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ON ALGEBRAIC SPECIFICATIONS OF COMPUTABLE ALGEBRAS WITH THE DISCRIMINATOR TECHNIQUE (*)

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Abstract. – *Algebraic specifications of computable data types with respect to the initial and the final algebra semantics are discussed.*

The method consists of using a special equational theory, well known in universal algebra, which axiomatizes the behaviour of an operation called discriminator.

Résumé. – *On discute des spécifications algébriques des types de données calculables par rapport aux sémantiques initiales et finales.*

La méthode consiste à utiliser une théorie équationnelle particulière, bien connue en algèbre universelle, qui axiomatise le comportement d'un opérateur appelé discriminateur.

0. INTRODUCTION

An abstract data type can be identified with an isomorphism class of a (multisorted) minimal algebra of finite signature. The finite equational specification problem with enrichment of hidden operation, without new sorts, goes back to pioneering work on algebraic specifications of data types [10].

This problem was investigated for the class of semicomputable data types and it was solved positively by Bergstra and Tucker [5] for the class of computable data types with respect to the initial and final algebra semantics and it was also solved for the class of cosemicomputable data types with respect to the final algebra semantics.

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In this paper we prove that a computable infinite algebra can be specified by a single equation with enrichment of four hidden operations and a hidden constant with respect to the initial and final algebra semantics. Moreover, we prove the following for finite algebras. Only one hidden operation suffices and the obtained specification is ω-complete. Finally, we discuss the ω-incompleteness for finite specifications of infinite computable algebras.

Our approach uses a particular technique well known in Universal Algebra under the name of the discriminator theory. This method consists of using a special finite equational theory which axiomatizes the behaviour of a ternary operation called a discriminator. One of the useful aspects of the discriminator theory is its extreme simplification of the structure of equational classes (*see* Fact 2.7 in the proof of Theorem 2.1 and Fact 4 in the proof of Theorem 3.1). A second aspect is the possibility of reducing properties defined by universal sentences to properties defined by equations [*see* (1.5)]. The discriminator has recently come under investigation in Computer Science by several authors, Mekler and Nelson [16], Blum and Tindel [7], Guessarian and Meseguer [11], in papers where they studied data algebras with an operation which models the "if-then-else" operation.

In Section 1 we will briefly recall the main definitions and results that we will use and we define the terminology.

In Section 2 we revisit the Bergstra and Tucker Theorem for the specification of computable data types in the light of the discriminator theory. We give a new proof of this Theorem and observe that this proof yields a specification which consists of a single equation.

In Section 3 we prove that a finite algebra can be specified by a single equation and one hidden operation with respect to the initial and final algebra semantics. This problem was solved in [4] with three hidden operations and two equations. Furthermore, our specification turns out to be ω-complete.

In Section 4 we discuss ω-incompleteness for the specification of infinite computable algebras. This property arises in many cases. In fact, we prove that all the specifications obtained along the line of Theorem 2.1 cannot be ω-complete.

1. PRELIMINARIES AND NOTATION

The terminology and notation are standard, *see* [10], [8], [6] as main references. We recall briefly the main definitions and the results we are going to use for the sake of clarity.

A *minimal algebra* A of (multisorted) signature Σ is an algebra without subalgebras, i.e. it is generated by the interpretations of the constants of Σ. A *specification* of A with respect to the *initial algebra semantics* is a pair (Σ, E), where E is a set of algebraic equations, alias identities, which proves all and only the ground equations, *i.e.* without variables, which are true in A. When this happens, A is isomorphic to the initial algebra $I(\Sigma, E)$ of the category $\mathrm{ALG}(\Sigma, E)$ of all minimal algebras of signature Σ which are models of E. A is specified as *final algebra* by (Σ, E) if the final object $F(\Sigma, E)$ in the category $\mathrm{ALG}(\Sigma, E)$, of the non trivial members of $\mathrm{ALG}(\Sigma, E)$, exists and A is isomorphic to $F(\Sigma, E)$. When A is specified as final algebra by (Σ, E), we have that a ground equation is consistent with E if and only if it is true in A.

