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C. P. RUPERT

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CROSSABILITY OF CANCELLATIVE KLEENE SEMIGROUPS (*)

by C. P. Rupert [1]

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Abstract. – *Every cancellative Kleene semigroup satisfies Eilenberg's theorem.*

Résumé. – *Si S est un semigroupe simplifiable de type Kleene, alors S satisfait le théorème d'Eilenberg.*

INTRODUCTION

A morphism $\varphi : T \to S$ of semigroups is called crossable if every rational subset R of T contains a rational cross-section R_0 for the restriction of φ to R or (in other words) if there exists for each rational subset R of T another rational subset R_0 of T satisfying:

(1) $R_0 \subset R$;

(2) $\varphi(R_0) = \varphi(R)$; and

(3) φ is injective on R_0.

The following classical crossability result is useful in the theory of rational relations.

Eilenberg's Theorem [1]: *If* Σ^* *and* Γ^* *are finitely generated free monoids, then every morphism* $\varphi : \Sigma^* \to \Gamma^*$ *is crossable.* ■

We say that a semigroup S satisfies Eilenberg's theorem, or that S is crossable, if every morphism $\varphi : \Sigma^+ \to S$ is crossable for every free semigroup Σ^+.

(*) Received September 1990, revised February 1991.
(1) Department of Mathematics and Computer Science, Robinson Science Building, North Carolina Central University, Durham NC 27707, U.S.A.

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Crossability results often have interesting consequences. For example, if S satisfies Eilenberg's theorem then every rational subset of S is unambiguously rational. Moreover, an effective proof that S satisfies Eilenberg's theorem enables us to decide whether a given rational expression over S is unambiguously rational.

Pelletier [3] introduced a technique for constructing congruences from equivalence relations, used it to produce various counter-examples in the theory of Kleene semigroups, and in this way showed that not all Kleene semigroups satisfy Eilenberg's theorem.

Our major result, Theorem 2 below, proves that every cancellative Kleene semigroup satisfies Eilenberg's theorem, by modifying a method used by Sakarovitch [5] (to prove a special case of Eilenberg's theorem) and by Johnson (to show that every deterministic rational equivalence relation has a rational cross-section, *cf*. Theorem 5.3 in [2]). The method produces rational cross-sections of equivalence relations by lexicographic minimalization, a tactic which does not work in general (*cf*. Theorem 8.2 in [2]) but does work here.

I. PRELIMINARIES

Recall some definitions and theorems.

A subset R (of a semigroup S), which is saturated by a congruence $\equiv$ of finite index on S, is called recognizable. Rec(S) denotes the set of recognizable subsets of S.

NERODE'S THEOREM: *A subset R of a semigroup S is recognizable iff there are only finitely many different quotient sets* $s^{-1}R := \{t \in S : st \in R\}$. ■

LEMMA 1: *Let R be a recognizable subset of a semigroup S; for each $s \in R$, define the set* $[s]_R := \{t : t^{-1}R = s^{-1}R\}$. *Then there are only finitely many sets $[s]_R$ and each of these sets is recognizable.* ■

Rational subsets of a semigroup S are defined as follows: the empty set $\varnothing$ is rational and so is every singleton $s \in S$; if U and V are rational, then so are the union $U \cup V$, product $UV := \{uv : u \in U, v \in V\}$, and subsemigroup $U^+ \subset S$ generated by U. Rat(S) denotes the collection of rational subsets of S.

In an arbitrary semigroup S, Rec(S) and Rat(S) are not closely related. However, the following result holds.

KLEENE'S THEOREM: *If Σ^+ is a finitely generated free semigroup, then every rational subset of Σ^+ is recognizable and conversely.* ■

Motivated by this result, we call a semigroup S Kleene if $\mathrm{Rat}(S) = \mathrm{Rec}(S)$. Clearly, a Kleene semigroup is finitely generated.

