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A NOTE ON THE ITERATION OF INFINITE MATRICES (*)

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Abstract. – *Sufficient conditions for the existence of the iteration (star) of an infinite matrix over an arbitrary semiring are presented.*

Résumé. – *On établit des conditions suffisantes pour l'existence de l'itération d'une matrice infinie sur un demi-anneau arbitraire.*

1. INTRODUCTION

Kuich and Urbaneck [3] have recently introduced infinite linear system over formal power series. Under a certain mild assumption such systems have a unique solution which may be obtained by the iteration of the associated coefficient matrix. Therefore, the problem arises how to compute this iteration.

Here we will deal with infinite matrices over arbitrary semirings. For infinite matrices in Jacobi form a general criterion for the existence of the iteration will be established. But only for special infinite matrices in Jacobi form called Dyck matrices the iteration can be computed if we suppose that a related finite system of equations is solvable.

2. BASIC NOTIONS

A **semiring** is an algebraic structure $(R, +, ., 0, 1)$ consisting of a carrier set R, two binary operations $+$ and $.$ and two nullary operations (constants)

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0 and 1 such that

1. $(R, +, 0)$ is a commutative monoid,

2. $(R, ., 1)$ is a monoid with 0 as multiplicative zero, i. e. $0.a=0=a.0$ for all a of R,

3. $a.(b+c)=a.b+a.c$ and $(a+b).c=a.c+b.c$ for all a, b, c of R.

Semirings shall be denoted by their carrier sets.

Examples. 1. The Boolean semiring $\mathbb{B}=\{0, 1\}$ is determined by $1+1=1$.

2. The natural numbers $\mathbb{N}$ form a semiring with respect to common operations.

3. Let M be a monoid. The power set $P(M)$ of M is a semiring if addition and multiplication of two subsets A and B of M are defined by

$$A+B=A \cup B \qquad \text{(set-theoretic union)}$$

and

$$A.B=\{ab \mid a \in A \text{ and } b \in B\}.$$

The constants are $0=\varnothing$ and $1=\{e\}$ where e is the unit of M. If we take the free monoid X^* generated by an alphabet X, then we obtain the semiring $P(X^*)$ of all formal languages over X. □

Let R be a semiring and $m, n \in \mathbb{N}$. $R^{m \times n}$ denotes the set of all (finite) $m \times n$-matrices $A=(a_{ij})$ with $a_{ij} \in R$ for $i=1, \ldots, m$ and $j=1, \ldots, n$. Addition and multiplication are defined as usually. Let $A=(a_{ij})$ and $B=(b_{ij})$ be two $m \times n$-matrices. The sum is given by $A+B=(a_{ij}+b_{ij})$. Let $A=(a_{ij})$ be an $l \times m$-matrix and $B=(b_{ij})$ be an $m \times n$-matrix. The product is given by $AB=\left(\sum_{k=1}^{m} a_{ik}.b_{kj}\right)$. Obviously, $R^{n \times n}$ forms a semiring with the null matrix O_n and the identity matrix E_n as constant elements.

Any $m \times n$-matrix A can be partitioned into submatrices. Assume $m=m_1+\ldots+m_k$ for some $k>1$ and $n=n_1+\ldots+n_1$ for some $l>1$. The partition of A into submatrices $A_{ij} \in R^{m_i \times n_j}$ for $i=1, \ldots, k$ and $j=1, \ldots, l$ will be denoted by $A=(A_{ij})$.

The tensor product of an $k \times l$-matrix $A=(a_{ij})$ and an $m \times n$-matrix $B=(b_{ij})$ is an $k.m \times l.n$-matrix $A \otimes B$ defined by submatrices as follows $A \otimes B=(a_{ij}B)$. Notice $A \otimes B=(Ab_{ij})$.

As **infinite matrix** $\mathfrak{A}$ over a semiring R is defined to be a mapping

$$\mathfrak{A}: \quad \mathbb{N} \times \mathbb{N} \to R$$

written as usually $\mathfrak{A}=(a_{ij})$ with $a_{ij}=\mathfrak{A}(i,j)$ for $i,j\in\mathbb{N}$. $R^{\mathbb{N}\times\mathbb{N}}$ denotes the set of all infinite matrices over R. Sum and product of infinite matrices can be generalized in a straightforward way. Given two infinite matrices $\mathfrak{A}=(a_{ij})$ and $\mathfrak{B}=(b_{ij})$, the sum $\mathfrak{A}+\mathfrak{B}=(a_{ij}+b_{ij})$ is well-defined, while the product is in general only definable under some restrictions. If $\mathfrak{A}$ is row finite, i. e. in each of its rows there is only a finite number of nonzero coefficients, or $\mathfrak{B}$ is column finite, i. e. in each of its columns there is only a finite number of nonzero coefficients, then $\mathfrak{A}\mathfrak{B}=(\sum_{k\in\mathbb{N}} a_{ik}.b_{kj})$ is well-defined.