R is a *recursive algebra of* (multisorted) signature Σ if the carriers of R are sets of natural numbers and if the interpretations of the operation symbols are total recursive functions. A pair (R, p) is a *recursive coordinatization* of the minimal algebra A if R is a recursive algebra and $p: R \to A$ is an epimorphism. A is called *computable* (*semicomputable* or *cosemicomputable*) if the partition, induced by p in every carrier, is recursive (r.e. or co-r.e.). This turns out to be independent of the recursive coordinatization (R, p) (*see* [14], [6]).

A computable infinite algebra of signature Σ cannot necessarily be specified as initial algebra or as final algebra by a finite set E of equations in the signature Σ (*see* [10], [6]). Bergstra and Tucker ([2], [5]), found finite specifications by allowing hidden operations. (Σ', E') is a specification for A of signature Σ with respect to the initial (or the final) algebra semantics with *enrichment of hidden operations* if the following hold. Σ' is a finite signature extending Σ with possible new operations, but no extra sorts, and (Σ', E') specifies an expansion A' to the signature Σ' of A.

An algebraic specification is ω-*complete* if and only if all equations, possibly containing variables, which are valid in its initial algebra $I(\Sigma, E)$ are also provable from E by the rules of the equational logic.

To simplify notation we treat single sorted algebras. All main results extend easily to multisorted signatures. Throughout the paper $\Sigma = \{F_1, \ldots, F_k, c_1, \ldots, c_r\}$ denotes a fixed signature in a single sort, where $F_1, \ldots, F_k$ are operation symbols and $c_1, \ldots, c_r$ are constant symbols. The symbols in $\{0, S, +, .\}$ will be called *arithmetical symbols* and they will be interpreted on the natural number N as zero, successor operation, sum and product, respectively. The arithmetical terms $S^n(0)$, defined by induction by $S^0(0) = 0$ and $S^{n+1}(0) = S(S^n(0))$, are called *numerals*. If $p(x_1, \ldots, x_k)$ is a polynomial with non negative integer coefficients, then there is an arithmetical term $P(x_1, \ldots, x_k)$ obtained by p with the substitution of the coefficients for the corresponding numerals. We are going to use the Davis-Matijasevic-Robinson [9] Theorem in the following form. For every total recursive function $f: N \to N$ there are two polynomials $p(\vec{x}, y, \vec{z})$, $q(\vec{x}, y, \vec{z})$ in the variables $\{x_1, \ldots, x_k, y, z_1, \ldots, z_h\}$ and non negative integer coefficients such that

$$\forall m_1, \ldots, m_k, n \in N \exists r_1, \ldots, r_h \in N (f(m_1, \ldots, m_k) = n \Leftrightarrow p(\vec{m}, n, \vec{r}) = q(\vec{m}, n, \vec{r})).$$

The last equality is true if and only if in any algebra expanding the natural numbers the equation $P(\vec{m}, n, \vec{r}) = Q(\vec{m}, n, \vec{r})$ is satisfied, where any natural number is replaced by the corresponding numeral.

We will call *Peano equations* the usual equations which define sum and product in Peano arithmetic, *i.e.*

$$\begin{gathered} x+0=x \\ x+S(y)=S(x+y) \\ x.0=0 \\ x.S(y)=x+(x.y) \end{gathered} \tag{1.1}$$

Now, we briefly recall the main tools of the discriminator theory (*see* [15], [8]). A ternary function d on a set A is called a (ternary) *discriminator* and quaternary function s on A is called a *switching function* if

$$\forall a, b, c \in A, \qquad \textit{if} \quad a=b \textit{ then } d(a, b, c)=c \textit{ else } d(a, b, c)=a.$$

$$\forall a, b, u, v \in A, \qquad \textit{if} \quad a=b \textit{ then } s(a, b, u, v)=u \textit{ else } s(a, b, u, v)=v.$$

Let $t(x, y, z)$ be a ternary term of a signature Σ, then we call the following set $E(t, \Sigma)$ *McKenzie equations*, for t with respect to Σ, (*see* [15]).

$$\begin{gathered} t(x, x, y)=y \\ t(x, y, x)=x \\ t(x, y, y)=x \\ t(x, t(x, y, z), y)=y \\ t(x, y, F(z_1, \ldots, z_n))=t(x, y, F(t(x, y, z_1), \ldots, t(x, y, z_n))) \end{gathered} \tag{1.2}$$

for all operation symbols $F \in \Sigma$.

