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SEQUENTIAL MAPPINGS OF ω-LANGUAGES (*)

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Abstract. – *The present paper investigates how partial word-functions monotonic with respect to the initial word relation can be extended to partial mappings on the space X^ω of infinite sequences. It turns out that, to a certain extent, the infinite behaviour of these mappings can be described by the finite behaviour via two suitably defined limit-operators. We derive necessary and sufficient conditions for the validity of several translation formulae. Moreover, we investigate the special case of agsm-mappings and give an application to closure properties of families of ω-languages.*

Résumé. – *Cet article examine comment les fonctions partielles définies sur les mots et monotones par rapport à l'ordre préfixiel peuvent être étendues en des fonctions définies sur les mots infinis. Il apparaît que dans une certaine mesure, ces extensions peuvent être définies en utilisant deux types de limites. Nous donnons des conditions nécessaires et suffisantes pour la validité de formules définissant ces extensions. Nous étudions aussi le cas particulier des transductions séquentielles et nous en donnons une application aux propriétés de clôture des familles de ω-langages.*

0. INTRODUCTION

In the study of families of languages or ω-languages it is often useful to consider transductions, and it is important to have a certain scheme for extending transductions of languages to transductions of ω-languages if one deals with the joint study of families of languages and ω-languages (*cf.* [SW 1, BN]). A particular simply to extend case of transductions are sequential mappings, important subcases of which are *gsm*-mappings and processes (i. e. partial recursive functions monotonic with respect to the initial word relation).

The extension of *gsm*-mappings to the case of ω-languages has been used e. g. to derive subhierarchies of regular ω-languages [Wa 2] and results relating context-free languages to ω-languages [BN].

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In connection with recursive ω-languages [SW 1, St 4] the notion of recursive operator [Wa 1] is of some importance. It has been shown that recursive operators are simple limit-extensions of recursive functions monotonic with respect to the initial word relation [St 4]. The more general case of the above mentioned processes has been considered in connection with questions of program complexity and randomness of infinite sequences [Sc 1, 2]. In this paper we investigate the limit extension of general sequential mappings (i. e. partial functions monotonic w. r. t. the initial word relation). The extension of a sequential mapping φ will be denoted by $\bar{\varphi}$ and also called a sequential mapping. (It will be always clear from the notation or the context which kind of sequential mapping we have in mind.) In an earlier work [LS] we have considered in detail totally unbounded sequential mappings φ (i. e. fully defined mappings for which the limit extension $\bar{\varphi}$ is also fully defined). It has been shown that in this very case the infinite behaviour of the sequential mapping can be derived from the finite behaviour by means of the limit-operator ls via the translation formulae [St 1] (*cf.* also [LS]; and [BN] where this operator is called adherence Adh) :

$$\bar{\varphi}(\operatorname{ls} W)=\operatorname{ls} \varphi(W) \tag{I}$$

and

$$\bar{\varphi}^{-1}(\operatorname{ls} V)=\operatorname{ls} \varphi^{-1}(A(V)), \tag{II}$$

where W and V are arbitrary languages.

However, in the cases where the sequential mapping φ is not fully defined a translation of the finite behaviour to the infinite one by means of the ls-operator fails for topological reasons: the mappings $\bar{\varphi}$ need not be continuous, and the equations (I) and (II) imply to a certain kind the continuity of the mapping $\bar{\varphi}$. Thus it is impossible in the case of an arbitrary sequential mapping to obtain the translation formulae (I) and (II). It turns out that in this general case the δ-limit and a generalized inverse are useful tools for translating the behaviour of the sequential mapping φ to its counterpart $\bar{\varphi}$. In the case of translation formulae involving the δ-limit we have to confine to the case of equation (II), since G_δ-sets in the space X^ω are involved and it is well known that even simple projections of G_δ-sets may not belong to the Borel-hierarchy (*cf.* [Ku, St 3]). Basing on the translation formulae involving the ls-limit we are interested in the cases of validity of the translation formulae (I) and (II). Since we are concerned with not necessarily fully defined sequential mappings, in the sequel we consider equation (I) in the following slightly modified form

$$\bar{\varphi}(\mathrm{ls}\, W) = ls\, \varphi(A(W)) \tag{I}$$

In the final two sections we return to a point mentioned above. First we study the relations between a sequential mapping φ and its conterpart $\bar{\varphi}$ in the case when φ is an agsm-mapping. Then we apply these results to closure properties of families of ω-languages which are based on the closure properties of their underlying families of languages.