Every row and column finite infinite matrix $\mathfrak{A}$ can be represented in **Jacobi form**, that means there is a partition of $\mathfrak{A}$ into submatrices A_{ij}, $i,j\in\mathbb{N}$, such that

$$A_{ij}=0 \qquad \text{for} \quad |i-j|>1.$$

The set of all infinite matrices in Jacobi form shall be denoted by $(R)^{\mathbb{N}\times\mathbb{N}}$. Clearly, $(R)^{\mathbb{N}\times\mathbb{N}}$ forms a semiring with the infinite null matrix $\mathcal{O}$ and the infinite identity matrix $\mathfrak{E}$ as constant elements.

Let $\mathfrak{M}\in R^{\mathbb{N}\times\mathbb{N}}$. We denote by

$$\mathfrak{M}=\begin{vmatrix} A & \mathfrak{B} \\ \mathfrak{C} & \mathfrak{D} \end{vmatrix}$$

the partition of $\mathfrak{M}$ into submatrices where A is a finite quadratic matrix and $\mathfrak{D}$ is an infinite matrix. ($\mathfrak{B}$ resp. $\mathfrak{C}$ are infinite matrices but with only a finite number of rows resp. columns.)

Now we are going to describe infinite matrices in Jacobi form more detailed. Let $\mathfrak{M}\in(R)^{\mathbb{N}\times\mathbb{N}}$. By definition $\mathfrak{M}$ can be decomposed into finite submatrices $\mathfrak{M}=(M_{ij})$ such that $M_{ij}=0$ for $|i-j|>1$. Setting $A_n=M_{n,n}$, $B_n=M_{n,n+1}$ and $\bar{B}_n=M_{n+1,n}$ for $n\in\mathbb{N}$, $\mathfrak{M}$ can be represented as follows

$$\mathfrak{M}=\left|\begin{matrix} A_0 & B_0 & 0 & 0 & \cdots \\ \bar{B}_0 & A_1 & B_1 & 0 & \cdots \\ 0 & \bar{B}_1 & A_2 & B_2 & \cdots \\ 0 & 0 & \bar{B}_2 & A_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{matrix}\right.$$

Moreover, $\mathfrak{M}$ is the sum $\mathfrak{M}=\mathfrak{A}+\mathfrak{B}$ where $\mathfrak{A}$ is the quasidiagonal matrix

$$\mathfrak{A}=\left|\begin{matrix} A_0 & 0 & 0 & 0 & \dots \\ 0 & A_1 & 0 & 0 & \dots \\ 0 & 0 & A_2 & 0 & \dots \\ 0 & 0 & 0 & A_3 & \dots \\ \vdots & \vdots & \vdots & \vdots & \end{matrix}\right.$$

which may be regarded as an infinite direct sum $\mathfrak{A}=\bigoplus_{n\in\mathbb{N}} A_n$ of the finite matrices A_n, $n\in\mathbb{N}$, and $\mathfrak{B}$ is given by

$$\mathfrak{B}=\left|\begin{matrix} 0 & B_0 & 0 & 0 & \dots \\ \bar{B}_0 & 0 & B_1 & 0 & \dots \\ 0 & \bar{B}_1 & 0 & B_2 & \dots \\ 0 & 0 & \bar{B}_2 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \end{matrix}\right.$$

$\mathfrak{B}$ is equivalently defined by an infinite sequence $(\mathfrak{B}_n)_{n\in\mathbb{N}}$ of infinite matrices as follows $\mathfrak{B}=\mathfrak{B}_0$ and

$$\mathfrak{B}_n=\left|\begin{matrix} 0 & b_n \\ \bar{b}_n & \mathfrak{B}_{n+1} \end{matrix}\right|, \qquad n\in\mathbb{N}$$

where b_n is an infinite row vector with B_n at the first place and 0 else and $\bar{b}_n$ is an infinite column vector with $\bar{B}_n$ at the first place and 0 else.

