous quantales and the representation theorem for such quantales by means of the non commutative phase quantale obtained from the power set of a monoid.

4.1. The phase space closed poset

We show the relationship between the abstract, algebraic version and the concrete first order set theoretic version of the "central" property. Then we show the existence of a central element for the power set closed poset constructed from an autonomous poset.

4.1.1. FACT: Let M be a semigroup M, in the closed poset $(\mathscr{P}(M), \subseteq, \otimes)$ an element $C \in \mathscr{P}(M)$ is central if and only if

1) $\forall z \in M$: ${}^{C}(\{z\}^{CC}) \subseteq \{z\}^{C}$, and

2) $\forall z \in M$: $(({}^{CC}\{z\})^{C} \subseteq {}^{C}\{z\})$.

Proof: 1) and 2) are equivalent respectively to:

1′) $\forall z \in M$: $(\forall x (\forall y (\forall t (zt \in C \Rightarrow ty \in C) \Rightarrow xy \in C) \Rightarrow zx \in C))$ and

2′) $\forall z \in M$: $(\forall x (\forall y (\forall t (tz \in C \Rightarrow yt \in C) \Rightarrow yx \in C) \Rightarrow xz \in C))$.

Let us just show in detail that condition 2′) is equivalent to:

(ii) for every $F \in \mathscr{P}(M)$: $({}^{CC}F)^{C} \subseteq {}^{C}F$.

We obtain the following equivalences:

$$\forall z \in M: \quad (({}^{CC}\{z\})^{C} \subseteq {}^{C}\{z\})$$

$$\text{iff } \forall z \in M: \quad (\forall x ((x \in ({}^{CC}\{z\})^{C}) \Rightarrow (x \in {}^{C}\{z\})))$$

$$\text{iff } \forall z \in M: \quad (\forall x ((\forall y (y \in {}^{CC}\{z\} \Rightarrow yx \in C) \Rightarrow (\forall u \in \{z\}: xu \in C)))$$

iff

$$\forall z \in M: \quad (\forall x ((\forall y ((\forall t (t \in {}^{C}\{z\} \Rightarrow yt \in C)) \Rightarrow yx \in C)) \Rightarrow (\forall u \in \{z\}: xu \in C))).$$

iff $\forall z\in M$: $(\forall x((\forall y((\forall t((\forall u(u\in\{z\}\Rightarrow tu\in C))\Rightarrow yt\in C))\Rightarrow yx\in C))$
$\Rightarrow(\forall u\in\{z\}: xu\in C)))$

iff for every

$$\{z\}\in\mathscr{P}(M):\quad (\forall x((\forall u(u\in\{z\}\Rightarrow tu\in C)))\Rightarrow(\forall u\in\{z\}: xu\in C)))$$

iff for every

$$F\in\mathscr{P}(M):\quad (\forall x((\forall u(u\in F\Rightarrow tu\in C)))\Rightarrow(\forall u\in F: xu\in C)))$$

iff for every

$$F\in\mathscr{P}(M):\quad ((^{CC}F)^{C}\subseteqq {}^{C}F)$$

Similarly we can have condition (i) defining a central element.

Let us show now that the subset $\downarrow d=\{x\in Q\mid x\leqq d\}$ satisfies the needed property: it is a central element.

4.1.2. Proposition: *Let $(P, \leqq, \otimes, d)$ be an autonomous poset. The closed poset $(\mathscr{P}(P), \subseteqq, \otimes)$ has a central element $\downarrow d=\{x\in Q\mid x\leqq d\}$, denoted D.*

Proof: From fact 4.1.1, we have to prove that

(i) for every $z\in P$, $(^{DD}\{z\})^{D}\subseteqq {}^{D}\{z\}$ and

(ii) for every $z\in P$, $^{D}(\{z\}^{DD})\subseteqq\{z\}^{D}$

Let us prove (i).

For every $\alpha\in P$, the following equivalences are obtained: $\alpha\in{}^{D}\{z\}$ iff for every $u\in\{z\}:\alpha\otimes u\in\downarrow d$ iff $\alpha\otimes z\leqq d$ iff $\alpha\leqq{}^{d}z$; so $\alpha\in{}^{D}\{z\}$ iff $\alpha\leqq{}^{d}z$.