1. PRELIMINARIES

The set $\{0, 1, 2, \ldots\}$ of natural numbers is denoted by N, and for a finite alphabet X^* (X^ω) denotes the set of finite words (infinite sequences) on X. For a word $w \in X^*$ and a string $b \in X^* \cup X^\omega$ let wb be their concatenation. This in an obvious way defines a product $W \cdot B$ of sets $W \subseteq X^*$ and $B \subseteq X^* \cup X^\omega$. We introduce into $X^* \cup X^\omega$ a partial ordering

$$w \sqsubseteq b: \quad \Leftrightarrow \quad w \cdot b' = b \quad \text{for some } b' \in X^* \cup X^\omega.$$

By

$$A(b) := \{w : w \in X^* \text{ and } w \sqsubseteq b\}$$

and

$$A(B) := \bigcup_{b \in B} A(b)$$

we denote the set of initial words of $b \in X^* \cup X^\omega$ and $B \subseteq X^* \cup X^\omega$ resp.

For a word w its length is $|w|$, and $X^n := \{w : W \in X^* \text{ and } |w| = n\}$ where n is a natural number. By e we denote the empty word in X^*. We extend the operations * and $^\omega$ to arbitrary subsets $W \subseteq X^*$ in the usual way: $W^* := \bigcup_{n \in N} W^n$ where $W^0 := \{e\}$, and $W^\omega := \{w_0 \cdot w_1 \cdot \ldots \cdot w_i \cdot \ldots : i \in N$ and $w_i \in W \setminus \{e\}\}$ is the set of (infinite) sequences in X^ω formed by concatenating members of W.

We will refer to subsets of X^* (X^ω) as languages (ω-languages).

In X^ω we will consider the (natural) product topology which is defined by the basis $\{w \cdot W^\omega : w \in X^*\}$ or otherwise by the closure operator $\mathbf{C}$, where for

$F \subseteq X^\omega$

$$\mathbf{C}(F):=\{\beta : A(\beta) \subseteq A(F)\}$$

is the smallest closed set containing the set F.

As usual we define the Borel hierarchy in $X^\omega : G_\delta(F_\sigma)$ is the class of denumerable intersections (unions) of open (closed) sets. Then $G_{\delta\sigma}$, $G_{\delta\sigma\delta}$, ... and $F_{\sigma\delta}$, $F_{\delta\sigma\delta}$, ... and $F_{\sigma\delta}$, $F_{\sigma\delta\sigma}$ are defined in the usual manner.

Closed sets and G_δ-sets in X^ω can be characterized by languages in X^* using the following limits

Definition: For $W \subseteq X^*$ we will refer to

$$\operatorname{ls} W:=\{\beta : \beta \in X^\omega \text{ and } A(\beta) \subseteq A(W)\}$$

as the limit (in [BN]: adherence) of the language (*cf.* [St 1, LS]), and to

$$W^\delta :=\{\beta : \beta \in X^\omega \text{ and } A(\beta) \cap W \text{ infinite}\}$$

as the δ-limit of the languages (*cf.* [Da]).

The following properties of the operators ls and $^\delta$ are easily derived (*cf.* [LS]):

$$W^\delta \subseteq A(W)^\delta = \operatorname{ls} W = \operatorname{ls} A(W) \tag{1 a}$$

$$A(W^\delta) \subseteq A(\operatorname{ls} W) \subseteq A(W) \tag{1 b}$$

$$\operatorname{ls} W \cup \operatorname{ls} V = \operatorname{ls}(W \cup V) \tag{1 c}$$

$$\operatorname{ls} W \cap \operatorname{ls} V = \operatorname{ls}(A(W) \cap A(V)). \tag{1 d}$$

Property 1 : *If* $\beta \in \operatorname{ls} W$, *then there is a subset* $U \subseteq W$ *such that* $\{\beta\} = \operatorname{ls} U$.

Property 2 : *Let* X *be finite,* $W \subseteq X^*$. *Then* ls $W = \emptyset$ *iff* W *is finite.*

For $F \subseteq X^\omega$ the closure $\mathbf{C}(F)$ equals (*cf.* [LS])

$$\mathbf{C}(F) = \operatorname{ls} A(F) = A(F)^\delta. \tag{2}$$

Moreover, we have

Proposition 3 [Da] if: *A subset* $F \subseteq X^\omega$ *is a* G_δ*-set iff there is a* $W \subseteq X^*$ *such that* $F = W^\delta$.