If the infinite sequence $(\mathfrak{B}_n)$ is constant, i. e. $\mathfrak{B}=\mathfrak{B}_n$ for all $n\in\mathbb{N}$, then $\mathfrak{B}$ is called a Dyck matrix of order 1.

To define Dyck matrices of order $k>1$ some preparations are necessary. Let A be a finite matrix and $\mathfrak{M}$ be an infinite matrix in Jacobi form with $\mathfrak{M}=(M_{ij})$. The tensor product $A\otimes\mathfrak{M}$ is defined by $A\otimes\mathfrak{M}=(A\otimes M_{ij})$. Now, $\mathfrak{B}$ is said to be a **Dyck matrix of order** k, $k\geqq 1$, if $\mathfrak{B}_{n+1}=E_k\otimes\mathfrak{B}_n$ for all $n\in\mathbb{N}$, where E_k is the identity $k\times k$-matrix. A Dyck matrix is completely determined by $B=B_0$ and $\bar{B}=\bar{B}_0$. Therefore, we will write $\mathfrak{B}=\Delta_k(B,\bar{B})$.

Example. Given a set X, we consider the matrix $\mathfrak{B}$ over the semiring $P(X^*)$ of formal languages over $X=\{a, \bar{a}\}$:

$$\mathfrak{B}=\left|\begin{array}{ccccc} 0 & a & 0 & 0 & \ldots \\ \bar{a} & 0 & a & 0 & \ldots \\ 0 & \bar{a} & 0 & a & \ldots \\ 0 & 0 & \bar{a} & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \end{array}\right.$$

$\mathfrak{B}$ is a Dyck matrix of order 1, where a and $\bar{a}$ are regarded as 1×1-matrices. Latter on we will see that $\mathfrak{B}^*$ exists and its coefficient at the place (0,0) equals the Dyck language of order 1 with a and $\bar{a}$ as generators. □

3. ITERATION OF INFINITE MATRICES

A semiring R is called **iterative** if R is additionally equipped with a partial unary operation * satisfying the following conditions:

1. Let $r \in R$. If r^* exists, then $r^* = 1 + r . r^*$.
2. Assume that $s \in R$ is a solution of the equation $x = a + r . x$ for $a, r \in R$, that means $s = a + r . s$. Then r^* exists and $s = r^* . a$ holds.

In case r^* exists for a given element r of R, r^* is said to be the **iteration** of r.

Since for a semiring R the set of all finite $n \times n$-matrices over R forms a semiring too, the notion of iteration for any finite square matrix is well defined now. Notice that $R^{n \times n}$ is iterative whenever R is iterative which may be proved by Conway's method [2]. In contrast to this finite case the set of all infinite matrices over R does not form a semiring. But nevertheless we will also say without any further explanation that an infinite matrix $\mathfrak{A}^*$ is the iteration of a given infinite matrix $\mathfrak{A}$ if $\mathfrak{A}^* = \mathfrak{E} + \mathfrak{A}\mathfrak{A}^*$ and $\mathfrak{B} = \mathfrak{E} + \mathfrak{A}\mathfrak{B}$ implies $\mathfrak{B} = \mathfrak{A}^*$. Observe that necessarily $\mathfrak{A}$ must be row finite.

By an easy calculation one proves the following propositions.

PROPOSITION 1: *For infinite matrices $\mathfrak{A}$ and $\mathfrak{B}$ in Jacobi form*

1. $(\mathfrak{A} + \mathfrak{B})^* = \mathfrak{A}^* (\mathfrak{B}\mathfrak{A}^*)^*$
2. $(\mathfrak{A}\mathfrak{B})^* = \mathfrak{E} + \mathfrak{A} (\mathfrak{B}\mathfrak{A})^* \mathfrak{B}$

hold if one of the both sides of each equation exists. □

Generalizing Conway Theorem [2] for finite matrices one gets

PROPOSITION 2: *Let $\mathfrak{M}$ be an infinite matrix in Jacobi form partitioned as follows*

$$\mathfrak{M}=\begin{vmatrix} A & \mathfrak{B} \\ \mathfrak{C} & \mathfrak{D} \end{vmatrix}$$

