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ON SEARCHING FOR ROOTS OF A POLYNOMIAL IN A CIRCULAR ANNULUS (*)

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Abstract. – *If a circular annulus contains some zeroes of a polynomial, we try to discover in which part of the annulus the roots are located. We will first study a test which permits to assert that one sector of the annulus contains no zero; then an algorithm based on this test is built. An algorithm is then proposed, which permits a count of the zeroes contained in any sector of the annulus. It needs the computation of two Sturm sequences and can be used for every polynomial in* $\mathbb{C}[X]$.

Résumé. – *Connaissant une couronne circulaire qui contient certains zéros d'un polynôme, nous recherchons dans quelle partie de celle-ci ils se trouvent. Nous étudions d'abord un test permettant d'affirmer qu'un secteur de couronne ne contient pas de zéro. Un algorithme d'isolation des arguments des racines s'en déduit. On construit ensuite un algorithme permettant de compter le nombre de zéros contenus dans un secteur de couronne quelconque. Il nécessite le calcul de deux suites de Sturm et est applicable à tout polynôme de* $\mathbb{C}[X]$.

1. INTRODUCTION

Let $P(z)=\sum_{i=0}^{d} a_i z^i$ be a polynomial with complex coefficients; we denote by $C(r, R)$ the opened annulus centered in O $\{z\in\mathbb{C}\,|\,r<|z|<R\}$. Many methods can be used to find an annulus containing all the zeroes of P: for

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example the method proposed by Davenport and Mignotte [DM], which gives a very good upper bound of the moduli of the roots, can be applied to the reciprocal polynomial of P to obtain a lower bound of the same set and thus delimitate the annulus. It is also possible to determine the number of zeroes contained in any annulus for example by using the Schur-Cohn algorithm twice (*see* Henrici [HE]). Other algorithms, such as Graeffe's or that of the "*qd*" may also allow us to delimit several annuli, each one containing some zeroes of P. Let $C(r, R)$ be such an annulus.

It is natural to look in which part(s) of $C(r, R)$ the considered roots can be found. More precisely, a *sector* of the annulus $C(r, R)$ is any set such as $\{z \in C(r, R) \mid \alpha < \operatorname{Arg}(z) < \beta\}$, where $\operatorname{Arg}(z)$ represents the principal argument of z; we denote such a sector by $C(r, R; \alpha, \beta)$. The *angular width* of the sector is the quantity $\beta - \alpha$. A sector is defined as *suspect* if it contains some roots of the studied polynomial, and as *free* if it does not contain any.

In 2 below we construct a test to show where a sector of an annulus is free. We then propose an exclusion algorithm to isolate the arguments of the roots contained in $C(r, R)$: a problem which is the stumbling block of Graeffe's method. Later we provide a procedure to compute the number of roots in a given sector $C(r, R; \alpha, \beta)$. To do that we need the computation of two Sturm's sequences. This method is to be compared with the computation of the number of roots in a rectangle using Hurwitz's algorithm as presented by Collins and Pinckert [CO], which requires the computation of four Sturm's sequences.

All the modern methods to approximate the zeroes of a polynomial need to be initiated with a set of disks, each one containing a zero. Up to now we only know bisection processes to obtain this first set of disks, all very slow. For a description of these methods, one can consult Petkovic [PE]. Therefore, our aim, while presenting new algorithms, is to present new ways of isolating the roots of a polynomial, a preliminary step to the problem of approximation. We shall use the tools of Computer Algebra and all our results are true for any polynomial in $\mathbb{C}$.

We note the following immediate lemma:

LEMMA 1: *If the distance from a point z to the set of roots of P is larger than ρ, then*

$$|P(z)| \geqq |a_d| \rho^d.$$

2. A TEST OF EXCLUSION

2.1. Description of the test T

We denote by $|P|(z)$ the polynomial $\sum_{i=0}^{n} |a_i| z^i$.

PROPOSITION: *Let $C(r-\varepsilon, r+\varepsilon; \alpha, \beta)$ be a sector of the annulus studied. We note:*

$$a=\ln\left(1+\frac{\varepsilon}{r}\right), \qquad a'=\ln\left(1-\frac{\varepsilon}{r}\right), \qquad \theta=\frac{\beta+\alpha}{2}, \qquad \delta=\sqrt{a'^2+\left(\frac{\beta-a}{2}\right)^2}$$

If

$$|P(re^{i\theta})|>\frac{\varepsilon}{a}|P|'(r+\varepsilon)\,\delta \tag{$*$}$$

then $C(r-\varepsilon, r+\varepsilon; \alpha, \beta)$ is free.