With equation (1) we obtain

Corollary 4: *A subset* $F \subseteq X^\omega$ *is closed iff there is a* $W \subseteq X^*$ *such that* $F = \operatorname{ls} W$.

2. SEQUENTIAL MAPPINGS

Troughout this paper let X and Y be finite alphabets containing at least two letters each.

We regard the initial word relation « $\sqsubseteq$ » in X^* and Y^*. A mapping $\varphi : X^* \to Y^*$ is called sequential iff φ is monotonic with respect to « $\sqsubseteq$ », i. e. for words w, v in its domain dom (φ), $w \subseteq v$ implies $\varphi(w) \sqsubseteq \varphi(v)$.

Every sequential mapping $\varphi: X^* \to Y^*$ yields a sequential mapping $\bar{\varphi}$: $X^\omega \to Y^\omega$ the domain of which is

$$\operatorname{dom}(\bar{\varphi}) = \{\beta : \varphi(A(\beta)) \text{ is infinite}\} \tag{3}$$

and the values of which are given by

$$A(\bar{\varphi}(\beta)) = A(\varphi(A(\beta))) \quad \text{for} \quad \beta \in \operatorname{dom}(\bar{\varphi}) \tag{4}$$

Utilizing the δ-limit introduced above we may define $\bar{\varphi}$ alternatively via

$$\{\bar{\varphi}(\beta)\} = \varphi(A(\beta))^\delta \quad \text{when} \quad \beta \in X^\omega \tag{5}$$

(where $\{\bar{\varphi}(\beta)\} = \varnothing$ means that $\bar{\varphi}(\beta)$ is not defined).

In the sequel we shall often make use of the following property of sequential mappings.

PROPERTY 5 : *If* $U \subseteq \operatorname{dom}(\varphi) \cap A(\beta)$ *is infinite and* $\beta \in \operatorname{dom}(\bar{\varphi})$ *then* $\varphi(U)$ *is also infinite.*

Next, we introduce the upper quasiinverse $\mathbf{U}_\varphi$ of a sequential mapping φ.

To this end let Min W: $= W \setminus W \cdot X \cdot X^*$ be the set of all minimal elements with respect to « $\sqsubseteq$ » of $W \subseteq X^*$, and for $v \in Y^*$ and $V \subseteq Y^*$ we set

$$\mathbf{U}_\varphi(v) : = \operatorname{Min}\{w : w \in \operatorname{dom}(\varphi) \text{ and } v \sqsubseteq \varphi(w)\},$$

and

$$\mathbf{U}_\varphi(V) : = \bigcup_{v \in V} \mathbf{U}_\varphi(v).$$

If $w \in \mathbf{U}_\varphi(v)$ i. e. $v \sqsubseteq \varphi(w)$ and $v \not\sqsubseteq \varphi(w')$ for every $w' \sqsubset w$, then we will call w a φ-least upper bound (φ — l. u. b.) on v. One easily obtains the following equation.

$$\mathbf{U}_{\varphi}(Y^*)=\{w : w \in \operatorname{dom}(\varphi) \text{ and } \varphi u) \neq \varphi(w) \text{ for all } u \sqsubset w\} \tag{6}$$

In contrast to the inverse mapping φ^{-1} the upper quasiinverse $\mathbf{U}_{\varphi}$ has the following properties.

PROPERTY 6 : There is a $u \in \mathbf{U}_{\varphi}(v)$ such that

$$u \sqsubseteq w \text{ provided } v \sqsubseteq \varphi(w).$$

This property implies that a word $v \in Y^*$ has a φ-l. u. b. iff $v \in A(\varphi(X^*))$.

PROPERTY 7. : *Let* $w \in \mathbf{U}_{\varphi}(v)$ *and* $w' \in \mathbf{U}_{\varphi}(v')$.

Then $w \sqsubset w'$ *implies* $v \sqsubseteq \varphi(w) \sqsubset v' \sqsubseteq \varphi(w')$.

These two properties establish that the correspondence

$$\varphi\colon\ A(\beta) \cap \mathbf{U}_{\varphi}(Y^*) \to \varphi(A(\beta))$$

is one-to-ono and onto for every $\beta \in X^{\omega}$. The following theorem relates the upper quasiinverse $\mathbf{U}_{\varphi}$ of a sequential mapping φ to the inverse $\bar{\varphi}^{-1}$.