The main properties that we are going to use are the following (*see* [15], [8]).

(1.3) If a term $t(x,y,z)$ is interpreted as a discriminator function in an algebra A, then the term $sw(x,y,u,v)$, defined by $t(t(x,y,u),\ t(x,y,v),v)$, is interpreted in the switching function on A.

(1.4) A non trivial algebra A of signature Σ is a subdirectly irreducible (abbr. *SI*) model of $E(t,\Sigma)$ if and only if the interpretation t^A of t in A is the discriminator function.

(1.5) For every universal sentence Φ in the signature Σ it is possible to compute an equation ε, in the same signature, such that ε is logically equivalent to Φ on the *SI* models of $E(t,\Sigma)$.

(1.6) Every finite set E of equations in the signature Σ which contains $E(t,\Sigma)$ is logically equivalent to a single equation computable from E.

2. ON SPECIFICATIONS OF INFINITE COMPUTABLE ALGEBRAS

In this Section we apply the discriminator technique to the problem of specifying computable algebras. We obtain a new proof of a noteworthy Theorem of Bergstra and Tucker [2] about the existence of a finite specification using hidden operations and no extra sorts. Also here as in [5] we make use of the Davis-Matijasevic-Robinson Theorem about the Diophantine representation of the r.e. sets. However, by the use of the discriminator as hidden operation we may reduce the number of specifying equations to one. The revisited version of Bergstra and Tucker Theorem is the following

THEOREM 2.1: *Let A be a minimal algebra of finite signature Σ, then the following are equivalent*

(i) *A is computable;*

(ii) *A has a specification (Σ', E') with enrichment of hidden operations under initial algebra and under final algebra semantics. E' is a single equation; the hidden operations are the arithmetical operations and the discriminator.*

Proof: The proof of (ii) $\rightarrow$ (i) is by the well known fact that the initial algebra $I(\Sigma', E')$ is semicomputable and the final algebra $F(\Sigma', E')$ is co-semicomputable (*see* [5]). To prove (i) $\rightarrow$ (ii) we assume that A is infinite. The finite case is easier and it will be discussed in the next section (Theorem 3.1).

By the representation Lemma (*see* [14], [6]) A is isomorphic to a recursive algebra $R=(N, f_1, \ldots, f_k, m_1, \ldots, m_r)$. Now, we consider the signature $\Sigma' = \Sigma \cup \{0, S, +, ., D\}$, where the new symbols added to Σ are intended to be interpreted on the natural numbers as zero, successor operation, addition, multiplication and ternary discriminator. Let R' be the expansion of R to Σ' with the given interpretation of the new symbols.

We consider the following set E'_0 of equations in the signature Σ'.

(2.1) The Peano's equations for $\{0, S, +, .\}$;

(2.2) The McKenzie's equations for the term $D(x, y, z)$ with respect to the signature Σ';

(2.3) $c_i = S^{m_i}(0)$ for $i = 1, \ldots, r$;

(2.4) $sw(P_i(\vec{x}, y, \vec{z}),\quad Q_i(\vec{x}, y, \vec{z}),\quad F_i(\vec{x}, y) = y,$ for $i = 1, \ldots, k$, where $P_i(\vec{x}, y, \vec{z})$, $Q_i(\vec{x}, y, \vec{z})$ are the arithmetical terms corresponding to the polynomials which are given by the Davis-Matijasevic-Robinson Theorem for defining the graph of the recursive function f_i.

It is easy to see that the algebra R' is a model of the equations E'_0. Now, we are going to prove the following.

(2.5) The kern, $ker(ev)$, of the evaluation $ev: T(\Sigma') \to R'$ is the congruence on $T(\Sigma')$ determined by the equality modulo the equational theory E'_0.

(2.6) Every equation of signature Σ' is consistent with E'_0 if and only if it is true in R'.

The statements (2.5) and (2.6) follow by the Theorems of completeness and subdirect representation of G. Birkhoff and the following

Fact 2.7: Let B be a *SI* non trivial model of E'_0, then there exists a monomorphism $\Phi: R' \to B$.