where A is a finite square matrix. Then

$$\mathfrak{M}^*=\begin{vmatrix} (A+\mathfrak{B}\mathfrak{D}^*\mathfrak{C})^* & (A+\mathfrak{B}\mathfrak{D}^*\mathfrak{C})^*\mathfrak{B}\mathfrak{D}^* \\ \mathfrak{D}^*\mathfrak{C}(A+\mathfrak{B}\mathfrak{D}^*\mathfrak{C})^* & (\mathfrak{D}+\mathfrak{C}A^*\mathfrak{B})^* \end{vmatrix}$$

if all occuring iterations on the right hand side exist. □

If $\mathfrak{M}$ is an infinite matrix in Jacobi form, then $\mathfrak{M}=\mathfrak{A}+\mathfrak{B}$ with $\mathfrak{A}=\bigoplus_{n\in\mathbb{N}} A_n$ and $\mathfrak{B}$ is recursively defined by $\mathfrak{B}=\mathfrak{B}_0$ and

$$\mathfrak{B}_n=\begin{vmatrix} 0 & b_n \\ \bar{b}_n & \mathfrak{B}_{n+1} \end{vmatrix}, \qquad n\in\mathbb{N}$$

By Proposition 1, $\mathfrak{M}^*=\mathfrak{A}^*(\mathfrak{B}\mathfrak{A})^*$. Assume that all A_n^* exist, then $\mathfrak{A}^*$ exists too and $\mathfrak{A}^*=\bigoplus_{n\in\mathbb{N}} A_n^*$. Hence, it remains to consider the iteration of $\mathfrak{B}\mathfrak{A}^*$ in order to iterate $\mathfrak{M}$.

Thus the iteration of $\mathfrak{B}$ shall be studied firstly.

PROPOSITION 3: *Let $\mathfrak{B}$ be defined as above. If there is an infinite sequence $(D_n)_{n\in\mathbb{N}}$ of finite square matrices such that*

1. *D_n^* exists*
2. *$D_n=B_n D_{n+1}^*\bar{B}_n$*

for all $n\in\mathbb{N}$, then $\mathfrak{B}^*$ exists.

Proof: Define $\mathfrak{B}^*=(B_{m,n}^{(*)})$ as follows

$$B_{0,0}^{(*)}=D_0^*$$

$$B_{n,n}^{(*)}=D_n^*+D_n^*\bar{B}_{n-1}B_{n-1,n-1}^{(*)}B_{n-1}D_n^* \qquad \text{for} \quad n\geqq 1$$

$$B_{m,m+n}^{(*)}=B_{m,m}^{(*)}B_m D_{m+1}^*\ldots B_{m+n-1}D_{m+n}^* \qquad \text{for} \quad m,n\geqq 0$$

$$B_{m+n,m}^{(*)}=D_{m+n}^*\bar{B}_{m+n-1}\ldots D_{m+1}^*\bar{B}_m B_{m,m}^{(*)} \qquad \text{for} \quad m,n\geqq 0.$$

Under the assumptions (1) and (2) we may show by induction that $\mathfrak{B}^*$ fulfils the equation $\mathfrak{B}^*=\mathfrak{E}+\mathfrak{B}\mathfrak{B}^*$. □

As a conclusion we derive an existence criterion for the iteration of an infinite matrix $\mathfrak{M}$ in Jacobi form represented as follows

$$\mathfrak{M}=\left|\begin{array}{lllll} A_0 & B_0 & 0 & 0 & \dots \\ \bar{B}_0 & A_1 & B_1 & 0 & \dots \\ 0 & \bar{B}_1 & A_2 & B_2 & \dots \\ 0 & 0 & \bar{B}_2 & A_3 & \dots \\ \vdots & \vdots & \vdots & \vdots & \end{array}\right.$$

THEOREM 4: *Let $\mathfrak{M}$ be an infinite matrix in Jacobi form presented as above. The iteration $\mathfrak{M}^*$ of $\mathfrak{M}$ exists if*

1. *A_n^* exists for all $n \in \mathbb{N}$.*

There is an infinite sequence $(D_n)_{n \in \mathbb{N}}$ of finite square matrices D_n such that

2. *D_n^* exists for all $n \in \mathbb{N}$ and*
3. *$D_n = B_n A_{n+1} D_{n+1}^* \bar{B}_n A_n$ for all $n \in \mathbb{N}$.* □

We are now going to investigate the iteration of Dyck matrices. Let $\mathfrak{B} = \Delta_k(B, \bar{B})$ be a Dyck matrix of order k. Since, by definition, B belongs to $R^{n \times n \cdot k}$ and $\bar{B}$ belongs to $R^{n \cdot k \times n}$ for some $n \geqq 1$, both matrices B and $\bar{B}$ can be decomposed into k quadratic submatrices $A_1, \dots, A_k$ and $\bar{A}_1, \dots, \bar{A}_k$, respectively. This shall be expressed by $\mathfrak{B} = \Delta_k(A_1, \dots, A_k, \bar{A}_1, \dots, \bar{A}_k)$.

