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ON THE POSITIVE AND THE INVERSION COMPLEXITY OF BOOLEAN FUNCTIONS (*)

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Abstract. – *We study unbounded fan-in circuits over the bases* $P = M \cup \{\neg f : f \in M\}$ *and* $M' = M \cup \{\neg\}$ *with the inputs* $x_1, \ldots, x_n$ (positive *and* exmonotone *circuits respectively*). *It was proved by Santha and Wilson* [5] *that any depth-d exmonotone circuit computing* $PARITY_n$ *requires at least* $\Omega(dn^{1/d})$ *negations. Using similar arguments we establish relations between the* chain complexity $ch(f)$ *(defined by Wegener in* [6]) *of a Boolean function f, the size of the optimal positive circuit computing f and the minimal number of negations in any exmonotone circuit computing f. It allows us to unify and slightly improve the lower bounds of Markov* [3], *Santha and Wilson* [5] *and Wegener* [6]. *We also give almost matching upper bounds for symmetric functions.*

Résumé. – *Nous étudions les circuits à degré entrant borné sur les bases* $P = M \cup \{\neg f : f \in M\}$ *et* $M' = M \cup \{\neg\}$ *sur les entrées* $x_1, \ldots, x_n$, *où M est l'ensemble des fonctions booléennes monotones (que nous appelons circuits* positifs *et* exmonotones *respectivement*). *Santha et Wilson* [5] *ont prouvé que tout circuit exmonotone de profondeur d calculant* $PARITY_n$ *nécessitait au moins* $\Omega(dn^{1/d})$ *négations. En utilisant des arguments semblables, nous établissons des relations entre la* complexité en chaine $ch(f)$ *d'une fonction booléenne f (définie par Wegener en* [6]), *la taille d'un circuit positif optimal calculant f et le nombre minimum de négations dans tout circuit exmonotone calculant f. Ceci permet d'unifier et d'améliorer sensiblement les bornes inférieures de Markov* [3], *Santha et Wilson* [5] *et de Wegener* [6]. *Nous donnons aussi des bornes supérieures « presque exactes » pour les fonctions symétriques.*

1. INTRODUCTION

We study unbounded fan-in circuits over the bases $P = M \cup \{\neg f : f \in M\}$ and $M' = M \cup \{\neg\}$ (where M is a set of all monotone Boolean functions) with the inputs $x_1, \ldots, x_n$. We call them *positive* circuits and *exmonotone* (extended monotone) circuits respectively. The aim of this paper is to establish relations between the *chain complexity* $\mathrm{ch}(f)$ of an arbitrary Boolean

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function f, the size of the optimal positive circuit computing f and the minimal number of negations in any exmonotone circuit computing f. Let us at first introduce some notions.

An *edge* (α, β) is any pair of Boolean vectors $\alpha, \beta \in \{0, 1\}^n$ which differ in exactly one component. A *chain* $C=(\alpha^0, \ldots, \alpha^n)$ where $\alpha^i \in \{0, 1\}^n$, is a vector of vectors such that $\alpha^i < \alpha^{i+1}$. One may consider that any chain consists of n edges (α^i, α^{i+1}). Let $\mathrm{ch}(f, C)$ be the number of maximal constant subvectors (*intervals*) of the vector $(f(\alpha^0), \ldots, f(\alpha^n))$. The *chain complexity* $\mathrm{ch}(f)$ of f is the maximum of $\mathrm{ch}(f, C)$ over all chains C. Let $\mathrm{dec}(f, C)$ be the number of C edges (α^i, α^{i+1}) satisfying $f(\alpha^i)=1, f(\alpha^{i+1})=0$. The *decrease* $\mathrm{dec}(f)$ of f is the maximum of $\mathrm{dec}(f, C)$ over all chains C. It is easy to verify that

$$\mathrm{dec}(f)=\lfloor(\mathrm{ch}(f)-1+f(0^n))/2\rfloor.$$

We also define a *positive complexity* $P(f)$ (an *inversion complexity* $I(f)$) of f as the minimal number of gates (negations) in any positive (exmonotone) circuit computing f. In the usual way one may define $P_d(f)$ and $I_d(f)$.