By a regulator $\rho : \Sigma^+ \to \Sigma^+$, we mean a rationality-preserving relation: every rational subset $R \subset \Sigma^+$ has rational ρ-image $\rho(R)$.

LEMMA 2 [3]: *A semigroup S is Kleene iff S is isomorphic to the quotient Σ^+/κ of a finitely-generated free semigroup Σ^+ by a congruence κ which is also a regulator.* ■

LEMMA 3: *Any relation $\Sigma^+ \to \Sigma^+$ which is rational in $\Sigma^* \times \Sigma^*$ is a regulator.* ■

LEMMA 4: *The set of regulators is closed under finite union and under composition. If ψ is a regulator and if P and Q are rational subsets of Σ^+ then $(P \times Q) \cap \psi$ is also a regulator.*

Proof: Suppose that ψ and θ are regulators; if $R \in \mathrm{Rat}(\Sigma^+)$, then $(\psi \cup \theta)(R) = \psi(R) \cup \theta(R)$ and $\psi \circ \theta(R) = \psi(\theta(R))$; so the first sentence holds. If R is rational in Σ^+, then

$$\Delta_R = \{ (r, r) : r \in R \}$$

is a rational relation $\Sigma^* \to \Sigma^*$. Now $(P \times Q) \cap \psi$ is simply $\Delta_Q \circ \psi \circ \Delta_p$; if P and Q are rational, this is a composite of regulators; so the second sentence holds. ■

We also use another closure property of regulators. Given any relations $\psi : \Sigma^+ \to \Sigma^+$ and $\varphi : \Sigma^+ \to \Sigma^+$, define the product relation $\varphi \wedge \psi : \Sigma^+ \to \Sigma^+$ by

$$\varphi \wedge \psi := \{ (ac, bd) : (a, b) \in \varphi, (c, d) \in \psi \}.$$

LEMMA 5: *If $\psi : \Sigma^+ \to \Sigma^+$ and $\varphi : \Sigma^+ \to \Sigma^+$ are regulators, then the product relation $\varphi \wedge \psi : \Sigma^+ \to \Sigma^+$ is also a regulator.*

Proof: We begin with the following claim.

Claim: For $R \subset \Sigma^+$, $\varphi \wedge \psi(\mathrm{R}) = \bigcup_{x \in \Sigma^+} \varphi([x]_R \cap R(\Sigma^+)^{-1}) \psi(x^{-1} R)$.

Explanation: Suppose $t \in \varphi \wedge \psi(R)$. Choose $(a, b) \in \varphi$, $(c, d) \in \psi$ with $ac = s \in R$ and $bd = t$. Then $b \in \varphi([a]_R \cap R(\Sigma^+)^{-1})$ (since $a \in [a]_R$ and $ac = s \in R$), and $d \in \psi(a^{-1} R)$ (since $c \in a^{-1} R$). So $t = bd \in \varphi([a]_R \cap R(\Sigma^+)^{-1}) \psi(a^{-1} R)$

and thus

$$\varphi \wedge \psi(R) \subset \bigcup_{x \in \Sigma^+} \varphi([x]_R \cap R(\Sigma^+)^{-1}) \psi(x^{-1} R).$$

For the opposite inclusion, suppose that

$$t \in \bigcup_{x \in \Sigma^+} \varphi([x]_R \cap R(\Sigma^+)^{-1}) \psi(x^{-1} R).$$

Then $t \in \varphi([p]_R \cap R(\Sigma^+)^{-1}) \psi(p^{-1} R)$ for some $p \in \Sigma^+$. So $t = bd$ for some $a \in [p]_R \cap R(\Sigma^+)^{-1}$, $b \in \varphi(a)$ and $d \in \psi(p^{-1} R)$. Since $a \in [p]_R$, $[a]_R = [p]_R$ and $a^{-1} R = p^{-1} R$; so $a \in [a]_R \cap R(\Sigma^+)^{-1}$ and $d \in \psi(a^{-1} R)$. Choose $c \in a^{-1} R$ with $d \in \psi(c) \subset \psi(a^{-1} R)$. As $(a, b) \in \varphi$, $(c, d) \in \psi$, and $ac \in R$, so $t = bd \in \varphi \wedge \psi(R)$, and therefore