Proof: First we study the case where $\alpha=-\beta$. Let $Q(z)=P(re^z)$. For each z in $\mathbb{C}$, $Q'(z)=re^z P'(re^z)$.

The function $z \to re^z$ represents a one-to-one mapping of the rectangle $]a', a[\times]-\beta, \beta[$ onto $C(r-\varepsilon, r+\varepsilon; -\beta, \beta)$, so that Q vanishes in $]a', a[\times]-\beta, \beta[$ if and only if P vanishes in $C(r-\varepsilon, r+\varepsilon; -\beta, \beta)$. We then have, for any z in $]a', a[x]-\beta, \beta[$,

$$\begin{aligned}|Q(z)-Q(0)| &= \left|\int_0^z re^t P'(re^t)\,dt\right| = \left|\int_0^1 re^{\lambda z} P'(re^{\lambda z})\, z\, d\lambda\right| \\ &\leqq r\delta\int_0^1 |e^{\lambda z}|.|P'(re^{\lambda z})|\,d\lambda \qquad \text{with} \quad \delta^2=a'^2+\beta^2 \\ &\leqq r\delta\,|P|'(r+\varepsilon)\int_0^1 e^{\lambda a}\,d\lambda \\ &\leqq \delta\,|P|'(r+\varepsilon)\frac{\varepsilon}{a}\end{aligned}$$

Then if $|Q(0)|>\frac{\varepsilon}{a}|P|'(r+\varepsilon)\,\delta$, $Q(z)$ cannot be equal to zero in $]a', a[\times]-\beta, \beta[$. In the general case, it is clear that P has a zero in

$C(r-\varepsilon, r+\varepsilon; \alpha, \beta)$ if and only if $R(z)=P(ze^{i\theta})$ has a zero in $C(r-\varepsilon, r+\varepsilon; -\gamma, \gamma)$ with $\gamma=(\beta-\alpha)/2$. We notice that $|R|=|P|$ and that $\delta^2=a'^2+\gamma^2$. Hence the result of the proposition. □

Remarks: 1. The term $K=\varepsilon/a|P|'(r+\varepsilon)$ is independent of the studied sector of the annulus. Therefore, if we want to apply the test T to several sectors of a same annulus, we need only one polynomial evaluation for each sector, K being computed only once. 2. The smaller ε, the easier the test T passes: the efficiency of this test depends on the narrowness $R-r$ of the annulus.

When the formula $(*)$ is valid, it will be said that the polynomial P *passes the test* T on the sector of annulus $C(r-\varepsilon, r+\varepsilon; \alpha, \beta)$ and that $C(r-\varepsilon, r+\varepsilon; \alpha, \beta)$ is T-free.

Formula $(*)$ *for a circle*: It is easy to slightly transform the proof to obtain the following result: let $C(r; \alpha, \beta)$ be an arc of the circle $C(O, r)$; let $\theta=(\beta+\alpha)/2$ then, if

$$|P(re^{i\theta})|>r|P'|(r)\frac{\beta-\alpha}{2} \qquad (**)$$

$C(r; \alpha, \beta)$ does not contain any zero of P.

T-free sectors centered in $re^{i\theta}$: knowing $|P(re^{i\theta})|$ and K, we can use formula $(*)$ to get the upper bound of the half-angular width γ of a sector that is free and centered in $re^{i\theta}$. If $|P(re^{i\theta})|>Ka'$, we denote

$$\gamma_\theta=\sqrt{\frac{|P(re^{i\theta})|^2}{K^2}-a'^2} \qquad (***)$$

And we define:

$$C(r-\varepsilon, r+\varepsilon; \theta-\gamma_\theta, \theta-\gamma_\theta)=\bigcup_{\gamma\in]0,\gamma_\theta[} C(r-\varepsilon, r+\varepsilon; \theta-\gamma, \theta-\gamma).$$

As every sector of the right side is T-free, $C(r-\varepsilon, r+\varepsilon; \theta-\gamma_\theta, \theta-\gamma_\theta)$ is free. And we cannot construct a larger free sector centered in $re^{i\theta}$ using $(*)$.

