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ON A TREE COLLISION RESOLUTION ALGORITHM IN PRESENCE OF CAPTURE (*)

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Abstract. – *We investigate some characteristic parameters of trees underlying a collision resolution with a simple tree-algorithm. An extension of the usual assumptions about the basic model provides the ability of treating capture effects: in case of a collision it is not necessary to assume the destruction of all packets involved. Our investigations are based on the analysis of complicated alternating sums of a certain type already known from the analysis of search trees. By using an extension of techniques applied in those studies, which are mainly based on Rice's approach, we obtain asymptotic expansions for a number of interesting quantities.*

Résumé. – *Cet article a pour objet l'examen de certains paramètres caractéristiques induits par les protocoles de résolution de collision en arbre. Le modèle classique d'analyse est étendu par la possibilité d'effets de capture : dans ce cas, un paquet peut éventuellement survivre à une collision. L'analyse met en jeu des sommes alternées voisines de celles que l'on rencontre dans l'étude des arbres digitaux. Par l'utilisation de tehniques fondées sur la méthode de Rice, l'on obtient diverses analyses asymptotiques des caractéristiques principales du protocole sous ce modèle modifié.*

1. INTRODUCTION

We study one of the simple tree-algorithms (Capetanakis, Hayes, Tsybakov, Mikhailov) for collision resolution in a random access broadcast system, where a lot of results (through-put, delay-characteristics, stability ...) are well known from the past, *see* for example [1] for a nice survey. Most of the investigations mentioned based on a model which is similar to the following:

(1) A (infinite) large population of identical transmitters is supposed to have access to a common time-slotted noiseless collision-type channel.

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(2) The transmitters are constrained to transmit independent information in the form of "packets" whose length is one time slot, and the packet generation in the whole system is according to a Poisson-process with fixed rate λ.

(3) The channel feedback is supposed to be identical for all transmitters, and in case of a collision all packets involved in that collision are completely lost.

Our intention is to study the behavior of the algorithm (or exactly, parameters of the underlying tree) in the presence of capture effects. In many real communication systems the "strongest" of the actually colliding packets is able to capture the receiver und thus be received without error. To handle this subject we have to change (3) of the common used model.

In [3] the population of transmitters is statically divided into two disjoint groups (the dominating and nondominating group). A transmitter of the dominating group is supposed to capture out one or more transmitter(s) of the other group. Since choosing the strongest of two or more colliding packets is not enough to determine the presence of an actual capture (e. g., in a radio system we have the influence of atmospheric effects like fading), we model our capture in a (very simple) different way: We assume a fixed probability p for the complete lost of all packets involved in a collision. This destruction probability does not depend on the multiplicity of the collision. Moreover, we assume disjoint packets, so $1-p$ is the probability that exactly one of two or more colliding packets is received successfully.

However, we should admit that our approach is not able to cover all possible varieties of capture effects sufficiently [1]. For instance, capture is sometimes a local phenomenon, *i. e.*, concerns not all receivers in the network in the same way. The assumption of all stations agreeing on the destruction/non destruction of all packets involved in a collision is therefore sometimes too optimistic. Another question concerns the assumption of a destruction probability which is independent of the multiplicity of the collision. In radio networks, the possibility of a capture is determined by the ratio between the strongest and the *sum* of the other signals, which is clearly not independent of the multiplicity of the conflict.

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[1] We should like to thank the anonymous referee for stressing some possible directions of further research.

2. BASICS

We study a Q-ary CRA (Collision Resolution Algorithm) with the obvious BAP (Blocked Access Protocol, *see* [1]), which works as follows:

(1) *No collision resolution.* If the system is in an idle state, each transmitter who has a ready packet transmits it in the very next slot.

(2) *Collision resolution.* If the system manages the resolution of a (previous) conflict, all transmitters not involved in the initial collision remain blocked until the complete resolution. They may contend for the following idle slot(s), thus eventually forcing a new initial collision. After a collision, each transmitter involved flips a fair "Q-sided coin" with values from $1, 2, \ldots, Q$. This value is used for determining the relative number of the slot in which the packet should be transmitted, e. g., all transmitters with 1 flipped transmit in the very next slot. If a new collision occurs, it is resolved immediately, suspending the resolution of the other values. Transmitters who are not involved in the current collision but who have already a value flipped have to keep track with the current resolution process, e. g., to add $Q-1$ to their relative slot number.

We may represent the resolution of each initial conflict by a Q-ary tree containing two types of nodes. We distinguish C-nodes (Capture-nodes, representing a successful transmission of a packet in a collision slot, or a single transmission), and NC-nodes (NonCapture-nodes, representing a collision slot with total destruction of all packets involved, or an empty slot), each labelled with the multiplicity of the corresponding conflict (0 is the label for empty slots, 1 for a slot used by a single transmitter). Since the coin-flipping process splits the set of collided transmitters in exactly Q subsets, we use the cardinality of them for labelling the Q successors of the root, recursively for each subtree.

If we examine the trees generated by the application of these rules, further denoted by p-triees (p mixed radix search trie and digital search tree), we obtain the following properties:

(1) For each resolution of a conflict of multiplicity n there exists a unique representation of the resolution process with exactly n C-nodes, and conversely (if we assume indistinguishable transmitters, of course).

(2) Traversing the tree in preorder, we obtain the "traffic" on the channel; each C-node represents a slot with a successful transmission.

(3) Nodes with no successors correspond to empty slots (label 0) or to single transmission slots (label 1).

(4) Nodes with successors correspond to collision slots of multiplicity equal to their label.

(5) Each such tree may be viewed as a "mixture" of a digital search tree and a radix search trie, *see* [4] for a survey. For we are interested in parameters investigated for both type of trees, one could argue, that it might be possible to extend techniques used in these studies, and in fact, this is true.

3. OUTLINE

We are mainly interested in studying parameters of the underlying p-triee rather than obtaining results concerning the performance of the algorithm. Of course, it is possible to derive results like throughput easily from our computations, *see* [2] for a very complete survey.

Assuming a p-triee as mentioned before with exactly n C-nodes, we are interested in the following questions:

(1) What is the expected number of nodes in the whole tree? This problem corresponds to the computation of the CRI-length (Collision Resolution Interval) of an initial collision of multiplicity n.

(2) What is the expected number of nodes with label 0? This problem corresponds to the computation of the number of empty slots in the CRI, when resolving an initial collision of multiplicity n.

What is the expected number of nodes with label $V \geqq 2$? This problem corresponds to the computation of the number of collision slots with given multiplicity V when resolving an initial collision of multiplicity n.

We use the same general outline for the derivation of all three results: First, we obtain a recurrence relation for the desired parameter, say L_n. Starting from a functional equation for the corresponding EGF (Exponential Generating Function) $L(z)$ we derive a simpler one by introducing the PoGF (Poisson Generating Function) $H(z)$. Now we are able to obtain a simple linear recurrence for the Taylor-coefficients h_n of the PoGF, which leads to an explicit expression involving a sum of some partial products. Eventually, an explicit expression for the desired qantity L_n may be found, which is an alternating sum $\sum_k \binom{n}{k}(-1)^k f_k$ with f_k essentially h_k.

The remaining problem is to determine an asymptotic expression for L_n as n gets large. This task is done by means of the so called Rice's method, *see* for example [5], Exercise 5.2.2-54. Rice's method is based on a classical formula from the calculus of finite differences, which states an identity for alternating sums involving binomial coefficients and a special type contour integral. All we need to find the asymptotic expansion of our sum is a function $F(z)$ with the property $F(k)=f_k$ for all summation values k, analytically in a skinny region covering the (positive) real axis.

Unfortunately, easy computations show that $\lim_{n \to \infty} f_n = \infty$, thus no appropriate function could be found. Instead, we determine the asymptotic expansion of $f_n = c_1 n + c_2 +$ (exponential small terms) and investigate the alternating sum on $g_n = f_n - c_1 n - c_2$ with the approach mentioned. The evaluation of the contour integral is done by expanding the integration curve in an appropriate way and taking into account the residues of encounterd poles.