For every $\beta\in P$, we have: $\beta\in{}^{DD}\{z\}$ iff for every $\alpha\in{}^{D}\{z\}$, $\beta\otimes\alpha\in\downarrow d$ iff for every α ($\alpha\leqq{}^{d}z$ implies $\beta\otimes\alpha\leqq d$ iff for every α ($\alpha\leqq{}^{d}z$ implies $\alpha\leqq\beta^{d}$) iff ${}^{d}z\leqq\beta^{d}$ so $\beta\in{}^{DD}\{z\}$ iff ${}^{d}z\leqq\beta^{d}$.

For every $w\in P$: $w\in(^{DD}\{z\})^{D}$ iff for every $\beta\in{}^{DD}\{z\}$, $\beta\otimes w\in\downarrow d$ iff for every β (${}^{d}z\leqq\beta^{d}$ implies $w\leqq\beta^{d}$) iff for every β ($\beta\leqq{}^{dd}z$ implies $\beta\leqq{}^{d}w$) iff ${}^{dd}z\leqq{}^{d}w$ iff $w\leqq{}^{d}z$, since P is an autonomous poset.

So for every $w\in P$: $w\in(^{DD}\{z\})^{D}$ iff $w\leqq{}^{d}z$ iff $w\in{}^{D}\{z\}$. Hence $(^{DD}\{z\})^{D}\subseteqq{}^{D}\{z\}$. The proof of (ii) is obtained in a similar way.

4.2. The phase autonomous quantale

Let us conclude by studying the autonomous quantale obtained from a monoid by the power set construction, so obtaining the main results of the paper.

4.2.1. FACT: Let M be a monoid and $C \in \mathcal{P}(M)$ be a central element of the quantale $\mathcal{P}(M)$, then the quantic quotient of Q w.r.t. j_C is an autonomous quantale called *the non commutative phase autonomous quantale* with dualizing element C, and $\sup_i A_i = ({}^c(\bigcup_i A_i))^c = {}^c((\bigcup_i A_i)^c)$

4.2.2. PROPOSITION: *Let $(Q, \leqq, \otimes, d)$ be an autonomous quantale.*

1) *The quantale $(\mathcal{P}(Q), \subseteq, \otimes)$ has a central element $\downarrow d = \{x \in Q \mid x \leqq d\}$ denoted D.*

2) $({}^D(-))^D = {}^D((-)^D)$: $\mathcal{P}(Q) \to \mathcal{P}(Q)$ *is a quantic nucleus, denoted j_D.*

3) *The quantic quotient of $\mathcal{P}(Q)$* w.r.t. j_D *is an autonomous quantale with dualizing element D.*

Proof:

1) From proposition 3.1.2.

2) From proposition 1.2.6 and point 1) above.

3) From fact 3.3.3 2) and point 2) above.

Let us prove now a completion result for autonomous posets.

4.2.3. PROPOSITION: *Let $(P, \leqq, \otimes, d)$ be an autonomous poset. For every $A \in \mathcal{P}(P)$ let $L(A)$ be the set of all lower bounds of A and $U(A)$ be the set of all upper bounds of A.*

(1) *The closure operator $L(U(-))$: $\mathcal{P}(P) \to \mathcal{P}(P)$ is equal to $({}^D(-))^D = {}^D((-)^D)$, where D denotes $\downarrow d$.*

(2) *The quantic quotient of $\mathcal{P}(P)$* w.r.t. $L(U(-))$ *is an autonomous quantale with dualizing element $\downarrow d$, called the Dedeking Mcneille completion of the autonomous poset P.*

Proof: Let us prove that for every $A \in \mathcal{P}(P)$

$$A = ({}^D(A))^D = {}^D((A)^D) \quad \text{iff } A = L(U(A))$$

Step 1.