The first general lower bound was obtained by Markov [3]. He proved that any Boolean function f requires at least $\lceil\log(\mathrm{dec}(f)+1)\rceil$ negations. Two results of this kind were obtained by Wegener [6]. He considered threshold circuits with the gates $T^m_{\geqq k}$ and $T^m_{\leqq k}$. Notice that a threshold circuit is a special case of a positive circuit. Wegener proved that any Boolean function of n variables requires at least $\lceil\log(n+1)-\log(n+1-l_{\min}(f))\rceil$ threshold gates where the *minimum sensitive complexity* $l_{\min}(f)$ is the minimal number of variables which have to be replaced by constants in order to obtain a constant subfunction of f. He also showed that for any Boolean function f with $\mathrm{ch}(f)\geqq 3$ the size of synchronous threshold circuits computing f is bounded below by $\mathrm{ch}(f)$. Santha and Wilson [5] proved that $I_d(\mathrm{PARITY}_n)=\Omega(dn^{1/d})$. Using similar arguments we prove two almost evident combinatorial lemmas which allow us to unify and slightly improve the results mentioned above. In particular, we prove that any Boolean function requires at least $\lceil\log \mathrm{ch}(f)\rceil$ positive gates. The following proposition and example show that in some cases our bound is much better than Wegener's.

PROPOSITION 1: $\mathrm{ch}(f)\geqq(n+1)/(n+1-l_{\min}(f))$.

Proof: Let $C=(\alpha^0, \ldots, \alpha^n)$ be a chain satisfying $\mathrm{ch}(f)=\mathrm{ch}(f, C)$. Let $I_{\max}=(f(\alpha^i), \ldots, f(\alpha^{i+k}))$ be an interval of C of maximal length $l(I_{\max})=k+1$. Since C has maximal number of intervals, for any β satisfying $\alpha^i \leqq \beta \leqq \alpha^{i+k}$ we have $f(\beta)=\mathrm{const}$. Then, by definition of $l_{\min}(f)$,

$l_{\min}(f) \leqq n-k$. Hence, $l(I_{\max}) \leqq n+1-l_{\min}(f)$. But

$$\operatorname{ch}(f, C) \geqq (n+1)/l(I_{\max}). \quad \square$$

Example 1: Let $f(X^n) = \text{PARITY}_n(X^n)$ for all X^n satisfying $\sum_{i=1}^{n} x_i \leqq n/2$ and $f(X^n) \equiv 1$ for the remaining X^n.

Then $\operatorname{ch}(f) \geqq n/2$. On the other hand,

$$\log(n+1) - \log(n+1-l_{\min}(f)) \leqq 2.$$

For bounded depth circuits we prove that $P_{d+1}(f) \geqq d(\operatorname{ch}(f)/2)^{1/d} - d + 1$ and $I_d(f) \geqq t(\operatorname{dec}(f))^{1/t} - t$ where $t = \lfloor d/2 \rfloor$. Note that positive complexity and threshold complexity are related by the inequality $I_{2d}(f) \leqq P_d(f)$. So the lower bounds for the positive complexity, a bit worse than ours, may be obtained from Markov's and our lower bounds for the inversion complexity. However, we prefer the direct proof because it gives more tight lower bounds. In section 3 we prove their tightness for $f = \text{PARITY}_n$. We also give almost matching upper bounds for the size (number of negations) of threshold circuits computing *symmetric functions.*

2. LOWER BOUNDS ON THE POSITIVE AND THE INVERSION COMPLEXITY

We say that a Boolean function $f(X^n)$ *intersects* an edge (α, β) where $\alpha, \beta \in \{0, 1\}^n$ if $f(\alpha) \neq f(\beta)$. Let $\alpha < \beta$. We call an intersection *positive* if $f(\alpha) = 0$ and *negative* if $f(\alpha) = 1$. If $C = (\alpha^0, \ldots, \alpha^n)$ is a chain then any subvector $(\alpha^i, \ldots, \alpha^{i+k})$ of C is called a *subchain* of C $(0 \leqq i \leqq n, 0 \leqq k \leqq n-i)$. As usual, we may consider any gate g of a Boolean circuit as Boolean function $g(X^n)$.