Here we present our results stated as theorems:

THEOREM (1): *The average number L_n of nodes in a p-triee with exactly n C-nodes is*

$$L_n = n\,Q\left(\frac{q_0}{\log Q} + (1-p/Q)\,\beta_1(p) - q_1 + (1-q_0)\frac{p/Q}{1-p/Q}\right) + n\,Q\,q_0\,P(\log_Q n) + O(1).$$

The function $P(u)$ is periodic with periode 1, *has very low amplitude, mean* 0 *and its Fourier expansion is given by*

$$P(u) = \frac{1}{\log Q}\sum_{k \neq 0} \chi_k \Gamma(-1-\chi_k)\, e^{2k\pi i u} \qquad \textit{with} \quad \chi_k = \frac{2k\pi i}{\log Q}.$$

THEOREM (2): *The average number E_n of nodes with label* 0 *in a p-triee with exactly n C-nodes is*

$$E_n = n(Q-p)\left(\frac{q_0}{\log Q} + (1-p/Q)\,\beta_1(p) - q_1 + (1-q_0)\frac{p/Q}{1-p/Q}\right) + n(Q-p)\,q_0\,P(\log_Q n) + O(1).$$

The function $P(u)$ is periodic with periode 1, *has very low amplitude, mean* 0 *and its Fourier expansion is given by*

$$P(u) = \frac{1}{\log Q}\sum_{k \neq 0} \chi_k \Gamma(-1-\chi_k)\, e^{2k\pi i u} \qquad \textit{with} \quad \chi_k = \frac{2k\pi i}{\log Q}.$$

THEOREM (3): *The average number $I_n = I_n(V)$ of nodes with label $V \geqq 2$ in a p-triee with exactly n C-nodes is*

$$I_n = n(-1)^V \left(\frac{1}{\log Q} \sum_{k=2}^{V} (-1)^k \frac{q_{V-k}}{k(k-1)} + \frac{q_{V-1}}{\log Q} - q_V - q_{V-1} + K(V, p) \right)$$
$$+ n \left(\sum_{j=0}^{V-1} \frac{(-1)^j q_j}{(V-j)!} P_{V-j}(\log_Q n) - (-1)^V q_{V-1} P_0(\log_Q n) \right) + O(1).$$

The functions $P_t(u)$ are periodic with periode 1, *have very low amplitude, mean* 0 *and the Fourier expansions are given by*

$$P_t(u) = \frac{1}{\log Q} \sum_{k \neq 0} \Gamma(t - 1 - \chi_k) e^{2k\pi i u} \qquad \textit{with} \quad \chi_k = \frac{2k\pi i}{\log Q}.$$

The constants refered to in the theorems are defined as

$$q_0 = \prod_{j \geqq 1} \frac{1 - p Q^{-j}}{1 - Q^{-j}}$$

$$q_k = 1/k \sum_{j=0}^{k-1} q_j \sum_{l \geqq 1} \left(\frac{1}{(Q^l - l)^{k-j}} \frac{p^{k-j}}{(Q^l - p)^{k-j}} \right) \qquad \text{for} \quad k \geqq 1$$

$$\beta_0(p) = \frac{q_0 - 1}{1 - p/Q}$$

$$\beta_t(p) = \sum_{k > t} \left(\frac{1}{Q^k - 1} - \frac{p}{Q^k - p} \right) \sum_{l=t}^{k-1} \prod_{j=1}^{l} \frac{1 - p Q^{-j}}{1 - Q^{-j}} \binom{l}{t} Q^{-l} \qquad \text{for} \quad t \geqq 1$$

$$K(V, p) = (1 - p/Q) \beta_V(p) + (1 - 2p/Q) \beta_{V-1}(p) - p/Q\, \beta_{V-2}(p).$$

At last, we list some numerical results for different values of Q and p. The first table shows the major term in the asymptotic expansion of the quantity L_n/n, *i.e.*, the average number of nodes in a p-triee with exactly n C-nodes:

TABLE I

Major term of L_n/n.

Q	p										
	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
2	1.25	1.32	1.41	1.52	1.64	1.76	1.92	2.10	2.30	2.56	2.86
3	1.57	1.64	1.72	1.81	1.90	2.00	2.11	2.24	2.38	2.54	2.72
4	1.86	1.94	2.02	2.10	2.18	2.28	2.38	2.48	2.60	2.72	2.86

The next table shows the major term of ratio E_n/n, that is the average number of label 0 nodes in a p-triee with exactly n C-nodes:

TABLE II

Major term of E_n/n.

Q	p										
	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
2	0.26	0.27	0.28	0.30	0.31	0.33	0.34	0.36	0.38	0.41	0.44
3	0.57	0.59	0.61	0.63	0.66	0.68	0.70	0.73	0.76	0.79	0.82
4	0.88	0.90	0.93	0.95	0.98	1.02	1.04	1.07	1.10	1.13	1.17

4. PROOF (1)

Denoting by L_n the average number of nodes in a p-triee with n C-nodes, we have the following basic recurrence

$$L_n = 1 + p \sum_{\Sigma i_l = n} \binom{n}{i_1, \ldots, i_Q} Q^{-n} \sum_{l=1}^{Q} L_{i_l}$$
$$+ (1-p) \sum_{\Sigma j_l = n-1} \binom{n-1}{j_1, \ldots, j_Q} Q^{1-n} \sum_{l=1}^{Q} L_{j_l}$$
$$= 1 + pQ \sum_{i=0}^{n} \binom{n}{i} Q^{-i} (1-1/Q)^{n-i} L_i$$
$$+ (1-p) Q \sum_{j=0}^{n=1} \binom{n-1}{j} Q^{-j} (1-1/Q)^{n-1-j} L_j$$

for $n \geqq 2$. The initial values resulting from the physical model are

$$L_0 = 1 \quad \text{and} \quad L_1 = 1.$$

This comes from the following easily established facts. First, the number of nodes in the triee is 1 plus the sum of the nodes in the Q subtriees. Second, with Probability p the sum of C-nodes in the subtriees is n, with Probability $(1-p)$ it is only $n-1$, because the root is a C-node. Third, the splitting of the n resp. $n-1$ C-nodes in Q subsets is according to a multinomial probability-distribution. At last, the subtriees themselves are built in the same manner. Proving the simple multinomial identity is given to the reader.

We first introduce some notational conviences. $\mathscr{D}$ denotes the ordinary differential operator, $\mathscr{N}$ the 0-substitution and $\mathscr{U}$ the 1-substitution operator, all with respect to a variable clear from the context or explicitely given, e. g., the differential operator w.r.t. t is denoted by $\mathscr{D}_t$. Let $L(t)$ be the EGF of L_n, it is clear that $L_n = \mathscr{N}\mathscr{D}^n * L(t)$, all operators w.r.t. t. Using this in our recurrence, we obtain for $n \geqq 2$

$$L_n = 1 + p\,Q\,\mathscr{N}(1 - 1/Q + \mathscr{D}/Q)^n * L(t) + (1-p)\,Q\,\mathscr{N}(1 - 1/Q + \mathscr{D}/Q)^{n-1} * L(t),$$

the $*$ means application of the operator to the left on the function to the right. Multiplying both sides with $z^{n-1}/(n-1)!$ and summing over $n \geqq 2$ with mentioning the fact $\mathscr{N}_t e^{x\mathscr{D}_t} * f(t) = f(x)$ yields

$$L'(z) = pe^{z(1-1/Q)}\,L'(z/Q) + Q(1 - p/Q)\,e^{z(1-1/Q)}\,L(z/Q) + e^z - Q.$$

Introducing the PoGF of the sequence L_n

$$H(z) = L(z)\,e^{-z} = \sum_{n \geqq 0} h_n \frac{z^n}{n!}$$

which induces the following inverse pair

$$h_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} L_k \quad \text{and} \quad L_n = \sum_{k=0}^{n} \binom{n}{k} h_k \tag{4-1}$$

yields a simpler functional equation

$$H'(z) + H(z) = 1 + QH(z/Q) + p\,H'(z/Q) - Q\,e^{-z}$$
$$h_0 = 1 \quad \text{and} \quad h_1 = 0.$$

We will investigate a little generalisation of the above, which helps us to treat the proof of theorem (2) with the same approach. We look at the functional equation

$$H'(z) + H(z) = 1 + QH(z/Q) + p\,H'(z/Q) - A\,e^{-z}$$

with some h_0 and h_1. In the previous case we have $A = Q$, $h_0 = 1$ and $h_1 = 0$. Starting our treatment by equating the coefficients of $z^n/n!$ on both sides yields a simple recurrence for the h_n with $n \geqq 1$.

$$h_{n+1} = -h_n \frac{1 - Q^{1-n}}{1 - p\,Q^{-n}} + (-1)^{n+1} \frac{A}{1 - p\,Q^{-n}}. \tag{4-2}$$

Let $P_n = \prod_{k \geqq n} (1 - Q^{1-k})/(1 - p\,Q^{-k})$, then $P_1 = 0$, because the numerator of the product for $n = 1$ has a zero factor. Proving the convergence of the product is easy and therefore suppressed in this paper. Moreover, let $a_n = h_n P_n$, then $a_1 = 0$ and multiplying both sides of the former recurrence with P_{n+1} shows a linear recurrence for the a_n. The iterated solution is

$$a_{n+1} = (-1)^{n+1} A \sum_{k=1}^{n} \frac{P_{k+1}}{1 - p\,Q^{-k}}.$$

After some algebraic manipulations we get the following explicit expression for the coefficients h_n with $n \geqq 2$.

$$h_n = (-1)^n \frac{A}{1 - p\,Q^{1-n}} \sum_{k=1}^{n-1} \prod_{j=k}^{n-2} \frac{1 - Q^{-j}}{1 - p\,Q^{-j}}.$$

The desired quantity L_n is expressible from (4-1) and yields

$$L_n = \sum_{k=0}^{n} \binom{n}{k} h_k = h_0 + n h_1 + A \sum_{k=2}^{n} \binom{n}{k} (-1)^k \frac{1}{1 - p\,Q^{1-k}} \sum_{l=1}^{k-1} \prod_{j=l}^{k-2} \frac{1 - Q^{-j}}{1 - p\,Q^{-j}}. \quad (4\text{-}3)$$

The alternating sum is treatable by the formulae of Rice, so we need a meromorphic function $F(z)$ with the property

$$F(n) = f_n = \frac{1}{1 - p\,Q^{1-n}} \sum_{k=1}^{n-1} \prod_{j=k}^{n-2} \frac{1 - Q^{-j}}{1 - p\,Q^{-j}} = \frac{(-1)^n}{A} h_n \quad (4\text{-}4)$$

for all $n \geqq 2$.