THEOREM 5: *Let $\mathfrak{B} = \Delta_k(A_1, \dots, A_k, \bar{A}_1, \dots, \bar{A}_k)$ be a Dyck matrix of order k. If the matrix equation*

$$X = E_n + \sum_{\varkappa=1}^{k} A_\varkappa X \bar{A}_\varkappa X \qquad (*)$$

has a unique solution, then $\mathfrak{B}^*$ exists.

Proof: Let $\mathfrak{B} = \Delta_k(A_1, \dots, A_k, \bar{A}_1, \dots, \bar{A}_k)$ be a Dyck matrix of order k. Then, by definition

$$\mathfrak{B}=\left|\begin{array}{lllll} 0 & B_0 & 0 & 0 & \dots \\ \bar{B}_0 & 0 & B_1 & 0 & \dots \\ 0 & \bar{B}_1 & 0 & B_2 & \dots \\ 0 & 0 & \bar{B}_2 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \end{array}\right.$$

with $B_{n+1} = E_k \otimes B_n$ and $\bar{B}_{n+1} = E_k \otimes \bar{B}_n$ for all $n \in \mathbb{N}$ and $B_0 = (A_1, \dots, A_k)$ as well as $\bar{B}_0 = (\bar{A}_1, \dots, \bar{A}_k)^T$.

Assume that the equation (*) has a unique solution D^*. Setting

$$D = \sum_{\varkappa=1}^{k} A_\varkappa D^* \bar{A}_\varkappa$$

we get $D^* = E_n + DD^*$.

Now define an infinite sequence $(D_n)_{n \in \mathbb{N}}$ of finite square matrices D_n as follows

$$D_0 = D$$

$$D_{n+1} = E_k \otimes D_n \qquad \text{for all} \quad n \in \mathbb{N}.$$

By means of Proposition 3 we will prove that $\mathfrak{B}^*$ exists. Taking into account

$$D^*_{n+1} = E_k \otimes D^*_n \qquad \text{for all} \quad n \in \mathbb{N}$$

we easily see that assumption (1) of Proposition 3

$$D^*_n \text{ exists for all } n \in \mathbb{N}$$

is fulfilled.

It remains to verify, by induction over n, that assumption (2) of Proposition 3

$$D_n = B_n D^*_{n+1} \bar{B}_n \qquad \text{for all} \quad n \in \mathbb{N}$$

is satisfied too.

For $n=0$ we have

$$\begin{aligned} D_0 &= \sum_{\varkappa=1}^{k} A_\varkappa D^*_0 \bar{A}_\varkappa \\ &= B_0 (E_k \otimes D^*_0) \bar{B}_0 \\ &= B_0 D^*_1 \bar{B}_0. \end{aligned}$$

Induction step: Assume $D_{n-1} = B_{n-1} D^*_n B_{n-1}$.
An easy calculation yields

$$\begin{aligned} D_n &= E_k \otimes D_{n-1} \\ &= E_k \otimes (B_{n-1} D^*_n \bar{B}_{n-1}) \\ &= (E_k \otimes B_{n-1})(E_k \otimes D^*_n)(E_k \otimes B_{n-1}) \\ &= B_n D^*_{n+1} \bar{B}_n. \end{aligned}$$

Hence, by Proposition 3, $\mathfrak{B}^*$ exists. □

Observe that the coefficients of $\mathfrak{B}^*$ can effectively be computed by means of Proposition 3 as indicated in the proof.

Remark: Let us consider the semiring of formal languages. If the matrices $A_1, \ldots, A_k, \bar{A}_1, \ldots, \bar{A}_k$ reduces to single letters $a_1, \ldots, a_k, \bar{a}_1, \ldots, \bar{a}_k$, then the solution of the equation (*) equals the Dyck languages of order k (*cf.* [1]).

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