THEOREM 8: *Let* φ *be a sequential mapping and* $V \subseteq Y^*$.
Then

$$\bar{\varphi}^{-1}(V^{\delta})=\mathbf{U}_{\varphi}(V)^{\delta}.$$

Proof: If $\beta \in \mathbf{U}_{\varphi}(\mathrm{V})^{\delta}$ then there is an infinite family $\{w_i : i \in N\} \subseteq A(\beta)$ such that every w_i is a φ-l. u. b. on some word $v_i \in V$.

Property 7 shows that $w_i \sqsubset w_j$ implies $v_i \sqsubset v_j$. Thus $\{v_i : i \in N\}$ is an infinite family in $A(\varphi(A(\beta))) \cap V$, which proves $\bar{\varphi}(\beta) \in V^{\delta}$. Conversely, let $\bar{\varphi}(\beta) \in V^{\delta}$, i.e. there is an infinite family $\{v_i : i \in N\} \subseteq A(\varphi(A(\beta))) \cap V$. According to Property 6 we consider for each v_i a φ-l. u. b. w_i in $A(\beta)$. If $|v_j| > |\varphi(w_i)|$ then the corresponding w_j satisfies $w_i \sqsubset w_j$. Hence, $\{w_i : i \in N\}$ is an infinite family of φ-l. u. b.s on words in V contained in $A(\beta)$, which implies $\beta \in \mathbf{U}_{\varphi}(V)^{\delta}$. □

Inserting $V=Y^*$ yields a characterization of the domain of

$$\operatorname{dom}(\bar{\varphi})=\varphi^{-1}(Y^{\omega})=\mathbf{U}_{\varphi}(Y^*). \tag{7}$$

Though sequential mappings $\bar{\varphi} : X^{\omega} \to Y^{\omega}$ in general are not continuous, their inverses preserve most classes of the Borel hierarchy. By theorem 8 $\bar{\varphi}^{-1}$

preserves G_δ-sets. Since

$$\bar{\varphi}^{-1}\Big(\bigcup_{i \in N} F_i\Big) = \bigcup_{i \in N} \bar{\varphi}^{-1}(F_i)$$

and

$$\bar{\varphi}^{-1}\Big(\bigcap_{i \in N} F_i\Big) = \bigcap_{i \in N} \bar{\varphi}^{-1}(F_i),$$

the mapping $\bar{\varphi}^{-1}$ preserves also $G_{\delta\sigma}$-, $G_{\delta\varphi\delta}$-, ...-sets. More-over $\operatorname{dom}(\bar{\varphi})$ is a G_δ-set, and since

$$\bar{\varphi}^{-1}(X^\omega \setminus F) = \operatorname{dom}(\bar{\varphi}) \setminus \bar{\varphi}^{-1}(F), \tag{8}$$

$\bar{\varphi}^{-1}$ preserves also $F_{\sigma\delta}$-, $F_{\sigma\delta\sigma}$-, ...-sets.

The following example shows that we cannot do any better.

Example 1 : Let $X = Y = \{0, 1\}$ and let $h: X^* \to Y^*$ be a homomorphism defined by $h(0) = 00$ and $h(1) = e$. Then $\bar{h}^{-1}(Y^\omega) = (X^* \cdot \{0\})^\omega$ is a G_δ-set but not an F_σ-set, for $X^\omega \setminus (X^* \cdot \{0\})^\omega = X^* \cdot \{1\}^\omega$ is a countable and dense in itself set, hence no G_δ-set (*cf.* [Ku]). □

For F_σ-sets equations (7) and (8) shows that the following is true.

PROPERTY 9 : *Let* φ *be a sequential mapping. The inverse* $\bar{\varphi}^{-1}$ *preserves* F_σ-*sets if and only if* $\operatorname{dom}(\bar{\varphi})$ *itself is an* F_σ-*set.*

It arises the question, under which conditions the inverse $\bar{\varphi}^{-1}$ preserves open and/or closed sets and when $\bar{\varphi}$ is extendable to a continuous mapping $\Phi: X^\omega \to Y^\omega$. We are going to answer these questions in the subsequent sections.

This section is finished by a proof that $\bar{\varphi}: \operatorname{dom}(\bar{\varphi}) \to Y^\omega$ is a continuous mapping, when we use an appropriately chosen natural topology in $\operatorname{dom}(\bar{\varphi})$, the topology in Y^ω being the same as indicated in section 1.