As already mentioned, we are not allowed to treat the sum with Rice's approach directly. We first have to determine the high-order terms of their asymptotic expansion and to manage the remaining sum of residual terms.

We define for $n \geqq 0$

$$Q_n(p) = \prod_{j=1}^{n} (1 - p\,Q^{-j}) \qquad \text{and} \qquad Q(p) = Q_\infty(p)$$

$$T_n(p)=\prod_{j=1}^{n}\frac{1-Q^{-j}}{1-pQ^{-j}}\qquad\text{and}\qquad T(p)=T_\infty(p).$$

An empty product is assumed to be equal to 1, so both $Q_0(p)=T_0(p)=1$ and further $T_n(p)=Q_n(1)/Q_n(p)$. Moreover, we introduce

$$\begin{aligned}P_n(p)&=T_n(p)\left(\frac{1}{T_0(p)}+\frac{1}{T_1(p)}+\ldots+\frac{1}{T_n(p)}\right)\\&=\sum_{k=1}^{n+1}\sum_{j=k}^{n}\frac{1-Q^{-j}}{1-pQ^{-j}}\end{aligned}$$

which enables us to express f_n in the following simple manner

$$f_n=\frac{P_{n-2}(p)}{1-pQ^{1-n}}.$$

We should note that $P_n(p)$ must not be confused with the similar products P_n, which we used for determining a closed expression for the h_n. Fortunately, we are able to provide the ordinary GF (Generating Function) $S_p(z)$ of the sequence $1/T_n(p)$, *see* Appendix (A-1). The GF of their partial sums, denoted by $R_p(z)$, evaluates to

$$R_p(z)=\frac{S_p(z)}{1-z}=\frac{1}{(1-z)^2}\frac{Q(pz)}{Q(z)}$$

and using the known Taylor-expansion of $Q(pz)/Q(z)$ at $z=1$ from (A-2) leads to

$$R_p(z)=\frac{1}{(1-z)^2}\frac{Q(p)}{Q(1)}-\frac{1}{1-z}\frac{Q(p)}{Q(1)}(\alpha(1)-p\,\alpha(p))+r(z)$$

where $r(z)$ has a radius of convergency of Q around $z=0$. Choosing an arbitrary but fixed $\varepsilon>0$ and $q=Q-\varepsilon$, we find the asymptotic expansion of the coefficients of $R_p(z)$

$$\sum_{k=0}^{n}\frac{1}{T_k(p)}=\frac{Q(p)}{Q(1)}\left(\binom{n+1}{n}-\alpha(1)+p\,\alpha(p)\right)+O(q^{-n}).$$

To obtain the asymptotics of f_n, we need two easily proved expansions

$$\left.\begin{aligned} \frac{1}{1-pQ^{1-n}}&=1+O(q^{-n}) \\ T_n(p)&=T(p)(1+O(q^{-n}))=\frac{Q(1)}{Q(p)}(1+O(q^{-n})). \end{aligned}\right\} \qquad (4\text{-}5)$$

Multiplication of all expansions concerned with f_n yields

$$f_n=n-1-(\alpha(1)-p\,\alpha(p))+O(q^{-n})=n+\gamma+O(q^{-n})$$

with the shorthand

$$\gamma=-1-(\alpha(1)-p\,\alpha(p)). \qquad (4\text{-}6)$$

Now it is time for the introduction of a new sequence g_n, defined by

$$g_n=f_n-(n+\gamma).$$

Before treating the new subject, we deal with manipulating the expression for the desired quantity L_n. From (4-3) we have

$$\begin{aligned} L_n=h_0+nh_1+A\sum_{k=2}^{n}\binom{n}{k}(-1)^k f_k \\ &=n(h_1+A(p\,\alpha(p)-\alpha(1)))+h_0 \\ &\quad+A(1+\alpha(1)-p\,\alpha(p))+A\sum_{k=2}^{n}\binom{n}{k}(-1)^k g_k \qquad (4\text{-}7) \end{aligned}$$

where we used the well known expressions for $\sum\binom{n}{k}(-1)^k$ and $\sum\binom{n}{k}(-1)^k k$ and (4-6) for replacing γ by its definition. The remaining task is the computation of the alternating sum involving the g_n with Rice's method.

It is clear that $g(n)=O(q^{-n})$, and remembering (4-2) and (4-4) yields after some algebraic manipulations a recurrence for the g_n with $n\geqq 2$.

$$g_n=g_{n+1}\frac{1-pQ^{-n}}{1-Q^{1-n}}+\frac{(n+\gamma-(n+\gamma+1)p/Q)\,Q^{1-n}}{1-Q^{1-n}}$$

It should be noted that the condition $\lim_{n\to\infty} g_n=0$ inhibits the existence of more than one solution, as can be shown with an indirect proof using iteration. We investigate a corresponding functional equation rather than the recurrence, because we need a function $G(z)$ with the property $G(n)=g_n$ for $n \geqq 2$. This is a simple task, of course. Let

$$G(z)=a(z)\,G(z+1)+b(z)$$

with

$$a(z)=\frac{1-pQ^{-z}}{1-Q^{1-z}}$$

$$b(z)=\frac{(z+\gamma-(z+\gamma+1)p/Q)\,Q^{1-z}}{1-Q^{1-z}}.$$

We may iterate it and together with the assumption that $G(z)$ vanishes if $\Re(z)\to\infty$ in a certain region of the complex plane we obtain a solution of the functional equation as desired

$$G(z)=\frac{Q^{1-z}}{1-Q^{1-z}}\sum_{k\geq 0}\prod_{j=1}^{k}\frac{1-pQ^{-z-j+1}}{1-Q^{-z-j+1}}(z+\gamma+k-(z+\gamma+k+1)p/Q)\,Q^{-k}.$$

The function is meromorphic in the complex plane with poles at most at

$$z=1-j+\chi_k \qquad \text{with} \quad j\geqq 0 \text{ and } \chi_k=\frac{2k\pi i}{\log Q} \text{ for all integers } k.$$

For we are interested only in asymptotic terms of higher order than $O(1)$, it is necessary to obtain the residues of

$$\Phi_n(z)\,G(z)=-\frac{\Gamma(n+1)\,\Gamma(-z)}{\Gamma(n+1-z)}G(z)$$

only in the strip $0<\Re(z)<2$, *see* (A-7). This fact, together with our previous statement about $G(z)$ guarantees potential poles only at

$$z=1+\chi_k \qquad \text{with} \quad \chi_k=\frac{2k\pi i}{\log Q} \text{ for all integers } k.$$

The point $z=1$ plays a special role, because if it were a pole of $G(z)$, our function $\Phi_n(z)\,G(z)$ would have a pole of order greater than 1. A deeper investigation of Rice's method shows, that logarithmic terms would occur in this case. Fortunately, easy computations show that the desired quantity L_n

is of order $O(n)$, so we could expect the existence of $G(1)$, e. g., $G(z)$ has no pole at $z=1$.

The key for our further investigations is the function

$$F(u, v)=\sum_{k \geqq 1} \prod_{j=1}^{k} \frac{1-pv Q^{-j}}{1-v Q^{-j}} Q^{-k} u^k$$

for which we derive some properties in the Appendix. Rewriting $G(z)$ in terms of this function yields

$$G(z)=\frac{Q^{1-z}}{1-Q^{1-z}}((z+\gamma-(z+\gamma+1)p/Q)(F(1, Q^{1-z})+1) \\ +(1-p/Q)F_1(1, Q^{1-z})). \quad (4\text{-}8)$$

We evaluate the first few coefficients of the Laurent-series at $z=1+\chi_k$, which expresses to

$$\left.\begin{aligned} F(1, Q^{1-z})&=F(1, 1)-\log Q F_2(1, 1)(z-1-\chi_k)+O(z-1-\chi_k)^2) \\ F_1(1, Q^{1-z})&=F_1(1,1)-\log Q F_{12}(1, 1)(z-1-\chi_k)+O((z-1-\chi_k)^2). \end{aligned}\right\} \quad (4\text{-}9)$$

The indices of the function indicate parital derivatives w.r.t. the first (index 1) or the second (index 2) argument. Furthermore, an easy computation shows

$$\frac{Q^{1-z}}{1-Q^{1-z}}=\frac{1}{\log Q} \frac{1}{z-1-\chi_k}-1/2+O(z-1-\chi_k) \qquad \text{for} \quad z \to 1+\chi_k$$

so we are able to determine the expansion of (4-8). Paying attention to (A-4.2) when investigating the principle part of the expansion shows (after some tedious computations) that it vanishes.