To prove Fact 2.7 we define Φ by Φ: $n \mapsto ev_B(S^n(0))$. Since B is model of E'_0, hence of (2.1) and (2.3), Φ preserves the arithmetic operations and the interpretations of the constants $c_1, \ldots, c_n$. Moreover, for (2.2), (1.3) and the assumption that B is *SI*, the interpretation of the switching term *sw* on B must be the switching function. This, together with the Davis-Matijasevic-Robinson Theorem and (2.4) proves that Φ preserves the operations which interprete $F_1, \ldots, F_k$. We prove, now, that Φ is injective. Assume the contrary. Then Φ must be a constant function since R' is a simple algebra for the presence of the discriminator. Hence, $ev_B(0) = ev_B(S(0))$. This would imply that B be trivial since the equations $x.0 = 0$, $x.S(0) = x$ are provable from E'_0 and therefore they are true in B.

Now, by the McKenzie technique [*see* (1.6)] it is possible and easy to compute an equation ε which is logically equivalent to the set of equations in E'_0. Then, for (2.5) and (2.6), $E' = \{\varepsilon\}$ fulfils the request in the statement (ii) of the Theorem.

3. ω-COMPLETE SPECIFICATIONS FOR FINITE ALGEBRAS

Let (Σ, E) be an ω-complete and r. e. specification. Then, also the set of identities valid in the initial algebra $I(\Sigma, E)$ is r. e. On this account Heering [12] raised the following problem: *Suppose that the set of equations valid in a minimal algebra A of finite signature is recursively enumerable. Then, is it true that A posses an ω-complete finite specification with hidden operations and no extra sorts with respect to the initial algebra semantics?*

A remark in [3] shows that every finite minimal algebra has a finite specification with respect to the initial algebra semantics. This method, which is substantially a collection of the operation tables, was called *graph enumeration*. However, a finite algebra does not necessarily have a finite ω-complete specification without hidden operations. This was known in Universal Algebra a long time ago [13] and started investigations on the finite basis problem for identities (*see* [8]).

In this Section we exhibit a finite ω-complete specification with respect to the initial and final algebra semantics for every finite algebra of finite signature. We use the discriminator as unique hidden operation and we prove that the given specification consists of a single equation.

THEOREM 3.1: *Let A be a finite minimal algebra of finite signature Σ. Then there exists a finite ω-complete specification (Σ', E') of A with only one hidden ternary operation with respect to the initial and final algebra semantics. E' can be choosen to contain a single equation.*

Proof: Expand A to A' by adding the ternary discriminator as new operation. From Baker Theorem (*see* [1], [8]) we can get a finite basis E' for the identities of the equational class generated by A'. This yield a finite ω-complete specification for A with the discriminator as hidden operation. However, the number of identities in E' is large if compared with the cardinality of A. We exhibit the required specification for A' using the properties of the discriminator and the graph enumeration method to specify A (*see* [3]).

Let $\Sigma' = \Sigma \cup \{D\}$, where D is a ternary operation symbol. Denote the cardinality of A by n and take $R = \{t_1, \ldots, t_n\}$ a system of representatives (transversal) in the partition of $T(\Sigma)$ induced by the kern of the canonical epimorphism $ev_A : T(\Sigma) \rightarrow A$.

Now, consider the set E' of equations in the signature Σ'.

E 1. $c_j = t_{i_j}$ for $j = 1, \ldots, r$, where $t_{i_j} \in R$ and $ev_A(c_j) = ev_A(t_{i_j})$.

E 2. $f_j(t_{i_1}, \ldots, t_{i_s}) = t_{h_j}$ for all $j = 1, \ldots, k$ and $t \in R$ such that the equation holds in A.

E 3. $D(t_h, t_m, y) = t_h$, where t_h, $t_m \in R$, $h \neq m$ and y is a fixed variable.

E 4. The McKenzie's equations for the term $D(x, y, z)$ with respect to the signature Σ'.

E 5. The equation which is equivalent on the SI models of E 4 to the following universal sentence. Note that this is possible for (1.5).

$$\forall x \left(\bigvee_{1 \leq k \leq n} (x = t_k) \right)$$

We prove that (Σ', E') is a ω-complete specification of A' with respect to the initial and final algebra semantics. After that we can take E' of a single equation for (1.6) since E' contains E 4. To this end consider the following Facts.