To this end we define for a subset $U \subseteq X^*$ the following metric ρ_U in X^ω:

$$\rho_U(\beta, \xi) := \begin{cases} 0, & \text{if } \beta = \xi \\ 2^{1 - \operatorname{card}(A(\beta) \cap A(\xi) \cap U)}, & \text{if } \beta \neq \xi. \end{cases}$$

We mention that in case $U = X^*$ the metric $\rho := \rho_{X^*}$ is a standard metric inducing the usual product topology in X^ω (*cf.* [BN]).

Clearly, $\operatorname{card} A(\beta) \cap U = n < \infty$ implies $\rho_U(\beta, \xi) \geqq 2^{1-n}$ for all $\xi \in X^\omega$, $\xi \neq \beta$. Hence, $\beta \notin U^\delta$ implies that β is an isolated point in the space (X^ω, ρ_U). Consequently, U^δ is closed in (X^ω, ρ_U).

Now we can prove the announced theorem.

THEOREM 10: *Let* φ *be a sequential mapping, and let* $U^\delta = \mathrm{dom}(\bar{\varphi})$. *Then* $\bar{\varphi} : (\mathrm{dom}(\bar{\varphi}), \rho_U) \to (Y^\omega, \rho)$ *is a continuous mapping.*

Proof: Let $\beta \in \mathbf{U}^\delta = U_\varphi(Y^*)^\delta$, and let $\varepsilon = 2^{1-n}$. We choose a word $w \in A(\beta)$ such that $\mathrm{card}(A(w) \cap \mathbf{U}_\varphi(Y^*)) = n$. Since every $\xi \in U^\delta$ with $\rho_U(\xi, \beta) \leqq 2^{-|w|}$ has at least $|w| + 1$ initial words in $A(\beta) \cap U$, we have $A(\beta) \cap A(\xi) \supseteqq A(w)$. From this inequality we obtain via equation (4)

$$A(\bar{\varphi}(\beta)) \cap A(\bar{\varphi}(\xi)) \supseteqq \varphi(A(\beta)) \cap \varphi(A(\xi)) \supseteqq \varphi(A(w)).$$

Now, property 7 yields

$$\mathrm{card}\,\varphi(A(w)) = \mathrm{card}(A(w) \cap U_\varphi(Y^*)). \tag{9}$$

Thus, $\rho(\bar{\varphi}(\beta), \bar{\varphi}(\xi)) \leqq \varepsilon$ whenever $\rho_U(\beta, \xi) \leqq 2^{-|w|}$, and the assertion is proved. □

3. INVERSE MAPPINGS AND THE EQUATION (II)

The aim of this section is to show a direct relation betwen $\bar{\varphi}^{-1}$ and φ^{-1} instead of $\mathbf{U}_\varphi$ as in the preceding section. Furthermore, we derive a necessary and sufficient condition under which the equation (II) holds true. To this end we introduce a special class of subsets of X^* and investigate in certain cases the functional equations relating $\bar{\varphi}^{-1}$ and φ^{-1} together. First we derive an auxiliary result.

LEMMA 11: *Let* $\beta \in X^\omega$ *and* $V \subseteqq Y^*$. *Then* $\varphi(A(\beta)) \cap V$ *is infinite if and only if* $A(\beta) \cap \varphi^{-1}(V)$ *and* $\varphi(A(\beta))$ *both are infinite.*

Proof: The relation $\varphi(A(\beta) \cap \varphi^{-1}(V)) = \varphi(A(\beta)) \cap V \subseteqq \varphi(A(\beta))$ makes the only-if-part obvious. Now let $\varphi(A(\beta))$ and $A(\beta) \cap \varphi^{-1}(V)$ be infinite. Since $A(\beta) \cap \varphi^{-1}(V) \subseteqq \mathrm{dom}(\varphi)$ we can apply property 5 and obtain that $\varphi(A(\beta) \cap \varphi^{-1}(V)) = \varphi(A(\beta)) \cap V$ is infinite. □

As a consequence of lemma 11 we obtain an inequality being the first step to the investigation of equation (II). To this end, we consider the equations

$$\varphi^{-1}(V)^\delta \cap \mathrm{dom}(\bar{\varphi}) = \{\beta : \varphi(A(\beta)) \cap V \text{ is infinite}\}, \tag{10}$$

which is a consequence of lemma 11, and

$$\bar{\varphi}^{-1}(V^\delta) = \{\beta : A(\varphi(A(\beta))) \cap V \text{ is infinite}\}, \tag{11}$$

which is immediate by the definitions of $\bar{\varphi}$ and the δ-limit. Equations (10) and (11) yield the following one:

$$\bar{\varphi}^{-1}(V^{\delta}) \supseteq \varphi^{-1}(V)^{\delta} \cap \operatorname{dom}(\bar{\varphi}). \tag{12}$$

Inclusion in equation (12) may be proper as the following example shows.