$$\bigcup_{x \in \Sigma^+} \varphi([x]_R \cap R(\Sigma^+)^{-1}) \psi(x^{-1} R) \subset \varphi \wedge \psi(R),$$

which completes the proof of the claim. □

We now show that $\varphi \wedge \psi$ is a regulator. Suppose $R \in \mathrm{Rat}(\Sigma^+)$. Then the sets $[x]_R$, $R(\Sigma^+)^{-1}$, and $x^{-1} R$ are also rational; hence so is each set $\varphi([x]_R \cap R(\Sigma^+)^{-1}) \psi(x^{-1} R)$. There are but finitely many distinct sets $x^{-1} R$ and similarly only finitely many sets $[x]_R$. It follows that

$$\varphi \wedge \psi(R) = \bigcup_{x \in \Sigma^+} \varphi([x]_R \cap R(\Sigma^+)^{-1}) \psi(x^{-1} R)$$

actually reduces to a finite union of rational sets. Thus, $\varphi \wedge \psi(R)$ is rational and so $\varphi \wedge \psi$ is a regulator. ■

II. A METHOD OF SAKAROVITCH

By an order on a set X, we understand a binary relation $>$ on X which is asymmetric (no element $x \in X$ satisfies $x > x$) and transitive. A linear order is an order verifying trichotomy:

$$\forall x \in X \quad \forall y \in X \quad x = y \quad \text{or} \quad x > y \quad \text{or} \quad y > x.$$

If $>$ is an order on X and R is a subset of X, then by *a* $>$-minimal element of $R \subset X$ we mean any $r \in R$ with

$$\{s \in R : r > s\} = \varnothing.$$

When κ is a relation on X, $\Lambda = \Lambda(>, \kappa)$ denotes the relation

$$\kappa \cap >^{-1} = \{(u, v) \in \kappa : v > u\}.$$

If κ is an equivalence relation, $\mathrm{Min}(R) = \mathrm{Min}(>, \kappa, \mathrm{R})$ denotes the set

$$\{r \in R : r \text{ is a } > \text{-minimal element of } [r]_\kappa \cap R\},$$

where $[r]_\kappa$ denotes the κ-class of $r \in X$.

Lexicographic orders on a free semigroup Σ^+ are constructed as follows: fix a linear order $. >$ on the alphabet Σ; for distinct words $u \in \Sigma^+$ and $v \in \Sigma^+$, $v > u$ means either that u is a proper prefix of v or that there exist (possibly empty) words w, x, and y over the alphabet Σ and letters $\sigma . > \tau$ in Σ such that $u = w \tau x$ and $v = w \sigma y$. Any lexicographic order is linear.

LEMMA 6: *If κ is a relation and $>$ a lexicographic order on Σ^+ then $\Lambda(>, \kappa)$ is a union $\Lambda_1 \cup \Lambda_2$, where Λ_1 denotes the relation*

$$\{(x \sigma u, x \tau v) \in \kappa : x \in \Sigma^*, u \in \Sigma^*, v \in \Sigma^*, \sigma \in \Sigma, \tau \in \Sigma, \tau > \sigma\}$$

and Λ_2 denotes the relation

$$\{(x \sigma, x \sigma v) \in \kappa : x \in \Sigma^*, \sigma \in \Sigma, v \in \Sigma^+\}.$$

Proof: Obvious from the definition of lexicographic order. ■

Now the method of Sakarovitch [5] is essentially this: for any lexicographic order $>$ and any morphism $\pi : \Sigma^+ \to \Gamma^*$ from the free semigroup Σ^+ to the free monoid Γ^*, $\Lambda(>, \pi^{-1}\pi)$ is a rational relation $\Sigma^* \to \Sigma^*$, and the set $\mathrm{Min}(>, \pi^{-1}\pi, R)$ is therefore rational whenever $R \subset \Sigma^+$ is; when π is non-erasing, $\mathrm{Min}(>, \pi^{-1}\pi, R)$ is a cross-section for the restriction of $\pi^{-1}\pi$ to R, and Eilenberg's theorem follows easily.