LEMMA 1: *Let g be a monotone gate of some positive circuit, C be a chain and C' be a subchain of C without any edge intersected negatively by some input of g. Then g intersects positively no more than one edge of C' and cannot intersect negatively any edge of C'.*

Proof: If $C' = (\alpha^i, \ldots, \alpha^{i+k})$ has no negative intersections by inputs of g and g is monotone then $g(\alpha^i) \leqq \ldots \leqq g(\alpha^{i+k})$. $\square$

COROLLARY 1: *Let g be a monotone gate of some positive circuit and C be a chain. Then:*

1) *g makes no more positive intersections of C than the number of C edges intersected negatively by the inputs of g plus one;*

2) *g does not make any new negative intersection of C.*

Proof: Let the inputs of g intersect negatively exactly m edges $(\alpha^{i_1}, \alpha^{i_1+1}), \ldots, (\alpha^{i_m}, \alpha^{i_m+1})$ of C. Let C' be a subchain $(\alpha^{i_{k-1}+1}, \ldots, \alpha^{i_k+1})$ with one negatively intersected edge $(\alpha^{i_k}, \alpha^{i_k+1})$ $(k=1,\ldots, m;\ i_0 \overset{\text{def}}{=} -1)$. By Lemma 1, g intersects positively no more than one edge of the subchain $C''=(\alpha^{i_{k-1}+1}, \ldots, \alpha^{i_k})$. Obviously, g can also intersect positively the last edge $(\alpha^{i_k}, \alpha^{i_k+1})$ of C'. But if $g(\alpha^{i_k})=0$ then, by monotonicity of g, $g(\alpha^{i_{k-1}+1})= \ldots = g(\alpha^{i_k})=0$. Hence, in any case g intersects positively no more than one edge of C'. In addition it can intersect one edge of the remaining subchain $(\alpha^{i_m+1}, \ldots, \alpha^n)$ of C. □

Remark 1: We will also use an analogous proposition for the negative gates $g=\neg h$ where h is monotone. Namely, a negative gate makes no more *negative* intersections of any chain C than the number of C edges intersected positively by the inputs of g plus one.

LEMMA 2: *Let S be a positive depth-d circuit computing f. Let the i-th level of S have l_i gates connected with some of the inputs $x_1, \ldots, x_n$ $(i=1, \ldots, d)$. Then*

$$\operatorname{ch}(f) \leqq \prod_{i=1}^{d} (1+l_i).$$

Proof: Let C be a chain with the maximal number of edges intersected by f, *i.e.* $\operatorname{ch}(f)=\operatorname{ch}(f, C)$. Then f intersects exactly $\operatorname{ch}(f)-1$ edges of C dividing it into $\operatorname{ch}(f)$ subchains. We will estimate how many edges can be intersected by any gate of S.

Obviously, any gate of the first level intersects no more than one edge of C. Altogether they divide C into $\leqq (l_1+1)$ subchains. By Corollary 1 and Remark 1, any gate of the second level intersects no more than one edge in any such subchain. Notice, that the gates unconnected with the inputs do not make any *new* intersection. Therefore, the gates of the first two levels altogether divide C into $\leqq (l_1+1)(l_2+1)$ subchains and so on. □

LEMMA 3: *Let S be a depth-d exmonotone circuit with minimal number of negations computing $f(X^n)$. Let n_i be the number of negations on the i-th level* *of* $S (i=1, \ldots, d)$, $t=\lfloor d/2 \rfloor$. *Then*

$$\operatorname{dec}(f)+1 \leqq \prod_{i=1}^{t} (1+n_{2i-1}+n_{2i}).$$

If d is odd and the output of S is a $\neg$-gate then the last multiplicand is $(1+n_{d-2}+n_d)$ *instead of* $(1+n_{d-2}+n_{d-1})$.

Proof: Let C be a chain with the maximal number of edges intersected *negatively* by f, *i.e.* $\operatorname{dec}(f)=\operatorname{dec}(f, C)$. We will estimate how many negative and positive intersections each gate of S can make.

Let us denote by N_k the number of different negative intersections of C made by all the gates from the levels1, $\ldots$, k. We can prove that

$$N_k \leqq N_{k-1}+n_k (N_{k-2}+1).$$

Indeed, by Corollary 1, each monotone gate does not make *new* negative intersections. All new negative intersections are generated by negations. Let us consider some negation on level k. By the optimality of the number of negations, its input is some monotone gate from level $i \leqq k-1$. Obviously, the negation intersects negatively the same edges which its input intersects positively. By Corollary 1, the number of such edges is no more than $N_{k-2}+1$.