PROPERTY: From the above demonstration it appears that, when P passes the test T, for each z in $]a', a[\times]-\gamma, \gamma[$, $Q(z)$ is contained in a circle centered in $Q(0)$ with a radius strictly inferior to $|Q(0)|$; Q being continuous, we deduce that $Q([a', a]\times[-\gamma, \gamma])$ is entirely contained in a closed half plane whose boundary goes through O and hence that the image by P of the closure of $C(r-\varepsilon, r+\varepsilon; \alpha, \beta)$ has the same property. Following Henrici's terminology [HE], we say that the test T is *one-sided.*

2.2. An algorithm of isolation of the arguments of the zeroes in an annulus

We first consider the case where only one zero is contained in the studied sector $C(r-\varepsilon, r+\varepsilon; \alpha, \beta)$. This is the case when P is a complex polynomial chosen at random (an annulus, thick enough, has generally a single root in its interior) or when P is a real polynomial with exactly two conjugated roots in $C(r-\varepsilon, r+\varepsilon)$ so that only one root is in $C(r-\varepsilon, r+\varepsilon; 0, \pi)$. We then construct the following sequences:

$$\alpha_1=\alpha \quad \text{and} \quad \alpha_n=\sqrt{\left(\frac{|P(re^{i\alpha_{n-1}})|}{K}\right)^2-a'^2}+\alpha_{n-1}$$

$$\beta_1=\beta \quad \text{and} \quad \beta_n=-\sqrt{\left(\frac{|P(re^{i\beta_{n-1}})|}{K}\right)^2-a'^2}+\beta_{n-1}$$

Obviously the sequence (α_n) is increasing and the argument Θ of the only root contained in the sector is an upper bound for that sequence: suppose that there exists an n such as $\alpha_n<\Theta<\alpha_{n+1}$, therefore the sector $C(r-\varepsilon, r+\varepsilon; \alpha_n, \lambda)$, for λ such as $\Theta<\lambda<\alpha_{n+1}$, should be T-free, which it is not. Beside $\Theta=\alpha_{n+1}$ is impossible, otherwise no sector centered in $re^{i\alpha_{n+1}}$ would be T-free, and in this case α_{n+1} ends the sequence. Therefore the sequence (α_n) either furnishes a lower bound for Θ after a finite number of steps or converges towards a limit lower or equal to Θ. In the same way, with (β_n) we get an upper bound of Θ.

Sharpness of the bounds obtained: In practice we stop the process when $\alpha_n-\alpha_{n-1}$ becomes lower than a fixed quantity t. Lemma 1 shows that, if $\alpha_{\max}$ denotes the last computed term, we have $re^{i\alpha_{\max}}$ at a distance smaller than $\Delta=(K^2(a'^2+t^2)/|a_d|^2)^{1/2d}$ from a root of P.

DEFINITION: *If* $C(r-\varepsilon, r+\varepsilon)$ *is a suspect annulus of* P, *we say that it is* well separated *when the only zeroes contained in* $C(r-\Delta, r+\Delta)$ *are in fact contained in* $C(r-\varepsilon, r+\varepsilon)$. (We can test whether an annulus is well separated or not by using the algorithm of Schur-Cohn.)

Therefore, if we suppose that $C(r-\varepsilon, r+\varepsilon)$ is well separated, we can assert that $re^{i\alpha_{\max}}$ is at a distance than Δ of the root contained in the annulus. We have a similar result for the sequence (β_n) and its last computed term $re^{i\beta_{\max}}$.

ALGORITHM: *Input*: – P complex polynomial

– r, ε, α, β, t such that $C(r-\varepsilon, r+\varepsilon; \alpha, \beta)$ contains only one root and is well separated

Loop: Compute α_{max} and β_{max}

Output: $C(r-\varepsilon, r+\varepsilon; \alpha_{max}, \beta_{max})$ countaining the root studied with $\beta_{max}-\alpha_{max}<2\Delta$.

As mentioned in the introduction, the aim here is to perform only a few steps of the algorithm in order to sharpen quickly the initial sector. Therefore the parameter t is taken as large as possible, the only condition it must verify being that the suspect annulus remains well separated. It is also useless to study problems of convergence and numerical stability.