The next lemmas isolate some key ideas of this method.

LEMMA 7: *Let $>$ and κ (respectively) be an order and an equivalence relation on the set X. Then the following are equivalent:*

(1) *For each $r \in R$, there exists at least one $>$-minimal element of the set $[r]_\kappa \cap R$; and*

(2) $\mathrm{Min}(>, \kappa, R)$ *intersects every κ-class intersecting R. If these conditions hold and $>$ is a linear order, then* $\mathrm{Min}(>, \kappa, R)$ *is a cross-section for the restriction of κ to R.*

Proof: (1) $\Leftrightarrow$ (2) is obvious. If in addition $>$ is a linear order then each set $[r]_\kappa \cap R$ has a unique $>$-minimal element, so $\mathrm{Min}(R)$ must be a cross-section for the restriction of κ to R. ■

LEMMA 8: *Suppose that $>$ and κ (respectively) are an order and an equivalence relation on the finitely generated free semigroup Σ^+, and that $\Lambda(>, \kappa)$ is a regulator. Then* $\mathrm{Min}(>, \kappa, R)$ *is rational for every rational set R.*

Proof: Suppose that Λ is a regulator; let $\Lambda_0 : \Sigma^+ \to \Sigma^+$ denote the relation $\Delta_R \circ \Lambda$, where $\Delta_R = \{(r, r) : r \in R\}$. Then

$$R \setminus \Lambda_0(R) = R \setminus \{r \in R : \exists s \in R (s, r) \in \Lambda\}$$
$$= \{r \in R : \forall s (s \in [r]_\kappa \cap R) \Rightarrow \text{not}(r > s)\} = \mathrm{Min}(R).$$

Whenever R is rational, Λ_0 is a regulator by Lemma 4 and so $\mathrm{Min}(R)$ is rational. ■

III. LEXICOGRAPHIC MINIMALIZATION

For the remainder of this article, we fix a finitely generated free semigroup Σ^+ and a lexicographic order $>$ on Σ^+. To generalize Sakarovitch's argument, we show that $\Lambda(>, \kappa)$ is a regulator when κ is a left-cancellative congruence.

A semigroup S is called left-cancellative if

$$xy = xz \quad \Rightarrow \quad y = z;$$

the notion right-cancellative is dually defined; cancellative means left- and right-cancellative.

We can now obtain our first result.

THEOREM 1: *If Σ^+/κ is a left-cancellative Kleene semigroup, and if $>$ is a lexicographic order on Σ^+, then $\Lambda(>, \kappa) : \Sigma^+ \to \Sigma^+$ is a regulator.*

Proof: Express $\Lambda = \Lambda_1 \cup \Lambda_2$ according to Lemma 6. To show that Λ is a regulator, it suffices (according to Lemma 4) to prove that Λ_1 and Λ_2 are both regulators. Now Λ_2 is the relation

$$\bigcup_{\sigma\in\Sigma}\{(x\sigma, x\sigma w)\in\kappa : x\in\Sigma^*, w\in\Sigma^+\}$$

$$=\bigcup_{\sigma\in\Sigma}(\{\sigma, \sigma w)\in\kappa : w\in\Sigma^+\}\cup\Delta\{(\sigma, \sigma w)\in\kappa : w\in\Sigma^+\})$$

$$=\bigcup_{\sigma\in\Sigma}((\sigma\times(\sigma\Sigma^+\cap[\sigma]_\kappa))\cup\Delta(\sigma\times(\sigma\Sigma^+\cap[\sigma]_\kappa))),$$

where $[\sigma]_\kappa$ denotes the κ-class of $\sigma\in\Sigma$ and Δ denotes the diagonal $\{(x, x) : x\in\Sigma^+\}$; note that we used the left-cancellativity of κ. Since the semigroup is Kleene, each set $\sigma\Sigma^+\cap[\sigma]_\kappa$ is rational; thus, Λ_2 is actually a rational relation $\Sigma^*\to\Sigma^*$ and hence a regulator.