Now we will estimate how many negative intersections can be generated by the output gate of S. Obviously, $N_1 \leqq n_1$, $N_2 \leqq n_1+n_2$. Using the inequality from above by mathematical induction we obtain

$$N_k \leqq (1+n_1+n_2) \ldots (1+n_{k-1}+n_k)-1$$

for even k and $N_k \leqq (1+n_1+n_2) \ldots (1+n_k)-1$ for odd k. If the output of S is monotone then, by Corollary 1, it does not make new negative intersections and, therefore, $\operatorname{dec}(f) \leqq N_{d-1}$. If the output is a $\neg$-gate then $\operatorname{dec}(f) \leqq N_{d-2}+1$. In both cases we obtain the desired result. For example let us consider the case where the output is a negation and d is odd. Then

$$\operatorname{dec}(f) \leqq N_{d-2}+1 \leqq (1+n_1+n_2) \ldots (1+n_{d-4}+n_{d-3})(1+n_{d-2}).$$

Since $n_d (1+n_1+n_2) \ldots (1+n_{d-4}+n_{d-3}) \geqq 1$, we obtain

$$\operatorname{dec}(f) \leqq (1+n_1+n_2) \ldots (1+n_{d-2}+n_d)-1. \quad \square$$

Lemmas 2 and 3 immediately yield lower bounds on the positive and the inversion complexity.

THEOREM 1: *For any Boolean function f:*

1) $P(f) \geqq \lceil \log \mathrm{ch}(f) \rceil$;

2) $I(f) \geqq \lceil \log(\mathrm{dec}(f)+1) \rceil$ (*Markov* [3]).

Proof: Let S compute f. We may divide the gates of S into levels in such a way that each level contains only one gate. If S is an optimal positive circuit then, by Lemma 2, $\mathrm{ch}(f) \leqq 2^{P(f)}$. If S is an exmonotone circuit with the minimal number of negations then, by Lemma 3, $\mathrm{dec}(f)+1 \leqq \prod_{i=1}^{t} (1+n_{2i-1}+n_{2i})$, $t = \lfloor \mathrm{depth}(S)/2 \rfloor$. Let k multiplicands $(1+n_{2i-1}+n_{2i})$ be equal 3 ($k \geqq 0$). Then $\mathrm{dec}(f)+1 \leqq 3^k 2^{I(f)-2k} \leqq 2^{I(f)}$. □

THEOREM 2: *For any Boolean function f:*

1) $P_{d+1}(f) \geqq d(\mathrm{ch}(f)/2)^{1/d} - d + 1$;

2) $I_d(f) \geqq t(\mathrm{dec}(f))^{1/t} - t$ *where* $t = \lfloor d/2 \rfloor$.

Proof: Let us prove only the first inequality. Let l_i be the number of gates on the i-th level of an optimal depth-$(d+1)$ positive circuit computing f where $i=1, \ldots, d+1$, $\sum_{i=1}^{d+1} l_i = P_{d+1}(f)$ and $l_{d+1}=1$. By Lemma 2,

$$\mathrm{ch}(f) \leqq \prod_{i=1}^{d+1} (l_i+1) = 2 \prod_{i=1}^{d} (l_i+1).$$

But

$$\left(\sum_{i=1}^{d} (l_i+1)\right) / d = (P_{d+1}(f) - 1 + d)/d.$$

Using an inequality between arithmetical and geometrical means we get the desired result. □

COROLLARY 2: *Almost all symmetric functions require depth-$(d+1)$ positive circuits of size $\Omega(d(n/4)^{1/d})$ and depth-d exmonotone circuits with $\Omega(d(n/4)^{2/d})$ negations.*

Proof: Let $f(X^n)$ be a symmetric function. It can be identified with its *alternation vector* $a(f) = (a_0, \ldots, a_n) \in \{0, 1\}^{n+1}$ where $a_0 = f(0^n)$ and $a_i = 1$ iff $f(1^i 0^{n-i}) \neq f(1^{i-1} 0^{n-i+1})$ ($i=1, \ldots, n$). Obviously,

$$|a(f)| = \mathrm{ch}(f) - 1 + f(0^n).$$

It is known that for almost all vertices $\alpha \in \{0, 1\}^n$ their weights $|\alpha| \sim n/2$. Hence, for almost all symmetric functions $\mathrm{ch}(f) \sim n/2$ and $\mathrm{dec}(f) \sim n/4$. $\square$