Fact 1: A' is a SI model of E'.

Fact 2: For every ground term $t' \in T(\Sigma')$ there exists a $t \in R$ such that $E' \vdash t' = t$.

Fact 3: The interpretation of D on the initial algebra $I(\Sigma', E')$ is the discriminator function.

Fact 4: If B' is a non trivial SI model of E' then the unique morphism from $I(\Sigma', E')$ to B' is an isomorphism.

Fact 1 is true by construction of A' and E'. Fact 2 can be proved by induction on the structural complexity of t' using E 1-E 2-E 3-E 4. Fact 3 follows from the following simple observation

$$E' \vdash t_h = t_m \text{ implies } E' \vdash D(t_h, t_m, t_q) = t_q;$$

$$E' \nvdash t_h = t_m \text{ implies } m \neq h \text{ and } E' \vdash D(t_h, t_m, t_q) = t_h.$$

To prove Fact 4, let B' be a SI non trivial model of E'. Then, from E 5 the unique morphism Φ from $I(\Sigma', E')$ to B' is surjective. But, since $I(\Sigma', E')$ is simple, for Fact 3, and since B' is of cardinality greater than 1, Φ must be injective.

Now, Fact 4 proves that A' is the unique, up to isomorphism, SI model of E'. Therefore, (Σ', E') is a ω-complete specification of A' with respect to the initial and the final algebra semantics.

4. ω-COMPLETE SPECIFICATIONS FOR INFINITE COMPUTABLE ALGEBRAS

In this Section we prove (Theorem 4.1) that the finite specification given in Theorem 2.1 is not ω-complete when A is infinite. The ω-completeness of specifications of infinite algebras is rare for incompleteness phenomena like the one discovered by Gödel. We will use, in the proof, the Gödel incompleteness Theorem in the form given by Davis-Matijasevic-Robinson [9].

THEOREM 4.1: *Let A be an infinite algebra of signature Σ' such that*

(i) *Σ' contains the arithmetical symbols $\{0, S\}$ and every $a \in A$ is the interpretation of a numeral,* i.e. *there exists k such that $a = ev(S^k(0))$.*

(ii) *Σ' contains the arithmetical symbols $\{+, .\}$ and A satisfies the Peano equations.*

(iii) *There exists a ternary term $D(x, y, z)$ of signature Σ' such that A satisfies the McKenzie equations for D with respect to Σ'.*

Then A has no ω-complete r.e. specification, even if we allow hidden operations, with respect to the initial algebra semantics.

We call an algebra which satisfies (i) and (ii) of Theorem 4.1 *peanian* if it has a binary operation, say, such that the following equations hold in A:

$$x \dot{-} 0 = x, \qquad 0 \dot{-} x = 0, \qquad S(x) \dot{-} S(y) = x \dot{-} y.$$

COROLLARY 4.2: *Every non trivial peanian algebra has no r.e. ω-complete specifications, even if we allow hidden operation, with respect to the initial algebra semantics.*

Before the proofs we need a technical lemma.

LEMMA 4.3: *Let A be an infinite algebra. Suppose that A satisfies* (i) *and* (iii) *of Theorem* 4.1. *Then, the interpretation D^A of the term D in A is the ternary discriminator on A.*

Proof: Let us consider a subdirect representation of A with SI algebras

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[dr, "q_i"'] & \prod_{i \in I} A_i \arrow[d, "p_i"] \\ & A_i \end{tikzcd}$$

Then, the interpretation D^{A_i} of D on every A_i must be the discriminator for (1.4). *A priori* we may distinguish three cases.

Case 1: There exists $j \in I$ such that A_j is infinite.

Case 2: There exists a positive integer k such that $|A_i| < \mathbf{k}$ for every $i \in I$.

Case 3: Neither Case 1 nor Case 2.

Assume first Case 1. Then, it is enough to prove that $q_j = h \circ p_j$ is injective. If it is not, then it would exist m, n with $m < n$ such that A_j satisfies the equation $S^m(0) = S^n(0)$. This would imply A_j finite, in fact $|A_j| \leqq n$.