To show that Λ_1 is a regulator, we first observe that each relation $(\sigma\Sigma^*\times\tau\Sigma^*)\cap\kappa$ (where $\sigma\in\Sigma$, $\tau\in\Sigma$, and $\tau>\sigma$) is a regulator by Lemmas 2 and 4. Thus the union of these relations is another regulator Λ_3. If we show that $\Lambda_1=\Lambda_3\cup\Delta\wedge\Delta_3$ then Λ_1 will be a regulator by Lemmas 4 and 5.

Now $(s, t)\in\Lambda_1$ means $s\kappa t$, $(s, t)=(x\sigma u, x\tau v)$ where $\tau>\sigma$ are letters in Σ, and x, u, and v lie in Σ^*. If x is actually the empty word, then

$$(s, t)=(\sigma u, \tau v)\in(\sigma\Sigma^*\times\tau\Sigma^*)\cap\kappa\subset\Lambda_3.$$

On the other hand, when $x\in\Sigma^+$ we conclude from $x\sigma u\ \kappa\ x\tau v$ (using left-cancellativity) that

$$(\sigma u, \tau v)\in(\sigma\Sigma^*\times\tau\Sigma^*)\cap\kappa\subset\Lambda_3,$$

whence it is immediate (by the definition of $\Delta\wedge\Lambda_3$) that

$$(s, t)=(x\sigma u, x\tau v)\in\Delta\wedge\Lambda_3.$$

Thus $\Lambda_1\subset\Lambda_3\cup\Delta\wedge\Lambda_3$; for the opposite inclusion, read backwards, using the left-compatibility

$$y\kappa x \quad\Rightarrow\quad xy\ \kappa\ xz$$

of κ instead of left-cancellativity. ■

Corollaries 1 and 2 below generalize results in [5].

COROLLARY 1: *If $\pi:\Sigma^+\to S$ is a morphism from Σ^+ to a left-cancellative Kleene semigroup S, and if $>$ is a lexicographic order on Σ^+, then* $\mathrm{Min}(>, \pi^{-1}\pi, R)$ *is rational for every rational subset $R\subset\Sigma^+$.*

Proof: Since S is Kleene, $\pi^{-1}\pi$ is a regulator by Lemma 2; moroever, a subsemigroup of a left-cancellative semigroup is left-cancellative; hence, the theorem guarantees that Λ is a regulator. The result now follows from Lemma 8. ■

LEMMA 9: *Let S be a left-cancellative semigroup, every singleton subset of which is recognizable. Then the following conditions are equivalent for any morphism $\pi : \Sigma^+ \to S$:*

(1) *For each $s \in \pi(\Sigma^+)$, the set $ss^{-1} \cap \pi(\Sigma^+) = \emptyset$; and*

(2) *Each $\pi^{-1}\pi$-class is finite.*

Proof: (1) $\Rightarrow$ (2): Suppose that some set $\pi^{-1}\pi(w)$ is infinite. Then $\pi^{-1}\pi(w)$ is rational because S has recognizable singletons and consequently $\pi^{-1}\pi(w)$ contains an infinite subset xy^+z by the pumping lemma. From $\pi(xyz) = (\pi(xy^2z)$, we conclude (by left-cancellativity) that

$$\pi(yz) = \pi(y^2 z) = \pi(y)\pi(yz) \text{ so } \pi(y)$$

belongs to the set $\pi(yz)\pi(yz)^{-1}$.