$(\beta-\alpha)/2t$ is an upper bound of the number of steps to perform, but it remains very rough. For example, according to Lehmer [LE], it is not easy to determine the arguments of the roots of the polynomial Z^5-1 by Graeffe's process; but α_5 and β_5 computed in $C(0.9999, 1.0001)$ with $\alpha_1=0.3$ and $\beta_1=2$ give an approximation of $2\pi/5$ better than 7.10^{-5}.

We can easily generalize this algorithm to the case where p roots are contained in the suspect sector ($p>1$). Applying the above method we obtain first a lower bound Θ_{min} for the smallest argument and an upper bound Θ_{max} for the greatest. Then we cut $[\Theta_{min}, \Theta_{max}]$ by the middle, and apply again the algorithm to each new interval $[\Theta_{min}, \Theta]$ and $[\Theta, \Theta_{max}]$. We stop the process when the obtained intervals have an amplitude less than $2p\Delta$.

3. EXACT COMPUTATION OF THE NUMBER OF ROOTS IN A SECTOR OF AN ANNULUS

3.1. Introduction

We apply the principle of argument to the border δC of $C(r-\varepsilon, r+\varepsilon; \alpha, \beta)$ after having verified that this border contains no root of P – we show below that it is not difficult to ensure that this last condition is fulfilled. We subdivide δC into four pieces: two straight lines, L_1 and L_2, and two arcs of circles, A_1 and A_2, orientated as shown in the *fig.* 1. The numbers of half-planes Π limited by the imaginary axis and through which $P(z)$ passes when z goes along L_1 or L_2, are computed by the technique of Cauchy's index, as described in Henrici [HE] or Marden [MA]; each of these indexes needs the computation of one Sturm sequence.

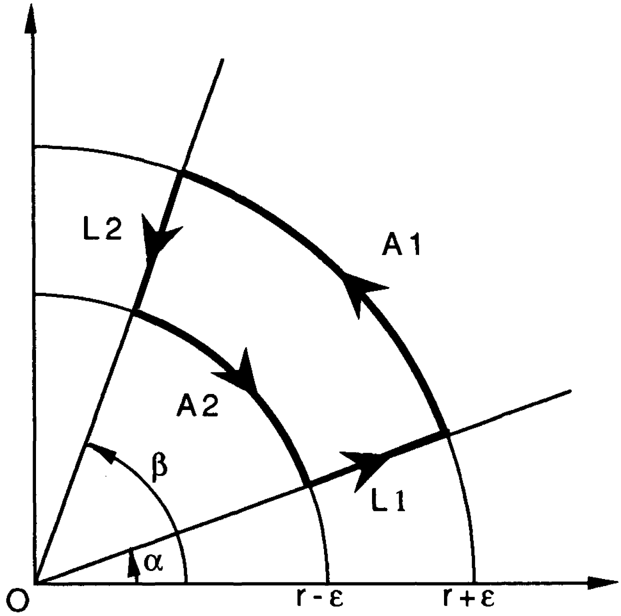

Figure 1

3.2. Computation of the number of half planes Π through which $P(z)$ passes when z runs along an arc of a circle

We first notice that, if an arc of the circle with extremities $re^{i\alpha}$ and $re^{i\beta}$ is T-free, then the variation of argument along this arc is equal to $\mathrm{Arg}(P(re^{i\beta})) - \mathrm{Arg}(P(re^{i\alpha}))$: the test T is one-sided. The following proposition allows us to conclude in the case of an arc which is not T-free.

PROPOSITION: *Every arc of a circle which doesn't contain any root of P can be covered by a finite number of T-free arcs.*

Proof: Since $z \rightarrow P(z)$ is continuous, the image by P of the circle $C(O, r)$ containing the arc studied is compact and does not go through O. Therefore there exists a real number 1, $1 \neq 0$ such that:

$$|z| = r \Rightarrow |P(z)| > 1$$

We then apply the test T on the circle C. It follows immediately by (**) that every arc with an angular width less than or equal to $21\, k/r\, |P|'(r)$ is

T-free. So we can cover every arc of C by at most $\pi r|P|'(r)/1k$ T-free arcs. □

Having so covered the arc A_1 with n arcs, $[re^{i\alpha_k}, re^{i\alpha_{k+1}}]$, the variation of argument of $P(z)$ when z runs along A_1 in the positive orientation is:

$$\sum_{k=1}^{n} \operatorname{Arg}(P(re^{i\alpha_{k+1}}) - \operatorname{Arg}(P(re^{i\alpha_k}))$$

Therefore the number of half-planes Π gone through is easily computed; we have then all the required elements to apply the argument principle and, compute the number of roots contained in $C(r-\varepsilon, r+\varepsilon; \alpha, \beta)$. We name this procedure the *T-procedure*.