We prove, now, that Case 2 cannot occur. Assume Case 2. Then, for every $i \in I$ there exist integers p_i, q_i such that $0 \leqq q_i < p_i + q_i \leqq k$ and A_i satisfies the equation

$$S^{q_i}(0) = S^{p_i + q_i}(0).$$

The bounded family $\{p_i : i \in I\}$ has a liest common multiple p. Then, in every A_i the equation

$$S^k(0) = S^{p+k}(0)$$

is true. Therefore, this equation is true in A. This would imply that A is finite, which is a contradiction.

Assume Case 3. We define $c : I \to N$ with $c(i) = |A_i|$. Let θ be the equivalence on I induced by c, *i.e.* $i \theta j$ if and only if $c(i) = c(j)$. It is easy to prove that the family

$$\mathscr{F} = \{ X \mid X \subseteq I \text{ and } I - X \text{ is a finite union of classes of } \theta \} \tag{4.4}$$

is a filter basis. Let $\mathscr{U}$ be an ultrafilter on I extending $\mathscr{F}$. Call the map which commutes the diagram q

$$\begin{array}{ccc} A & \xrightarrow{h} & \prod_{i \in I} A_i \\ & \overset{q}{\searrow} & \downarrow p \\ & & \prod_{i \in I} A_i / \mathscr{U} \end{array}$$

where p is the canonical projection on the ultraproduct. Observe that the interpretation of the term D on the ultraproduct must be the discriminator function since it is so on every factor. Then, to obtain the result, it is enough to prove that the morphism q is injective. Assume that q is not injective.

Then, there are m, n such that $m<n$ and there exists Y such that

$$Y=\{i \mid A_i \models S^m(0)=S^n(0)\} \in \mathcal{U} \tag{4.5}$$

Therefore, $i \in Y$ implies $|A_i| \leqq n$. Then,

$$Y \subseteq Z = \bigcup_{r \leqq n} c^{-1}\{r\} \tag{4.6}$$

But, from (4.4) and the choice of $\mathcal{U}$ we have that $I-Z \in \mathcal{U}$. This together with (4.6) and (4.5) leads to contradiction.

Remark 4.7: *The hypothesis "A infinite" in Lemma* 4.3 *is necessary. Let A be $A_2 \times A_3$, where $A_n=(\{0, \ldots, n-1\}, 0, succ, d)$ with d ternary discriminator and $succ(i)=(i+1)$ mod n. It is easy to prove that A satisfies all the hypothesis of Lemma* 4.3 *except to be infinite. However, A does not satisfy the thesis.*

PROOF OF THEOREM 4.1: Assume by hypothesis of contradiction that (Σ', E') is a r.e. ω-complete specification of A with hidden operations with respect to the initial algbra semantics. Then, from the Gödel incompleteness Theorem in the Davis-Matijasevic-Robinson form [9], there exist arithmetical terms, i.e. polinomials with numerals as coefficients, such that

$$A \models \forall \vec{x}(P(\vec{x}) \neq Q(\vec{x})) \tag{4.8}$$

$$E' \cup \{S(0) \neq 0\} \nvdash \forall \vec{x}(P(\vec{x}) \neq Q(\vec{x})) \tag{4.9}$$

Then, from (4.8), from Lemma 4.3, from (1.3) and (1.4), the equation $sw(P(\vec{x}), Q(\vec{x}), S(0), 0)=0$ is true in A. Hence, from the initial hypothesis to be contradicted

$$E' \vdash \forall \vec{x}(sw(P(\vec{x}), Q(\vec{x}), S(0), 0)=0) \tag{4.10}$$

Now, from (4.9) there exists a model B' of $E' \cup \{S(0) \neq 0\}$ and there exists a n-tuple $b_1, \ldots, b_n$ such that

$$B' \models P(\vec{b})=Q(\vec{b}) \tag{4.11}$$

Then, from (4.11), (4.10) and (1.2), we have that B' satisfies $S(0)=0$. This contradicts the choice of B'.

PROOF OF COROLLARY 4.2: It is easy to prove that every non trivial peanian algebra A is infinite. Moreover, there exists a term whose interpretation on A is the discriminator, namely

$$(1 \dot{-} (1 \dot{-} |x-y|))\, x + (1 \dot{-} |x-y|)\, z$$

where $|x-y|$ denotes $(x \dot{-} y)+(y \dot{-} x)$.

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