(2) $\Rightarrow$ (1) : If $\pi(v) \in \pi(u)\pi(u)^{-1}$ *for some words* u and v, then v^+u is an infinite subset of $\pi^{-1}\pi(u)$. ■

COROLLARY 2: *If S is a left-cancellative Kleene semigroup, $>$ a lexicographic order on Σ^+ and $\pi : \Sigma^+ \to S$ a morphism such that that $ss^{-1} \cap \pi(\Sigma^+) = \emptyset$ for each $s \in \pi(\Sigma^+)$, then* $\mathrm{Min}(>, \pi^{-1}\pi, R)$ *is a rational cross-section for the restriction of π to R, for each rational $R \subset \Sigma^+$; in particular, π is crossable.*

Proof: By hypothesis, every $\pi^{-1}\pi$-class is finite. Thus $\mathrm{Min}(R)$ is a rational cross-section by Theorem 1 and Lemmas 7 and 8, regardless of the rational set $R \subset \Sigma^+$. ■

For the next application of these ideas, we recall that an equivalence relation κ_1 is called locally-finite thinning of the equivalence relation $\kappa \subset \Sigma^+ \times \Sigma^+$ if κ_1 is a restriction of κ, if the domain of κ_1 intersects every κ-class, and if each κ_1-class is finite. The following result is due to Johnson.

JOHNSON'S THEOREM [2]: *Every rational equivalence relation has a rational locally-finite thinning.* ■

We also need the following result, which can be restated in various forms (*cf*. Proposition 1.4.3 in [3]).

CHOFFRUT'S THEOREM: *If the congruence κ on Σ^+ is rational as a subset of $\Sigma^* \times \Sigma^*$ and if κ has a rational cross-section, then the quotient Σ^+/κ satisfies Eilenberg's theorem.* ■

COROLLARY 3: *Suppose* $\kappa \subset \Sigma^+ \times \Sigma^+$ *is a left-cancellative congruence which is rational as a subset of* $\Sigma^* \times \Sigma^*$. *Then* Σ^+/κ *satisfies Eilenberg's theorem.*

Proof: According to the Johnson's Theorem, we can find a rational locally-finite thinning κ_1 for κ, or (in other words) we can find a rational set $D \subset \Sigma^+$ such that $\kappa_1 := \kappa \cap D \times D$ is a locally-finite thinning of κ. Fix any lexicographic order $>$ on Σ^+. Then $\mathrm{Min}(>, \kappa, D)$ is a rational cross-section for κ by Theorem 1 and Lemmas 7 and 8. The result now follows by Choffrut's theorem. ■

IV. CANCELLATIVE KLEENE SEMIGROUPS

In this section, we show that cancellative Kleene semigroups satisfy Eilenberg's theorem.

LEMMA 10 [4]: *Let S be a semigroup, every singleton subset of which is recognizable. Then every subgroup of S is finite. If S has an identity element* 1, *then every divisor of* 1 *actually belongs to the group of units of S.* ■

LEMMA 11: *Let S be a cancellative semigroup, every singleton subset of which is recognizable. Then the following conditions are equivalent for any morphism* $\pi : \Sigma^+ \to S$:

(1) *For each* $s \in \pi(\Sigma^+)$, *the set* $ss^{-1} \cap \pi(\Sigma^+) = \varnothing$;

(2) $\pi(\Sigma^+)$ *does not contain an idempotent;*

(3) *If S has an identity element* 1, *then* $1 \notin \pi(\Sigma^+)$; *and*

(4) *For each* $\sigma \in \Sigma$, *the set* $\pi^{-1}\pi(\sigma) \cap \sigma(\sigma^+) = \varnothing$.

Proof: (4) $\Rightarrow$ (3): Suppose $\pi(w)$ is an identity element for S; let $\sigma \in \Sigma$ be any letter appearing in w; then, as a divisor of the identity $\pi(w)$, $\pi(\sigma)$ belongs to the group of units of S; moreover, this is a finite group; hence for some $n > 1$, $\pi(\sigma)^n = \pi(\sigma^n) = \pi(\sigma)$ and therefore $\pi^{-1}\pi(\sigma) \cap \sigma(\sigma^+) \neq \varnothing$.