THEOREM 3 (Wegener [6]): *Let us denote by $P^s(f)$ the minimal number of gates contained in a synchronous positive circuit computing f. For any $f \notin P$, $P^s(f) \geqq \mathrm{ch}()$.*

Proof: In a synchronous circuit only the gates of the first level are connected with the inputs. Therefore, by Lemma 2, $\mathrm{ch}(f) \leqq l_1 + 1$. Since $f \notin P$, we have $P^s(f) \geqq l_1 + 1$. $\square$

3. UPPER BOUNDS

Our aim is to show that the lower bounds in section 2 are tight for PARITY_n and almost tight for arbitrary symmetric functions. We try to construct *threshold* circuits (positive or exmonotone), if possible.

THEOREM 4: 1) *For any Boolean function $f(X^n)$, $P(f) \leqq n+1$. If f is symmetric then $P(f) \leqq 2 \lceil \log(n+1) \rceil + 1$;*

2) *For any Boolean function $f(X^n)$, $I(f) \leqq \lceil \log(\mathrm{dec}(f)+1) \rceil$ (Markov* [3]).

Proof: The inequality $P(f) \leqq n+1$ is obvious because any function is a monotone function of the arguments $x_1, \neg x_1, \ldots, x_n, \neg x_n$. Now let f be symmetric and $N_f \stackrel{\text{def}}{=} \{i : f(X^n) = 1 \Leftrightarrow |X^n| = i\}$. Then $f(X^n) = \bigvee_{i \in N_f} E_i(X^n)$ where $E_i(X^n) = 1$ iff $|X^n| = i$.

Let $l = \lceil \log(n+1) \rceil$ and $s = (s_{l-1}, \ldots, s_0)$ be the binary representation of $|X^n|$. Let $B_j(X^n) \stackrel{\text{def}}{=} \neg s_j$. Then $E_i = \neg B_{l-1}^{i_{l-1}} \wedge \ldots \wedge \neg B_0^{i_0}$ where $(i_{l-1}, \ldots, i_0)$ is the binary representation of i. Therefore, any symmetric Boolean function is a monotone function of $B_0, \neg B_0, \ldots, B_{l-1}, \neg B_{l-1}$. Obviously, $B_0 = \neg\, \mathrm{PARITY}_n$. Wegener [6] proved that any B_j for $j = l-1, \ldots, 0$ is a threshold function of the inputs $x_1, \ldots, x_n$ and $B_{l-1}, \ldots, B_{j+1}$:

$$B_j = 1 \quad \Leftrightarrow \quad \sum_{i=1}^{n} x_i + \sum_{i=j+1}^{l-1} 2^i B_i < 2^l - 2^j.$$

Hence, all the functions $B_0, \neg B_0, \ldots, B_{l-1}, \neg B_{l-1}$ can be computed by $2l$ positive gates. $\square$

The optimal synchronous threshold circuits computing symmetric functions are described by Wegener in [6]. It remains to consider bounded depth circuits. Let $\mathrm{PTC}_d(f)$ be the minimal size of any positive threshold circuit of depth-d computing f. Let also denote $\mathrm{ITC}_d(f)$ the minimal number of negations in any depth-d circuit of monotone threshold gates and negations which computes f. The following result shows the tightness of our lower bound for bounded depth circuits.

PROPOSITION 2 (Dičiūnas [1]): *We may construct a positive threshold circuit of depth-$(d+1)$ and size $O(d(n/2)^{1/d})$ computing $PARITY_n$. Replacing in this circuit each negative gate $g = \neg h$ $(h \in M)$ by two gates we obtain an exmonotone circuit of depth-$(2d+1)$ with the optimal number of negations.*

Proof: Let $k \overset{\text{def}}{=} \lceil ((n+1)/2)^{1/d} \rceil - 1$. The circuit we describe will have $(d+1)$ level, each level will consist of no more than $2(k+1)$ monotone threshold gate. The gates of the first level will divide an interval $[0, n]$ into smaller intervals of length r_1, the gates of the second level will divide these smaller intervals into intervals of length r_2 and so on. Let $r_0 \overset{\text{def}}{=} n+1$ and let us define r_j for $j=1, \ldots, d$. Let $a_j \overset{\text{def}}{=} \lfloor r_{j-1}/(k+1) \rfloor$. When $a_j \geqq 2$ we choose $r_j \overset{\text{def}}{=} a_j$ if a_j is even and $r_j \overset{\text{def}}{=} a_j - 1$ if a_j is odd. If $a_j < 2$ then $r_j \overset{\text{def}}{=} 2$. Let also $r_{d+1} \overset{\text{def}}{=} 1$ and $p_j \overset{\text{def}}{=} \max(\lceil r_j/r_{j+1} \rceil - 1, 1)$ $(j=1, \ldots, d)$, $p_{d+1} \overset{\text{def}}{=} 1$. The i-th gate G_i^j of level j will be