Remark: The only similar result we can find in literature, is the computation of the number of roots of a polynomial in an angular sector by Marden [MA]. It is not difficult to see that the T-procedure is a generalization of Marden's result.

3.3. Further considerations

Three remarks guide our steps in the use the T-procedure:

1. Lemma 1 suggests that the further we are from the roots of P, the more assured we are that the test T is passed for a given sector. Therefore, instead of studying the sector of the annulus $(abcd)$ – see *fig.* 2 –, we can enlarge it using (ABCD), taking for moduli the radii of the circles which cut through the middle the non-suspect annuli bordering $C(r-\varepsilon, r+\varepsilon; \alpha, \beta)$. By doing this, we do not change the number of roots contained in the sector studied, we decrease the number of the covering T-free arcs and we are assured that no roots are on the border of the new sector, which is a necessary condition to apply the T-procedure. (Zeros on the segments [AD] and [BC] are revealed during the computation of the Cauchy's indexes by sub-resultant sequences [LO]).

2. It is not necessary to cover the arcs [AB] and [CD] with arcs of the same angular width. We can use a method similar to those proposed in 2.2 by choosing the extremities α_n of the covering arcs using:

$$\alpha_n = \frac{|P(re^{i\alpha_{n-1}})|}{r|P|'(r)} + \alpha_{n-1}$$

3. We only need to cover the arcs of the borders of the sectors once; every subsequent use of the T-procedure in certain sub-sectors uses the same set of

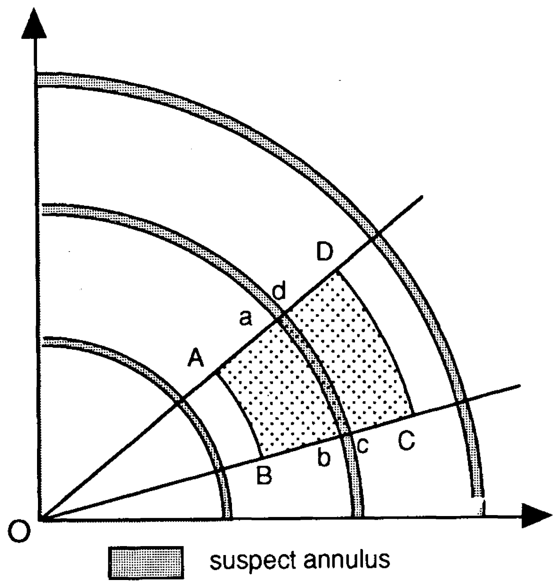

Figure 2

covering T-free arcs, except perhaps the two extremities, which may have to be adapted.

One can ask for a good estimate of the number of T-free arcs needed to cover the arcs A_1 and A_2, because the bound presented in the demonstration above, $\pi r |P|'(r)/1 k$, has only a theoretical value. Furthermore we can find in [SP] tools for evaluating a lower bound for 1 in term of the norm and measure of P. However the theoretical results remain very far from experimentation. By combining the algorithm presented in 2 to sharpen the suspect sector and the T-procedure to compute the number of roots in the sector delimited, the set of covering T-free arcs needed contains very few elements. We tested 400 integer polynomials chosen at random with a degree between 5 and 20. On an average, using remarks 1 and 2 just above, 2 T-free arcs were needed to cover each circular border of the suspect sector obtained, whatever the root considered.

The T-procedure is performed using Computer Algebra. Indeed, we know of no paper describing the behavior of Sturm's sequences in numerical analysis, a subject which in itself deserves a deep study.

Further work involves a study of different strategies that are well adapted to Computer Algebra and parallel architecture for isolating the roots of a polynomial, isolating first the moduli of the roots, then the arguments.

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