Let us show that: ${}^D A = L({}^*A)$ where ${}^*A = \{{}^d a \mid a \in A\}$ and $A^D = L(A^*)$ where $A^* = \{a^d \mid a \in A\}$

Indeed

$$^{D}A = A \multimap\!\!\bullet_{l} \downarrow d = \{x \mid \text{for every } a \in A,\ x \otimes a \leqq d\}$$
$$= \{x \mid \text{for every } a \in A,\ x \leqq {}^{d}a\} = \{x \mid \text{for every } {}^{d}a \in {}^{*}A,\ x \leqq {}^{d}a\} = L({}^{*}A).$$

Similarly

$$A^{D} = A \multimap\!\!\bullet_{r} \downarrow d = \{x \mid \text{for every } a \in A,\ a \otimes x \leqq d\}$$
$$= \{x \mid \text{for every } a \in A,\ x \leqq a^{d}\} = \{x \mid \text{for every } a^{d} \in A^{*},\ x \leqq a^{d}\} = L(A^{*}).$$

Step 2.

$$L({}^{*}A) = {}^{*}(U(A)) \qquad \text{and} \qquad L(A^{*}) = (U(A))^{*}.$$

Indeed

$${}^{*}(U(A)) = {}^{*}(\{x \in P \mid \text{for every } a \in A,\ a \leqq x\})$$
$$= \{{}^{d}x \in P \mid \text{for every } a \in A,\ a \leqq x\} = \{{}^{d}x \in P \mid \text{for every } a \in A,\ {}^{d}a \leqq {}^{d}x\}$$
$$= \{{}^{d}x \in P \mid \text{for every } {}^{d}a \in {}^{*}A,\ {}^{d}a \geqq {}^{d}x\} = L({}^{*}A),$$

since every element of P is equal to ${}^{d}x$ for some x (take $x = y^{d}$)

Similarly we obtain $L(A^{*}) = (U(A))^{*}$.

Step 3.

$${}^{*}(A^{*}) = ({}^{*}A)^{*} = A$$
$${}^{*}(A^{*}) = {}^{*}(\{a^{d} \mid a \in A\}) = \{{}^{d}(a^{d}) \mid a \in A\} = \{a \mid a \in A\} = A$$
$$({}^{*}A)^{*} = (\{{}^{d}a \mid a \in A\})^{*} = \{({}^{d}a)^{d} \mid a \in A\} = \{a \mid a \in A\} = A$$

Step 4.

$$({}^{D}(A))^{D} = L(U(A)) \qquad \text{and} \qquad {}^{D}((A)^{D}) = L(U(A)).$$
$$({}^{D}(A))^{D} = L(({}^{D}(A)^{*}) = L((L({}^{*}A))^{*}) = L(({}^{*}(U(A)))^{*}) = L(U(A)).$$
$${}^{D}(A^{D}) = L({}^{*}(A^{D})) = L({}^{*}(L(A^{*}))) = L({}^{*}((U(A))^{*})) = L(U(A)).$$

Let us prove now a representation result for autonomous quantales.

4.2.4. PROPOSITION: *Every autonomous quantale* $(Q, \leqq, \otimes, d)$ *is isomorphic to a non commutative phase quantale.*

Proof: Let $(Q, \leqq, \otimes, d)$ be an autonomous quantale.

Step 1.

From 3.2.2, the quantic quotient of Q w.r.t. $({}^{\downarrow d}(-))^{\downarrow d} = {}^{\downarrow d}((-)^{\downarrow d})$, denoted by $\mathscr{P}(Q)_{\downarrow d}$, is an autonomous quantale with dualizing element $\downarrow d$.

Step 2.

There exists a bijection between Q and $\mathscr{P}(Q)_{\downarrow d}$.

Let us define the mapping i: $Q \to \mathcal{P}(Q)_{\downarrow d}$ by $i(x)=\downarrow x$, for every $x \in Q$. The mapping i is injective, let us show that it is surjective; *i.e.* for every $A \in \mathcal{P}(Q)_{\downarrow d}$ there exists $a \in Q$, $A=\downarrow a$. We show that such $a \in Q$ exists and is equal to sup A, *i.e.* $A=({}^D(A))^D$ iff $A=\downarrow \sup A$.