$$G(z)=\frac{1-p/Q}{\log Q}(F(1, 1)+1) \\ -(\gamma+1-(\gamma+2)p/Q)\left(F_2(1, 1)+\frac{1}{2}(F(1, 1)+1)\right) \\ -(1-p/Q)\left(F_{12}(1, 1)+\frac{1}{2}F_1(1, 1)\right)+O(z-1) \qquad \text{for} \quad z \to 1.$$

We should note the abbreviation $\beta(p)=\beta_1(p)$ to get the connection to the constants given in the outline. With the identities of (A-4) and resubstituting

γ this simplifies to

$$G(1)=(F(1,\ 1)+1)\left(\frac{1-p/Q}{\log Q}+(1-p/Q)(\gamma+1)-p/Q)\right.$$
$$\left.-(\gamma+1)+\frac{p/Q}{1-p/Q}+\beta(p)(1-p/Q)\right.$$
$$=\frac{Q(p)}{Q(1)}\frac{1}{\log Q}+\left(1-\frac{Q(p)}{Q(1)}\right)\left(\alpha(1)-p\,\alpha(p)+\frac{p/Q}{1-p/Q}\right)$$
$$+\beta(p)(1-p/Q).$$

Now we treat the pointz $z=1+\chi_k$ with $k\neq 0$ by multiplying the local expansions in the same manner as above. Here we may not expect the cancellation of the whole principal part, because $\Phi_n(z)$ does not have any poles at these points. In fact, only the real part vanishes similar to the former derivation.

$$G(z)=(1-p/Q)\,\chi_k\frac{F(1,\ 1)+1}{\log Q}\ \frac{1}{z-1-\chi_k}+O(1)\qquad \text{for}\quad z\to 1+\chi_k.$$

Using the identity of (A-4.1) we finally obtain for $k\neq 0$

$$\operatorname*{Res}_{z=1+\chi_k} G(z)=\frac{\chi_k}{\log Q}\ \frac{Q(p)}{Q(1)}.$$

The application of (A-7) makes it necessary to find a sequence of rectangular contours, which will be used for expanding the skinny one of the integral. We may select such a sequence with the property that $G(z)=O(z)$ for all z lying on such a contour as following.

$$\gamma_1^k:\Re(z)=\rho>0\quad \text{with } \rho \text{ arbitrary small but fixed}$$
$$\gamma_2^k:\Im(z)=-\frac{(2k+1)\pi}{\log Q}$$
$$\gamma_3^k:\Re(z)=k$$
$$\gamma_4^k:\Im(z)=\frac{(2k+1)\pi}{\log Q}.$$

For each z lying on such a contour the stated property results from easy established facts, *see* (4-8). First, $F(1,\ Q^{1-z})$ and all derivatives w.r.t. the first argument are meromorphic functions and no poles lie on the contours. Moreover, the second argument $Q^{1-z}/(1-Q^{1-z})=1/(Q^{z-1}-1)$ is limited even on γ_2^k and γ_4^k because of choosing the imaginary part as stated above. The order of the residual term would be $Q(n^p)$, but the fact that the next poles of the integrand lie at the vertical line $\Re(z)=0$ allows us to state $Q(1)$ instead.

The application of (A-7) yields the following expansion for the alternating sum involving the g_n.

$$-nG(1)=n\frac{Q(p)}{Q(1)}P(\log_Q n)+O(1)$$

with the function

$$P(u)=\frac{1}{\log Q}\sum_{k\neq 0}\chi_k\Gamma(-1-\chi_k)e^{2k\pi iu}.$$

Collecting all terms according to (4-7) gives

$$L_n=nA\left(\frac{Q(p)}{Q(1)}\frac{1}{\log Q}+(1-p/Q)\beta(p)\right.$$
$$\left.-\frac{Q(p)}{Q(1)}\left(\alpha(1)-p\alpha(p)+\frac{p/Q}{1-p/Q}\right)+\frac{p/Q}{1-p/Q}\right)$$
$$+nh_1+nA\frac{Q(p)}{Q(1)}P(\log_Q n)+O(1).$$

We simplify the expression by using the notation of (A-2), e. g.,

$$q_0=\frac{Q(p)}{Q(1)}\qquad\text{and}\qquad q_1=\frac{Q(p)}{Q(1)}(\alpha(1)-p\alpha(p))$$

which leads to

$$L_n=nA\left(\frac{q_0}{\log Q}+(1-p/Q)\beta_1(p)-q_1+(1-q_0)\frac{p/Q}{1-p/Q}\right)$$
$$+nh_1+nAq_0P(\log_Q n)+O(1).$$

The result as stated in section 3, theorem (1) is obtained by substituting $A=Q$, $h_0=1$ and $h_1=0$, as mentioned in the beginning of this section. ■

The technique used for the determination of the first term in the asymptotic expansion may be extended to obtain higher accuracy. This could be done by extending the integration path, e. g., shifting γ_1^k more to the left and taking into account the residues of newly encountered poles. For example, the computations for the linear term are more tedious, but manageable, for the sake of shortness we decided to suppress them in this paper.

5. PROOF (2)

Denoting by E_n the average number of empty nodes in a p-triee with n C-nodes, we have the following basic recurrence

$$E_n = p \sum_{\Sigma i_t = n} \binom{n}{i_1, \ldots, i_Q} Q^{-n} \sum_{l=1}^{Q} E_{i_l} + (1-p) \sum_{\Sigma j_t = n-1} \binom{n-1}{j_1, \ldots, j_Q} Q^{1-n} \sum_{l=1}^{Q} E_{j_l}$$

for $n \geqq 2$. The initial values again resulting from the physical model are

$$E_0 = 1 \qquad \text{and} \qquad E_1 = 0.$$

This come from similar reasoning as in the previous section with mentioning that the number of empty nodes in the triee is the sum of the empty nodes in the Q subtriees. Fortunately, we need not be concerned with treating this subject for its own, we may adapt the results of section 4 instead. Denoting by $E(z)$ the EGF of E_n and introducing the PoGF

$$W(z) = E(z)\, e^{-z} = \sum_{n \geqq 0} w_n \frac{z^n}{n!}$$

we obtain a functional equation

$$W'(z) + W(z) = Q\, W(z/Q) + p\, W'(z/Q) - (Q-p)\, e^{-z}$$

$$w_0 = 1 \qquad \text{and} \qquad w_1 = -1.$$

For using the previous results, we have to modify the functional equation making it looking like the universal one treated there. Let

$$H(z) = W(z) - \frac{1}{Q-1}$$

and therefore

$$h_0 = 1 - \frac{1}{Q-1} \qquad \text{and} \qquad h_1 = -1$$

we obtain the functional equation

$$H'(z)+H(z)=1+QH(z/Q)+p\,H'(z/Q)-(Q-p)\,e^{-z}$$

as desired for the application of the general solution from chapter 4. The connection between the L_n and E_n is given by

$$E(z)=L(z)+\frac{e^z}{Q-1} \quad \text{e. g.,}$$

$$E_n=L_n+\frac{1}{Q-1}.$$

Therefore, we must substitute $A=Q-p$, $h_0=1-1/(Q-1)$ and $h_1=-1$ and add $1/(Q-1)$ to the final expression in chapter 3 in order to obtain Theorem (2). ■

6. PROOF (3)

Denoting by $I_n=I_n(V)$ the average number of nodes with label $V\geqq 2$ in a p-triee with n C-nodes, we obtain the following basic recurrence

$$I_n=\delta_{n,V}+p\sum_{\Sigma i_t=n}\binom{n}{i_1,\ldots,i_Q}Q^{-n}\sum_{l=1}^{Q}I_{i_l}$$
$$+(1-p)\sum_{\Sigma j_t=n-1}\binom{n-1}{j_1,\ldots,j_Q}Q^{1-n}\sum_{l=1}^{Q}I_{j_l}$$

for $n\geqq V$. The initial values resulting from the physical model are

$$I_i=0 \qquad \text{for} \quad 0\leqq i\leqq V-1.$$

The reasons are similar to those in the sections before and not further mentioned. However, following the general strategy of section 4 we obtain the functional equation for the PoGF $H(z)=I(z)\,e^{-z}$ with $I(z)$ the EGF of the sequence I_n, e. g.,

$$H'(z)+H(z)=QH(z/Q)+p\,H'(z/Q)+\frac{z^{V-1}}{(V-1)!}e^{-z}$$

$$h_i=0 \qquad \text{for} \quad 0\leqq i\leqq V-1.$$

Extracting the coefficients of $z^n/n!$ for $n \geqq V-1$ yields the simple recurrence

$$h_{n+1} = -h_n \frac{1-Q^{1-n}}{1-pQ^{-n}} + (-1)^{n+1} \binom{n}{V-1} \frac{(-1)^V}{1-pQ^n}. \qquad (6\text{-}1)$$

Introducing again $P_n = \prod_{k \geqq n} (1-Q^{1-k})/(1-pQ^{-k})$ with $P_1 = 0$ and $a_n = h_n P_n$, we obtain a linear reccurence for the a_n by multiplying both sides of the former recurrence with P_{n+1}. The iterated solution is

$$a_{n+1} = (-1)^{n+1} \sum_{k=V}^{n+1} \frac{P_k}{1-pQ^{1-k}} \binom{k-1}{V-1} (-1)^V.$$