$$G_i^j(X^n) = 1 \quad \Leftrightarrow \quad \sum_{l=1}^{n} x_l + \sum_{t=1}^{j-1} \sum_{v=1}^{p_t} r_t G_v^t < \sum_{t=1}^{j-1} p_t r_t + i r_j$$

where $j=1, \ldots, d+1$; $i=1, \ldots, p_j$. It is easy to verify that $G_1^{d+1} = \neg\, \mathrm{PARITY}_n$. Since $r_{j+1} > r_j/(k+1) - 2$, the number of the gates in the level j, $p_j \leqq r_j/r_{j+1} < 2(k+1)$. By the definition of k, the size of the circuit is $O(d(n/2)^{1/d})$.

To prove the last part of Proposition 2 it is enough to replace any gate G_i^j $(j=1, \ldots, d)$ by a monotone threshold gate $H_i^j = \neg\, G_i^j$ and its negation. We obtain an exmonotone circuit of depth-$(2d+1)$ computing PARITY_n. It contains $O(d(n/2)^{1/d})$ negations. From the other hand, by Theorem 2, $I_{2d+1}(\mathrm{PARITY}_n) = \Omega(d(n/2)^{1/d})$. □

Now let us consider *arbitrary* symmetric functions.

PROPOSITION 3 (Dičiūnas [1]): *For any symmetric function f, $PTC_2(f) \leqq \lceil ch(f)/2 \rceil$.*

Thus, for depth 2 the lower bound from Theorem 2 is tight. For depth 3 Theorem 2 yields $P_3(f) \geqq \sqrt{2}\sqrt{\mathrm{ch}(f)}-1$. Using Redkin's [4] method we construct almost optimal depth-3 circuits for arbitrary symmetric functions.

THEOREM 5: *For any symmetric function f, $PTC_3(f) \leqq 2\sqrt{ch(f)}+1$.*

Proof: Let us at first construct a depth-3 circuit of size $2\sqrt{\mathrm{ch}(f)}-1$ for an arbitrary symmetric $f(X^n)$ satisfying $f(0^n)=0$ and $m \overset{\text{def}}{=} \mathrm{ch}(f)=4s^2$ $(s=1, 2, \ldots)$.

Let $v(f)=(v_0, \ldots, v_n)$ be the *value vector* (defined in [6]) of f and $L=\{l : v_l \neq v_{l-1}\} \cup \{0\}$. We choose $k=\sqrt{\mathrm{ch}(f)}$ and divide L into k groups of k numbers:

$$L=\{l_0^0, l_1^0, \ldots, l_{k-1}^0, \ldots, l_0^{k-1}, l_1^{k-1}, \ldots, l_{k-1}^{k-1}\}.$$

The j-th gate G_j of the first level will be $\sum_{t=1}^{n} x_t < l_0^j$ $(j=1, \ldots, k-1)$. The i-th gate H_i of the second level will be

$$\sum_{t=1}^{n} x_t + \sum_{j=1}^{k-1} (l_i^j - l_i^{j-1}) G_j \geqq l_i^{k-1} \; (i=1, \ldots, k-1)$$

where for even i we choose the sign $<$ instead of $\geqq$. Finally, in level 3 we take $\sum_{i=1}^{k-1} H_i \geqq k/2$.

In general (when $\mathrm{ch}(f)$ is arbitrary) we take $k=\lceil \sqrt{\mathrm{ch}(f)} \rceil$ and divide L into k groups of $h \leqq \sqrt{\mathrm{ch}(f)}$ numbers. If h is odd we add one additional gate on level 2. $\square$

By a theorem of Lupanov [2], we are not able to prove the tightness of our bound for threshold circuirs of depth $d>3$. He proved that almost all symmetric functions require threshold circuits of size $\Omega(\sqrt{n/\log n})$. However, we can construct almost optimal bounded depth *positive* (not threshold) circuits for almost all (by Corollary 2) symmetric functions. As for inversion complexity, its optimality can be achieved in the class of bounded depth *threshold* circuits.