1) First we prove that $A^{\downarrow d}=\downarrow(\sup A)^d$.

$x \in A^{\downarrow d}$ iff for every $\alpha \in A$, $x \leqq \alpha^d$ iff for every $\alpha \in A$, ${}^d x \geqq \alpha$ iff $\sup A \leqq {}^d x$ iff $x \leqq (\sup A)^d$.

2) Then we prove that ${}^{\downarrow d}(A^{\downarrow d})=\downarrow(\sup A)$.

$x \in {}^{\downarrow d}(A^{\downarrow d})$ iff for every $b \in A^{\downarrow d}$, $x \leqq {}^d b$ iff by 1), for every $b \leqq (\sup A)^d$, $x \leqq {}^d b$ iff for every ${}^d b \geqq \sup$ A, $x \leqq {}^d b$ iff $x \leqq \sup A$ iff $x \in \downarrow \sup A$.

Step 3.

The bijection i: $Q \to \mathcal{P}(Q)_{\downarrow d}$ preserves sups, *i.e.* for every $\{\alpha_t\} \subset Q$, $i(\sup\{\alpha_t\})=\sup\{i(\alpha_t)\}$.

We have: $\sup\{i(\alpha_t)\}=\sup\{\downarrow\alpha_t\}=({}^D(\bigcup\{\downarrow\alpha_t\}))^D={}^D((\bigcup\{\downarrow\alpha_t\})^D)$.

So $x \in \sup\{i(\alpha_t)\}$ iff $x \in {}^{\uparrow d}((\bigcup\{\downarrow\alpha_t\})^D)$.

iff for every $y \in (\bigcup\{\downarrow\alpha_t\})^{\downarrow d}$, $x \otimes y \leqq d$,

iff for every y: ((for every $z \in \bigcup\{\downarrow\alpha_t\}$ $z \otimes y \leqq d$) implies $x \otimes y \leqq d$),

iff for every y: ((for every $z \in \bigcup\{\beta \leqq \alpha_t\}$ $z \leqq {}^d y$) implies $x \leqq {}^d y$),

iff for every ${}^d y$: (($\sup\{\alpha_t\} \leqq {}^d y$) implies $x \leqq {}^d y$),

iff $x \leqq \sup\{\alpha_t\}$ iff $x \in \downarrow \sup\{\alpha_t\}=i(\sup\{\alpha_t\})$.

Step 4.

The bijection $i: Q \to \mathcal{P}(Q)_{\downarrow d}$ is a monoid homomorphism, *i.e.* for every $a, b \in Q$, $i(a \otimes b)=i(a) \otimes i(b)$, (where $A \otimes B={}^{\downarrow d}((A \otimes B)^{\downarrow d})=({}^{\downarrow d}(A \otimes B))^{\downarrow d}$ and $A \otimes B=\{a \otimes b \mid a, b \in Q\}$).

We have $i(a) \otimes i(b)=({}^{\downarrow d}(i(a) \otimes i(b)))^{\downarrow d}=({}^{\downarrow d}(\downarrow a \otimes \downarrow b)^{\downarrow d}$.

Let us denote $\downarrow a \otimes \downarrow b$ by F.

So $x \in i(a) \otimes i(b)$ iff $x \in ({}^{\downarrow d}F)^{\downarrow d}$ iff for every $y \in {}^{\downarrow d}F$: $x \leqq {}^d y$,

iff for every y: (for every z ($z \in F$ implies $z \leqq {}^d y$) implies $x \leqq {}^d y$),

iff for every ${}^d y$: ((${}^d y$ is an upper bound of F) implies $x \leqq {}^d y$),

iff $x=\sup$ F $=\sup(\downarrow a \otimes \downarrow b)=a \otimes b$ iff $x \in \downarrow(a \otimes b)=i(a \otimes b)$.

4.2.5. *Remark:* As corollary we obtain the representation theorem for cyclic autonomous quantales given in Rosenthal's book:

If Q is a Girard quantale, then it is isomorphic to a (cyclic) phase quantale.

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