After some algebraic manipulations we get the following explicit expression for the coefficients h_n with $n \geqq V$.

$$h_n = (-1)^{n-V} \frac{1}{1-pQ^{1-n}} \sum_{k=V-1}^{n-1} \prod_{j=k}^{n-2} \frac{1-Q^{-j}}{1-pQ^{-j}} \binom{k}{V-1}.$$

Let in analogy to Section 4 for $n \geqq V$

$$\begin{aligned} F(n) = f_n &= \frac{1}{1-pQ^{1-n}} \sum_{k=V-1}^{n-1} \prod_{j=k}^{n-2} \frac{1-Q^{-j}}{1-pQ^{-j}} \binom{k}{V-1}' \\ &= (-1)^{n-V} h_n. \end{aligned} \qquad (6\text{-}2)$$

for $n \geqq V-2$ we introduce

$$\begin{aligned} P_n(p) &= T_n(p) \left(\frac{(V-1)\dots(1)}{T_{V-2}(p)} + \frac{(V)\dots(2)}{T_{V-1}(p)} + \dots + \frac{(n+1)\dots(n-V+3)}{T_n(p)} \right) \\ &= \sum_{k=V-1}^{n+1} \prod_{j=k}^{n} \frac{1-Q^{-j}}{1-pQ^{-j}} k(k-1)\dots(k-V+2) \end{aligned}$$

which enables us to express f_n for $n \geqq V$ in the following simple manner

$$f_n = \frac{P_{n-2}(p)}{(1-pQ^{1-n})(V-1)!}. \qquad (6\text{-}3)$$

The products $T_n(p)$ are defined in section 4, but we should note that $P_n(p)$ must not be confused with the similar products P_n which we used for determining a closed expression for the h_n, and with the products $P_n(p)$ of the section 4, too.

Now we have a look at the ordinary GF of the sequence involved and obtain

$$W_p(z)=\sum_{n\geqq V-2}\frac{(n+1)\dots(n-V+2)}{T_n(p)}z^n$$
$$=z^{V-2}\mathscr{D}^{V-1}zS_p(z)\quad\text{with } S_p(z)\text{ from (A-1).}\quad(6\text{-}4).$$

From (A-1) it is clear that $W_p(z)$ has a pole of order V at $z=1$, so

$$W_p(z)=\frac{w_V}{(z-1)^V}+\dots+\frac{w_1}{z-1}+w_0+w(z).$$

The function $w(z)$ has radius of convergency Q around $z=0$ and $w(0)=0$. To evaluate the unknown coefficients we expand $zS_p(z)$ with respect to (A-2), which yields

$$zS_p(z)=\frac{Q(p)}{Q(1)}\frac{1}{1-z}-\sum_{n\geqq 0}(q_n+q_{n+1})(z-1)^n.$$

Using the operator approach (6-4) we find after a short algebraic manipulation

$$w_j=(-1)^V\frac{Q(p)}{Q(1)}(V-1)!\binom{V-2}{V-j}\qquad\text{for}\quad 2\leqq j\leqq V$$
$$w_1=0$$
$$w_0=-(V-1)!(q_{V-1}+q_V).$$

The GF of the partial sums, denoted by $R_p(z)$, is $W_p(z)/(1-z)$ and choosing an arbitrary but fixed $\varepsilon>0$ and $q=Q-\varepsilon$ we obtain the asymptotic expansion of the coefficients

$$\sum_{k=V-2}^{n}\frac{(k+1)\dots(k-V+3)}{T_k(p)}=(-1)^V\binom{n+V}{V}w_V$$
$$+(-1)^{V-1}\binom{n+V-1}{V-1}w_{V-1}+\dots+w_0+O(q^{-n}).$$

Multiplying all expansions concerned with f_n according to (6-3) with mentioning (4-5) yields

$$f_n = \frac{Q(1)}{Q(p)(V-1)!} \sum_{j=0}^{V} \binom{n+j-2}{j} (-1)^j w_j + O(q^{-n}).$$

Substituting the evaluated w_j and mentioning the combinatorial identity from [8], p. 11

$$\sum_k (-1)^{k+m} \binom{m}{k} \binom{n+p+k}{p+k} = \binom{n+p}{m+p}$$

finally yields the asymptotic expansion

$$f_n = \binom{n}{V} - T_{V-1} + O(q^{-n}) \qquad \text{with} \quad T_V = \frac{Q(1)}{Q(p)} (q_V + q_{V+1}).$$

We define a new sequence g_n by

$$g_n = f_n - \left(\binom{n}{V} - T_{V-1} \right).$$

Before treating this subject, we deal with manipulating the expression for the desired quantity I_n. In analogy to (4-3) we obtain after some straightforward computations and mentioning the identity $\binom{n}{k}\binom{k}{V} = \binom{n}{V}\binom{n-V}{k-V}$ an expression

$$I_n = \sum_{k=V}^{n} \binom{n}{k} (-1)^{k-V} f_k$$
$$= T_{V-1} \sum_{k=0}^{V-1} \binom{n}{k} (-1)^{k-V} + \sum_{k=V}^{n} \binom{n}{k} (-1)^{k-V} g_k. \tag{6-5}$$

The remaining task is the computation of the alternating sum involving the g_n with Rice's method. It is clear that $g(n) = O(q^{-n})$, and remembering (6-1) and (6-2) we obtain a recurrence for the g_n.

$$g_n = g_{n+1} \frac{1 - qQ^{-n}}{1 - Q^{1-n}} + \frac{\left(\binom{n}{V} - T_{V-1} - p/Q \left(\binom{n+1}{V} - T_{V-1} \right) \right) Q^{1-n}}{1 - Q^{1-n}}.$$

The condition $\lim_{n \to \infty} g_n = 0$ again inhibits the existence of more than one solution, as can be shown with an indirect proof using iteration. We treat the corresponding functional equation

$$G(z) = a(z)\, G(z+1) + b(z)$$

with

$$a(z) = \frac{1 - p\,Q^{-z}}{1 - Q^{1-z}}$$

$$b(z) = \frac{\left(\binom{z}{V} - T_{V-1} - p/Q\left(\binom{z+1}{V} - T_{V-1}\right)\right) Q^{1-z}}{1 - Q^{1-z}}$$

by iterating it, and mentioning the assumption that $G(z)$ vanishes if $\Re(z) \to \infty$ in a certain region of the complex plane yields a solution

$$G(z) = \frac{Q^{1-z}}{1 - Q^{1-z}} \sum_{k \geq 0} \prod_{j=1}^{k} \frac{1 - p\,Q^{-z-j+1}}{1 - Q^{-z-j+1}}$$
$$\times \left(\binom{z+k}{V} - T_{V-1} - p/Q\left(\binom{z+k+1}{V} - T_{V-1}\right)\right) Q^{-k}$$

meromorphic in the complex plane with poles at most at

$$z = 1 - j + \chi_k \qquad \text{with} \quad j \geq 0 \quad \text{and} \quad \chi_k = \frac{2k\pi i}{\log Q} \text{ for all integers } k.$$

For we are interested only in asymptotic terms of higher order than $O(1)$, it is necessary to obtain the residues of

$$\Phi_n(z)\, G(z) = -\frac{\Gamma(n+1)\,\Gamma(-z)}{\Gamma(n+1-z)}\, G(z)$$

only in the strip $0 < \Re(z) < V$, *see* (A-7). We start our treatment by investigating the function-values $G(x)$ for integer $2 \leq x \leq V-1$. At these points we obtain a simple pole from $\Phi_n(z)$, and looking at (6-5) with mentioning the fact that simple computations show $I_n = O(n)$, we could argue that these contributions yield to the cancellation of the higher-order terms.

For $m=0,1$ we define

$$J^{(m)}(z)=\sum_{k\geq 0}\prod_{j=1}^{k}\frac{1-pQ^{-z-j+1}}{1-Q^{-z-j+1}}\binom{z+k+m}{V}Q^{-k}$$

$$=\mathcal{U}_u\frac{\mathcal{D}_u^V}{V!}u^{z+m}(F(u,\,Q^{1-z})+1)$$

$$=[(u-1)^V]\,u^{z+m}(F(u,\,Q^{1-z})+1) \quad (6\text{-}6)$$

with the function $F(u,\,v)$ defined in section 4. $G(z)$ may be rewritten in terms of these functions, which yields to

$$G(z)=\frac{Q^{1-z}}{1-Q^{1-z}}(J^{(0)}(z)-p/QJ^{(1)}(z)-(1-p/Q)\,T_{V-1}(F(1,\,Q^{1-z})+1)). \quad (6\text{-}7)$$