THEOREM 6: *For any symmetric function $f(X^n)$, $P_{d+2}(f)=O(d(n/2)^{1/d})$ and $ITC_{d+2}(f) \leqq (d/2)(n+1)^{2/d}$.*

Proof: To prove the first inequality, according to the proof of Theorem 4, it is enough to construct a depth-$(d+1)$ positive circuit of size $O(d(n/2)^{1/d})$

computing the functions $B_0, \ldots, B_{l-1}$ where $l = \lceil \log(n+1) \rceil$. In that proof a circuit computing $B_0, \ldots, B_{l-1}$ was obtained from the unbounded depth threshold circuit computing PARITY_n. Now we will do the same with the bounded depth threshold circuit computing PARITY_n. In the proof of Proposition 2 such a circuit of depth-$(d+1)$ with no more than $2(k+1)$ gates on each level was described (where $k = \lceil ((n+1)/2)^{1/d} \rceil - 1$). Now we will construct the gates which divide the interval $[0, n]$ into smaller and smaller intervals in such a way that the length of the intervals all the time is some power of 2. We take the numbers r_i from the proof of Proposition 2 and define $r'_j \stackrel{\text{def}}{=} 2^m$ for $m = 0, 1, \ldots$ satisfying $2^m \leqq r_j < 2^{m+1}$. Then $r_j \geqq r'_j \geqq (r_j + 1)/2 \geqq ((k+1) r_{j+1} + 1)/2 > r_{j+1}$. The number of the gates on level j,

$$p'_j = \max(\lceil r'_j / r'_{j+1} \rceil - 1,\ 1) \leqq r'_j / r'_{j+1}$$
$$\leqq 2 r'_j / (r_{j+1} + 1) \leqq 2(k+1)(r_{j+1} + 2)/(r_{j+1} + 1) \leqq 3(k+1).$$

The gates G_i^j will be the same with the only exception that we replace r_t and p_t by r'_t and p'_t. After all changes we obtain a circuit computing PARITY_n such that any B_i is already computed by some gate or may be computed by one additional gate. The circuit has depth $(d+1)$ and size $O(d(n/2)^{1/d})$.

Let us prove the second inequality. We will construct a depth-$(2t+1)$ threshold circuit with $t(n+1)^{1/t}$ negations computing $E_0, \ldots, E_n$. Now we do it in a different way. We use the same functions F_i^k and H_j^k as Santha and Wilson [5] but we compute them more efficiently. The circuit has $(t+1)$ layers and each layer consists of two levels (the last layer of one).

Let $l \stackrel{\text{def}}{=} (n+1)^{1/t}$, $n_k \stackrel{\text{def}}{=} (n+1)^{1-k/t}$. For $0 \leqq k \leqq t$, $0 \leqq i \leqq l^k - 1$ and $1 \leqq j \leqq l - 1$ we define

$$F_i^k(X^n) = 1 \quad \text{iff} \quad i n_k \leqq |X^n| < (i+1) n_k$$

and

$$H_j^k(X^n) = 1 \quad \text{iff} \quad \exists\, i\,(0 \leqq i < l^k) : i n_k + j n_{k+1} \leqq |X^n| < (i+1) n_k.$$

Let us also define

$$H_0^k \stackrel{\text{def}}{=} 1,\ H_l^k \stackrel{\text{def}}{=} 0,\ \sum_{i=0}^{-1} \ldots \stackrel{\text{def}}{=} 0.$$

Then

$$F_0^0 \equiv 1,\ F_i^d = E_i,\ F_{il+j}^{k+1} = F_i^k \wedge H_j^k \wedge \bar{H}_{j+1}^k$$

and

$$H_j^{k+1}(X^n) = 1 \quad \Leftrightarrow \quad \sum_{m=1}^{n} x_m + \sum_{i=0}^{k} \sum_{j=1}^{l-1} n_{i+1} \bar{H}_j^i \geqq n + 1 - n_{k+1} + j n_{k+2}.$$

The circuit is of depth-$(2t+1)$ and consists of threshold gates and $t(l-1)$ negations. □

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