(3) $\Rightarrow$ (2): An idempotent in a cancellative semigroup must be the identity.

(2) $\Rightarrow$ (1): If $\pi(v) \in \pi(u)\pi(u)^{-1}$, then $\pi(v^2 u) = \pi(vu) = \pi(u)$ and $\pi(v)$ is idempotent by cancellativity.

(1) $\Rightarrow$ (4): If $\sigma \in \Sigma$ and $n > 1$ satisfy $\sigma^n \in \pi^{-1}\pi(\sigma)$, then

$$\pi(\sigma^{n-1}) \in \pi(\sigma)\pi(\sigma)^{-1}. \quad ■$$

THEOREM 2: *Let* $\pi : \Sigma^+ \to S$ *be a morphism from* Σ^+ *to the cancellative Kleene semigroup S. Then* π *is crossable.*

Proof: Fix a lexicographic order $>$ on Σ^+. According to part (4) of Lemma 11, we can easily test whether $\pi(\Sigma^+)$ contains an identity element for S. Our proof splits according to the outcome of this test; if $\pi(\Sigma^+)$ does not contain an identity element for S, and if R is any rational set, then (by Lemma 11 and Corollary 2) $\mathrm{Min}(>, \pi^{-1}\pi, R)$ is a rational cross-section for the restriction of π to R.

On the other hand, if S is actually a monoid with identity element $1 \in \pi(\Sigma^+)$, and if $R \subset \Sigma^+$ is any rational set, put $G := \pi^{-1}(1)$, and define $\varepsilon : \Sigma^+ \to \Sigma^+$ by

$$\varepsilon := (\Delta \cup \Theta) * \Theta (\Delta \cup \Theta) *$$

where $\Delta := \{ (x, x) : x \in \Sigma^+ \}$ and

$$\Theta := \bigcup_{\sigma \in \Sigma} ((\sigma G \times \sigma) \cup (G \sigma \times \sigma)).$$

Then ε is an order which is also a rational relation $\Sigma^* \to \Sigma^*$. If $(u, v) \in \varepsilon$, then v has length strictly less than the length of u, so there is no infinite chain

$$w_1 \varepsilon w_2 \varepsilon w_3 \varepsilon \ldots;$$

hence each $\pi^{-1}\pi$-class has an ε-minimal element. As $\varepsilon \subset \pi^{-1}\pi$, we have $\Lambda(\varepsilon, \pi^{-1}\pi) = \varepsilon^{-1}$, which is certainly a regulator. By Lemmas 7 and 8, $\mathrm{Min}(\varepsilon, \pi^{-1}\pi, R)$ is rational and $\pi(\mathrm{Min}(\varepsilon, \pi^{-1}\pi, R)) = \pi(R)$.

We claim no $\pi^{-1}\pi$-class contains infinitely many elements of $R_1 := \mathrm{Min}(\varepsilon, \pi^{-1}\pi, R)$. If indeed $R_1 \cap \pi^{-1}\pi(w)$ were infinite, then according to the pumping lemma this rational set would contain an infinite subset xy^+z with $y \in \Sigma^+$, by cancellativity, $\pi(y)$ is idempotent so $y \in G$, which implies that $(xy^2z, xyz) \in \varepsilon$; but this contradicts the fact that $xy^2z \in R_1 = \mathrm{Min}(\varepsilon, \pi^{-1}\pi, R)$. By Lemmas 7 and 8, $\mathrm{Min}(>, \pi^{-1}\pi, R_1)$ is therefore a rational cross-section for the restriction of π to R_1 and even for the restriction of π to R. ■

We remark that Theorem 2 is effective relative to the given Kleene semigroup S: if we have an explicit finite generating set for S, and an algorithm which produces for each $R \in \mathrm{Rat}(S)$ a congruence of finite index saturating R, then we can really produce the cross-sections described.

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