With $I(u,\,z)=F(u,\,Q^{-z})$ we obtain for integer $2\leq x\leq V-1$

$$J^{(m)}(x)=[(u-1)^V]\,u^{x+m}(I(u,\,x-1)+1)$$

and obviously

$$F(1,\,Q^{1-x})+1=I(1,\,x-1)+1.$$

First, we evaluate an explicit expression for $I(u,\,x)$, which leads after some tedious algebraic manipulations to

$$I(u,\,x)=\frac{Q(pu)}{Q(u)}\frac{u^{-x}}{1-pu/Q}Q^x\prod_{j=1}^{x}\frac{1-Q^{-j}}{1-pQ^{-j}}-\sum_{k=0}^{x}\prod_{j=x-k+1}^{x}\frac{1-Q^{-j}}{1-pQ^{-j}}Q^k u^{-k}$$

and therefore

$$u^{x+m}(I(u,\,x-1)+1)=\frac{Q(pu)}{Q(u)}\frac{u^{m+1}}{1-pu/Q}Q^{x-1}\prod_{j=1}^{x-1}\frac{1-Q^{-j}}{1-pQ^{-j}}$$

$$-\sum_{k=1}^{x-1}\prod_{j=x-k}^{x-1}\frac{1-Q^{-j}}{1-pQ^{-j}}Q^k u^{x+m-k}.$$

For an integer $t\geq 0$ we easily obtain the Taylor-expansion at $u=1$ of

$$\frac{u^t}{1-pu/Q}=\sum_{n\geq 0}w_n^{(t)}(u-1)^n$$

with

$$w_n^{(t)} = \frac{(p/Q)^n}{(1-p/Q)^{n+1}} \sum_{k=0}^{n} \binom{t}{k} \left(\frac{1-p/Q}{p/Q}\right)^k$$

$$w_n^{(t)} = \frac{(p/Q)^{n-t}}{(1-p/Q)^{n+1}} \qquad \text{for} \quad n \geqq t.$$

Moreover, for special values of t we find the property

$$w_n^{(1)} - p/Q\, w_n^{(2)} = \begin{cases} 1 & \text{for} \quad n=0 \\ 1 & \text{for} \quad n=1 \\ 0 & \text{for} \quad n \geqq 2. \end{cases} \tag{6-8}$$

Now we are ready with collecting the expansions at $u=1$ of all functions concerned with our $J^{(m)}(x)$, remembering Lemma (A-2), too. Extracting the V-th Taylor-coefficient yields after some straightforward but nasty computations the desired expressions

$$J^{(m)}(x) = Q^{x-1} \prod_{j=1}^{x-1} \frac{1-Q^{-j}}{1-p\,Q^{-j}} \sum_{k=0}^{V} q_k\, w_{V-k}^{(m+1)}.$$

Putting all things together we obtain an expression for $G(x)$ with $2 \leqq x \leqq V-1$ according to (6-7)

$$G(x) = T_{V-1}(1-p/Q)\, b_{x-1}$$

with

$$b_x = \frac{Q^{-x}}{1-Q^{-x}} \sum_{k=1}^{x} \prod_{j=z+1-k}^{x} \frac{1-Q^{-j}}{1-p\,Q^{-j}} Q^k.$$

LEMMA (Partition-Identity): *For all integers $x \geqq 1$ we have*

$$b_x = \frac{1}{1-p/Q} \quad \textit{independent of } x. \tag{6-9}$$

Proof: Mentioning the easy to find recurrence relation

$$b_x = \frac{1-Q^{1-x}}{1-p\,Q^{-x}} b_{x-1} + \frac{Q^{1-x}}{1-p\,Q^{-x}}$$

for $x \geqq 2$ the proof by induction is trivial. ■

Thus we finally obtain

$$G(x) = T_{V-1} \qquad \text{for} \quad 2 \leqq x \leqq V-1.$$

The previous computations could be done in principle even in the case $x=1$, but unfortunately this leads to an expression $0 . \infty$ and the only statement which we could derive is that it is of order $O(1)$, e. g., that $G(1)$ exists. Thus we have to take into account higher order terms of the functions involved in $G(z)$. We start this task by investigating the functions from (6-6) and using the Taylor-expansion of $F(u, v)$ from (4-9) to obtain the first few terms of

$$\begin{aligned} J^{(m)}(z) = [(u-1)^V]\, u^{z+m} (F(u, Q^{1-z})+1) &\\ = [(u-1)^V]((F(u, 1)+1)\, u^{1+m}) &\\ + [(u-1)^V](u^{1+m} \log u\, (F(u, 1)+1) &\\ - \log Q\, u^{1+m} F_2(u, 1))(z-1) + O((z-1)^2). \end{aligned}$$

Mentioning the cancellation of the principal part of the whole function $G(z)$ for $z \to 1$ we need not be concerned with the constant term of the functions $J^{(m)}(z)$. This fact becomes clearer when we recall (6-7). The potential pole comes from the function $Q^{1-z}/(1-Q^{1-z})$ and is cancelled by a zero of the bracketed expression. Therefore, the constant term of $G(z)$ only results from the coefficient of $(z-1)$ in the bracketed expression, e. g., from the functions actually investigated, times the residue of

$$\frac{Q^{1-z}}{1-Q^{1-z}} = \frac{1}{\log Q} \frac{1}{z-1-\chi_k} - 1/2 + O(z-1-\chi_k) \qquad \text{for} \quad z \to 1+\chi_k. \quad (6\text{-}11)$$

Now we start with the first term in (6-10) and use (A-4.1) to obtain

$$u^{1+m} \log u\, (F(u, 1)+1) = \log u \frac{u^{1+m}}{1-pu/Q} \frac{Q(pu)}{Q(u)}.$$

Using the well known Taylor-expansion $\log(1+z) = \sum_{n \geqq 1} (-1)^{n+1} z^n/n$ and the result from (A-2) we are able to derive the Taylor-series at $u=1$ of all functions concerned. After some computations we find the V-th Taylor-coefficient

$$[(u-1)^V]\, u^{1+m} \log u\, (F(u, 1)+1) = \sum_{n=0}^{V} \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k} w_{n-k}^{(1+m)} q_{V-n}.$$

Collecting the terms for $m=0$ and $m=1$ according to (6-7) yields after some computations and mentioning the property of $w_n^{(1)}-p/Q\,w_n^{(2)}$ from (6-8) and the expansion (6-11) the contribution

$$\frac{1}{\log Q}\left(\sum_{n=2}^{V}(-1)^n\frac{q_{V-n}}{n(n-1)}+q_{V-1}\right).$$

The second term in (6-10) is treatable by using the expansion of $F_2(u, 1)$ from (A-6) and the trivial one of u^{1+m} at the point $u=1$. The requested V-th Taylor-coefficient evaluates to

$$[(u-1)^V]\,u^{1+m}F_2(u, v)=\sum_{k=0}^{V}\binom{1+m}{k}s_{V-k}.$$

The upper limit of the sum could be replaced by $1+m$, because $1+m\leqq V$. Collecting this for $m=0$ and $m=1$ according to (6-7) yields after using (A-6) for replacing s_n by r_n and applying the recurrence from (A-5) the following contribution

$$-(\alpha(1)-p\,\alpha(p))(q_V+q_{V+1})+K(V, p)$$

with

$$K(V, p)=(1-p/Q)\,\beta_V(p)+(1-2p/Q)\,\beta_{V-1}(p)-p/Q\,\beta_{V-2}(p).$$

The derivation of the stated expression is straightforward, but a little tedious. The residue $\log Q$ of (6-11) cancels with the factor in the second investigated term of the Taylor-expansion of $J^{(m)}(z)$ in (6-10), so it disappears.

After treating the contributions of $J^{(m)}(z)$ we have to deal with the last term in the large bracket of (6-7), which is simply evaluated by mentioning the Taylor-expansion of $F(1, Q^{1-z})$. The whole contribution is

$$(1-p/Q)\,T_{V-1}F_2(1, 1)\qquad\text{with}\quad T_V=\frac{Q(1)}{Q(p)}(q_V+q_{V+1}).$$

Note again the cancellation of the residue $\log Q$. The Appendix provides an alternative expression for $F_2(1, 1)$, *see* (A-4-3).

Collecting all contributions yield after some algebraic manipulations the desired value

$$G(1)=\frac{1}{\log Q}\sum_{n=2}^{V}(-1)^n\frac{q_{V-n}}{n(n-1)}+\frac{q_{V-1}}{\log Q}+(q_V+q_{V-1})\left(\frac{Q(p)}{Q(1)}-1\right)+K(V, p).$$

The remaining problem is the determination of the residues of $G(z)$ at the points $z=1+\chi_k$ with $k \neq 0$. Remembering (6-7) we see that the poles come from (6-11), thus we need the values of all functions concerned at the points $z=1+\chi_k$. Remembering (A-4.1) and the definition of T_V there is no problem when treating the last part of the bracketed expression in (6-7). Moreover, for $m=0, 1$ we evaluate the remaining terms

$$J^{(m)}(1+\chi_k)=[(u-1)^V]((F(u, 1)+1)\, u^{1+m+\chi_k}).$$

Following the derivation of the previouse value $G(1)$, we compute the Taylor-expansion of the functions concerned at the point $u=1$ and extract the V-th coefficient of the whole product. Using (A-4.1) we obtain

$$(F(u, 1)+1)\, u^{1+m+\chi_k}=\frac{u^{1+m+\chi_k}}{1-pu/Q}\,\frac{Q(pu)}{Q(u)}.$$

Similar to (6-8) we find the Taylor-expansion

$$\frac{u^{1+m+\chi_k}}{1-pu/Q}=\sum_{n \geqq 0} y_n^{(m)}(u-1)^n$$

with

$$y_n^{(m)}=\sum_{j=0}^{n}\binom{1+m+\chi_k}{j}\frac{(p/Q)^{n-j}}{(1-p/Q)^{n-j+1}}$$

and the property

$$y_n^{(0)}-p/Q\, y_n^{(1)}=\binom{1+\chi_k}{n}$$

as can be seen by a straightforward computation. Therefore, we can state

$$J^{(m)}(1+\chi_k)=\sum_{n=0}^{V} q_n\, y_{V-n}^{(m)}$$

and collecting all contributions according to (6-7) we finally obtain the desired residue

$$\operatorname*{Res}_{z=1+\chi_k} G(z)=\frac{1}{\log Q}\sum_{n=0}^{V-1}\binom{1+\chi_k}{V-n} q_n-\frac{q_{V-1}}{\log Q}.$$

Now we have computed all values of $G(z)$ in order to apply Rice's method (A-7). The behaviour of $G(z)$ along the rectangular contours defined in section 4 is proved to be $O(z^V)$ with similar reasoning, so no problems occur on the application. Using (A-7) together with (6-5) and

$$T_V = \frac{Q(1)}{Q(p)}(q_V + q_{V+1})$$

yields after some cosmetic manipulations the statement of Theorem (3) in section 3.

A. APPENDIX

This is the place to establish some Lemmas refered to in the sections before. (A-1) and (A-2) are related with the products $Q_n(p)$ and $T_n(p)$, the following theorems (A-3) to (A-6) deal with properties of the function $F(u, v)$, all defined in section 4. Finally, theorem (A-7) is a simple version of the formula of Rice, tuned to the application on our problems.

(A-1) LEMMA (Generating Function $S_p(z)$): *The ordinary generating function of the sequence* $1/T_n(p)$ *is given by*

$$S_p(z) = \frac{1}{1-z}\,\frac{Q(pz)}{Q(z)}.$$

Proof : We state the following remarkable identity from the theory of partitions, *see* [6] for a proof and further details. For $|q|<1$ and $|z|<1$ we have

$$\prod_{n \geq 0} \frac{1-azq^n}{1-zq^n} = 1 + \sum_{n \geq 1} \prod_{j=1}^{n} \frac{1-aq^{j-1}}{1-q^i} z^n.$$

Recalling the definition of $T_n(p)$ from section 4, the substitution $a=p/Q$ and $q=1/Q$ yields to

$$S_p(z) = \sum_{n \geq 0} \prod_{j=1}^{n} \frac{1-pQ^{-j}}{1-Q^{-j}} z^n = \prod_{n \geq 0} \frac{1-pzQ^{-n-1}}{1-zQ^{-n}}.$$

Remembering the definition $Q(z)=\prod_{n\geqq 1}(1-zQ^{-n})$ yields the desired result. ■

(A-2) LEMMA (Taylor-Expansion $Q(pz)/Q(z)$): *The function*

$$\frac{Q(pz)}{Q(z)}=\prod_{j\geqq 1}\frac{1-pzQ^{-j}}{1-zQ^{-j}}$$

is analytic near the point $z=0$ with radius of convergency Q. Moreover, the Taylor-expansion at the point $z=1$ is

$$\frac{Q(pz)}{Q(z)}=\sum_{n\geqq 0}q_n(z-1)^n$$

and

$$q_0=\frac{Q(p)}{Q(1)}$$

$$q_n=\frac{1}{n}\sum_{j=0}^{n-1}q_j(\alpha(1, n-j)-p^{n-j}\alpha(p, n-j)) \qquad \textit{for} \quad n\geqq 1$$

where

$$\alpha(p, l)=\sum_{k\geqq 1}\frac{Q^{-kl}}{(1-pQ^{-k})^l} \quad \textit{with the abbreviation } \alpha(p)=\alpha(p, 1).$$

Proof: The maintained analyticity is clear by mentioning the fact, that the first pole lies at the point $z=Q$. Computation of q_0 is straightforward, the recurrence for the q_n with $n\geqq 1$ comes from using the so-called logarithmic derivation $\mathscr{D}\log f(z)=\mathscr{D}f(z)/f(z)$ for computing the derivative, e. g.,

$$\mathscr{D}\frac{Q(pz)}{Q(z)}=\frac{Q(pz)}{Q(z)}\sum_{k\geqq 1}\left(\frac{Q^{-k}}{1-zQ^{-k}}-\frac{pQ^{-k}}{1-pzQ^{-k}}\right)$$
$$=\frac{Q(pz)}{Q(z)}(\alpha(z, 1)-p\,\alpha(pz, 1)).$$

For $n\geqq 1$ we have

$$q_n=\frac{1}{n!}\mathscr{U}\mathscr{D}^n\frac{Q(pz)}{Q(z)}=\frac{1}{n!}\mathscr{U}\mathscr{D}^{n-1}\mathscr{D}\frac{Q(pz)}{Q(z)}$$

and using the Leibnitz-formula for derivations of higher order together with

$$\mathscr{D}^k \alpha(pz, 1) = k!\, p^k \alpha(pz, k+1)$$

leads to the desired result. ■

(A-3) THEOREM (Functional Equation $F(u, v)$): *The function*

$$F(u, v) = \sum_{k \geq 1} \prod_{j=1}^{k} \frac{1 - pv\,Q^{-j}}{1 - v\,Q^{-j}}\, Q^{-k} u^k$$

solves the functional equation

$$F(u, v) = \frac{u(1 - pv/Q)}{v(1 - pu/Q)} F(v, u).$$

Proof : Multiplying $F(u/Q, v)$ with v and subtracting it from $F(u, v)$ yields after some algebraic manipulations

$$F(u, v) - v\,F(u/Q, v) = u/Q\,(F(u, v) + 1 - pv/Q\,(F(u/Q, v) + 1)).$$

Iterating this functional equation with mentioning the fact

$$\lim_{\Re(u) \to +\infty} F(u, v) = 0$$

within a certain region of the complex plane yields

$$\begin{aligned} F(u, v) &= (1 - pv/Q)\frac{u/Q}{1 - u/Q} + v\frac{1 - pu\,Q^{-2}}{1 - u/Q} F(u/Q, v) \\ &= (1 - pv/Q)\frac{u/Q}{1 - u/Q} \\ &\qquad + (1 - pv/Q)\,u \sum_{k \geq 2} \frac{(1 - pu\,Q^{-2}) \ldots (1 - pu\,Q^{-k})}{(1 - u\,Q^{-1}) \ldots (1 - u\,Q^{-k})} v^{k-1} Q^{-k}. \end{aligned}$$

After shifting the range of summation of $k \geq 1$ and some algebraic manipulations we obtain the desired result. ■

(A-4) THEOREM (Values $F(u, v)$): *The function $F(u, v)$ has the following special values.*

$$F(u, 1) = \frac{Q(pu)}{Q(u)} \frac{1}{1 - pu/Q} - 1 \tag{1}$$

$$F_1(u, 1)=\mathcal{D}_u F(u, 1)=(F(u, 1)+1)\left(\alpha(u)-p\,\alpha(pu)+\frac{p/Q}{1-pu/Q}\right) \tag{2}$$

$$F_2(1, 1)=\mathcal{U}_v \mathcal{D}_v F(1, v)=\frac{1}{1-p/Q}+(F(1, 1)+1)(\alpha(1)-p\,\alpha(p)-1) \tag{3}$$

$$F_{12}(u, v)=\mathcal{D}_v \mathcal{D}_u F(u, v)=(\alpha(v)-p\,\alpha(pv))\,F_1(u, v)-\beta(u, v) \tag{4}$$

with

$$\beta(u, v)=\sum_{k\geq 1}\left(\frac{Q^{-k}}{1-v\,Q^{-k}}-\frac{p\,Q^{-k}}{1-pv\,Q^{-k}}\right)\sum_{l=1}^{k-1}\prod_{j=1}^{l}\frac{1-pv\,Q^{-j}}{1-v\,Q^{-j}}\,lu^{l-1}\,Q^{-l}$$

and the abbreviation $\beta(p)=\beta_1(p)=\beta(1, 1)$.

Proof: (1) This follows directly from the application of the identity stated in the proof of (A-1) with $z=u/Q$, $a=p/Q$ and $q=1/Q$. ■

(2) We apply the logarithmic derivation on (1) and mention the definition of the $\alpha(p)$ from (A-2), which yields to the desired result. ■

(3) Differentiating (A-3) with respect to u leads to

$$F_2(v, u)=\frac{v(1-pu/Q)}{u(1-pv/Q)}F_1(u, v)-\frac{1}{u(1-pu/Q)}F(v, u).$$

Substituting $u=v=1$ and using the result from (2) completes the proof. ■

(4) The function $F(u, v)$ is represented by a uniform convergent series, so we may exchange the order of differentiation and summation, which yields to

$$\begin{aligned}F_{12}(u, v)&=\mathcal{D}_v \mathcal{D}_u F(u, v)\\&=\sum_{k\geq 1}\prod_{j=1}^{k}\frac{1-pv\,Q^{-j}}{1-v\,Q^{-j}}\,Q^{-k}\,ku^{k-1}\,(\alpha_k(v)-p\,\alpha_k(pv)).\end{aligned}$$

Differentiating with respect to v is done in the same manner as in the proof of (A-2), with $\alpha_n(p)=\alpha_n(p, 1)$ and

$$\alpha_n(p, l)=\sum_{k=1}^{n}\frac{Q^{-kl}}{(1-p\,Q^{-k})^l}.$$

We should note that $\alpha(p, l)=\alpha_{\infty}(p, l)$, of course. Let t be an arbitrary parameter, we use Abels transformation, e. g.,

$$\sum_{k=c}^{n} a_k b_k = a_n \sum_{k=c}^{n} b_k - \sum_{k=c}^{n-1} (a_{k+1}-a_k) \sum_{j=c}^{k} b_j$$

for investigating the following expression.

$$\sum_{k\geqq 1} \prod_{j=1}^{k} \frac{1-pvQ^{-j}}{1-vQ^{-j}} Q^{-k} k u^{k-1} \alpha_k(t) = \alpha(t) F_1(u, v)$$

$$- \sum_{k\geqq 1} \frac{Q^{-k}}{1-tQ^{-k}} \sum_{l=1}^{k-1} \prod_{j=1}^{l} \frac{1-pwQ^{-j}}{1-vQ^{-j}} Q^{-l} l u^{l-1}.$$

Using this in the former equation and mentioning the abbreviation $\beta(u, v)$ yields to the desired result. ■

(A-5) Lemma (Taylor Expansion $F(u, 1)$): *The expansion of the function $F(u, 1)$ at the point $u=1$ is*

$$F(u, 1) = \sum_{n\geqq 0} r_n (u-1)^n \qquad and \qquad r_n = \sum_{k=0}^{n} q_k \frac{(p/Q)^{n-k}}{(1-p/Q)^{n+1-k}} - \delta_{n,0}$$

with q_n from (A-2) *and $\delta_{n,k}$ denoting the Kronecker-symbol. Moreover, the coefficients solve the following recurrence relation*

$$r_n = r_{n-1} \frac{p/Q}{1-p/Q} + \frac{q_n}{1-p/Q} + \delta_{n-1,0} \frac{p/Q}{1-p/Q} \qquad for \quad n \geqq 1$$

$$r_0 = \frac{q_0}{1-p/Q} - 1.$$

Proof: The Taylor-expansion easily follows from extracting the coefficients of the Cauchy-product from (A-4.1)

$$F(u, 1) = \frac{Q(pu)}{Q(u)} \frac{1}{1-pu/Q} - 1.$$

The recurrence relation comes from a trivial direct manipulation of the explicit expression, so the proof is completed. ■

(A-6) LEMMA (Taylor Expansion $F_2(u, 1)$): *The expansion of the function* $F_2(u, 1)$ *at the point* $u=1$ *is*

$$F_2(u, 1) = \sum_{n \geq 0} s_n (u-1)^n$$

and

$$s_n = r_n(\alpha(1) - p\,\alpha(p)) - \beta_n(p) + \delta_{n,0}(\alpha(1) - p\,\alpha(p))$$

with r_n *from* (A-5), $\delta_{n,k}$ *denoting the Kronecker-symbol and*

$$\beta_n(p) = \sum_{k \geq n} \left(\frac{Q^{-k}}{1-Q^{-k}} - \frac{p\,Q^{-k}}{1-p\,Q^{-k}} \right) \sum_{l=n}^{k-1} \prod_{j=1}^{l} \frac{1-p\,Q^{-j}}{1-Q^{-j}} \binom{l}{n} Q^{-l} \qquad \textit{for} \quad n \geq 1$$

$$\beta_0(p) = \left(\frac{Q(p)}{Q(1)} - 1 \right) \frac{1}{1-p/Q}.$$

Proof : Remembering the fact that $s_0 = F_2(1, 1)$ is known from (A-4.3) the case $n=0$ is trivial to show. For $n \geq 1$ we have

$$s_n = \frac{1}{n!} \mathcal{U} \mathcal{D}^{n-1} F_{12}(u, 1).$$

Using (A-4.4) and

$$\beta_n(p) = \frac{1}{n!} \mathcal{U} \mathcal{D}^{n-1} \beta(u, 1)$$

we achieve the desired result mentioning the Taylor-expansion of $F(u, 1)$ from (A-5). ■

(A-7) THEOREM (Rice's Method): *The asymptotic expansion of the alternating sum*

$$A_n = \sum_{k=b}^{n} \binom{n}{k} (-1)^k f_k \qquad \textit{with} \quad b \geq 0$$

is given by

$$A_n = - \sum_{\rho < p < b} \; \operatorname*{Res}_{z = p + \chi_k} (\Phi_n(z) F(z)) + O(n^{\rho})$$

on the premises

(1) *$F(z)$ is a meromorphic function with at most simple poles at $z=p+\chi_k$ with integers $p<b$ and $\chi_k=2k\pi i/\log Q$ for all integers k, except pointz $z=p$ with $0\leqq p<b$, where no poles are allowed. Moreover, $F(z)$ has to fulfill the condition $F(k)=f_k$ for all $b\leqq k\leqq n$.*

(2) *There exists a sequence of closed, rectangular contours γ_l with the left margin fixed on $\Re(z)=\rho$, enclosing the whole halfplane $\Re(z)\geqq\rho$ as $l\to\infty$, and the property that for all z lying on such a contour $F(z)=O(z^a)$ with an arbitrary but fixed constant $a\geqq 0$.*

(3) *The function $\Phi_n(z)$ is expressible in terms of the Gamma-function in the following manner.*

$$\Phi_n(z)=\frac{(-1)^n n!}{z(z-1)\ldots(z-n)}=-\frac{\Gamma(n+1)\Gamma(-z)}{\Gamma(n+1-z)}.$$

The contributions to the sum of residues are

(1) *For the integers $0\leqq p<b$ we obtain*

$$\operatorname*{Res}_{z=p}(\Phi_n(z)F(z))=(-1)^p\binom{n}{p}F(p).$$

(2) *For all integers $p<0$ we have*

$$\operatorname*{Res}_{z=p}(\Phi_n(z)F(z))=-\frac{1}{|p|\binom{n+|p|}{|p|}}\operatorname*{Res}_{z=p}F(z).$$

(3) *The remaining points with nonzero imaginary part yield*

$$\sum_{k\neq 0}\operatorname*{Res}_{z=p+\chi_k}(\Phi_n(z)F(z))=-n^p P(\log_Q n)+O(n^{p-1})$$

$$P(u)=\sum_{k\neq 0}\operatorname*{Res}_{z=p+\chi_k}F(z)\Gamma(-p-\chi_k)e^{2k\pi iu}$$

supposing the convergence of the infinite sum, of course.

Proof (*Sketch*): Rice's method bases, itself as already mentioned, on an old identity from the calculus of finite differences, *see* [7] for example. Let $F(z)$ be a meromorphic function with finite $F(k)=f_k$ for $0\leqq b\leqq k\leqq n$, and γ a positive oriented contour enclosing the points b, $b+1$, $\ldots$, n but no other poles of the integrand below, we have

$$\sum_{k=b}^{n}\binom{n}{k}(-1)^k f_k=\frac{1}{2\pi i}\int_\gamma \frac{(-1)^n n!}{z(z-1)\ldots(z-n)}F(z)\,dz.$$

Extending the skinny contour γ to our rectangular ones, we obtain

$$\int_{\gamma_l}\Phi_n(z)F(z)\,dz=\int_\gamma \Phi_n(z)F(z)\,dz+\sum \text{Residues of newly encountered poles.}$$

The computation of the first integral shows that it contributes $O(n^\rho)$ for $l\to\infty$. This follows from the following facts. First, the properties of the Gamma-function ensures that the integration along the horizontal part of γ_l is of order $O(l^d)$ with a constant $c<0$. Second, with the same argument one can prove, that the contribution of the integration along the right vertical part of the contour is of order $O(l^d)$ with a constant $d<0$. In the limiting case, all these terms vanish. The main term comes from the left vertical part, which lies on $\Re(z)=\rho$. Using the so-called limes relation of the Gamm-function and other estimations of the function $\Phi_n(z)$ yields the stated contribution after some tricky valuations.

Evaluating the contributions (1) and (2) to the sum of residues is straightforward, the expression for (3) is obtained by using the limes relation of the Gamma-function again. We should mention that the latter mainly comes from poles with small imaginary part, say $|\chi_k|<n^\varepsilon$ with some fixed $\varepsilon>0$.

Actually, the complete computation is too long for this paper and not very interesting for practical applications. Most of the "usual" appearing functions simply allow to deal with the sum of residues only, neglecting all estimations above. A simple pole, say ζ with $\Re(\zeta)=r$ yields a contribution $O(n^r)$ to the asymptotic expansion of the desired quantity. If the order of the pole is greater than 1, logarithmic terms occur in the expansion, but we will not treat this case here.

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