Example 1 (continued): Consider $W=\{00\}^*\cdot\{0\}$. Then $W \cap h(X^*)=\varnothing$. Hence $h^{-1}(W)^{\delta}=\varnothing$, but $W^{\delta}=\{0\}^{\omega}=\bar{h}(X^{\omega})$ which implies

$$\bar{h}^{-1}(W^{\delta})=\operatorname{dom}(\bar{h})=(X^*\cdot\{0\})^{\omega}.$$

On the other hand, if we consider $U:=\{00\}^*=h(X^*)$, we have $X^{\omega}=h^{-1}(U)^{\delta\frac{1}{2}}$ $\supset \bar{h}^{-1}(U^{\delta})=\operatorname{dom}(\bar{h})$. □

This example might lead to conclusion, that $\varphi^{-1}(V)^{\delta} \supseteq \bar{\varphi}^{-1}(V^{\delta})$ holds provided only $V \subseteq \varphi(X^*)$. This need not be true as the following example shows.

Example 2: Let $Y:=X:=\{0, 1\}$, and h be the (doubling) homomorphism defined by $h(0):=00$, $h(1):=11$, and define φ via

$$\varphi(e):=\varphi(0):=\varphi(1):=e$$
$$\varphi(0w):=h(w)$$
$$\varphi(1wx):=h(w)x \qquad \text{where} \quad w \in X^*, x \in X.$$

Then $\varphi^{-1}((X^2)^*)^{\delta}=0\cdot X^{\omega}$ whereas

$$\bar{\varphi}^{-1}(((X^2)^*)^{\delta})=\bar{\varphi}^{-1}(X^{\omega})=X^{\omega}. \quad \square$$

Next, we exhibit a class of languages for which in equation (12) equality holds. To this end we derive some properties of the δ-limit.

$$(U \cup W)^{\delta}=U^{\delta} \cup W^{\delta}. \tag{13}$$

From this identity one easily obtains $(U \cap W)^{\delta} \subseteq U^{\delta} \cap W^{\delta}$ as well as $(U \cup W)^{\delta}=U^{\delta}=(U \setminus W)^{\delta}$ provided $W^{\delta}=\varnothing$.

DEFINITION: We will refer to a language as a (σ, δ)-subset of X^* iff for every $\beta \in X^{\omega}$ either $A(\beta) \cap W$ or $A(\beta) \setminus W$ is finite. Clearly, this condition is equivalent to $W^{\delta} \cap (X^* \setminus W)^{\delta}=\varnothing$, which in turn implies that the complement of a (σ, δ)-subset is again a (σ, δ)-subset.

Examples of (σ, δ)-subsets are finite languages (and their complements). Further examples are provided by the languages of the forms $A(U)$ and $W\cdot X^*$. In [St2] those languages were termed closed and open languages, respectively, for their images under δ-limit $A(U)^\delta$ and $(W \cdot X^*)^\delta = W \cdot X^\omega$ are exactly the closed (*cf.* equation (1 *a*) and corollary 4) and open subsets of X^ω.

The following lemma explains the term (σ, δ)-subset.

LEMMA 12 [St2]: *A subset $F \subseteq X^\omega$ is simultaneously an F_σ- and a G_δ-set if and only if there is a (σ, δ)-subset W of X^* such that $F = W^\delta$.*

Proof: Let W be a (σ, δ)-subset of X^*. Then $W^\delta \cap (X^* \setminus W)^\delta = \emptyset$ and $W^\delta \cup (X^* \setminus W)^\sigma = X^\omega$. Consequently, $W^\delta = X^\omega \setminus (X^* \setminus W)^\delta$ is also an F_σ-set.

In order to prove the only-if-part, according to proposition 3 we assume $F = W'^\delta$ and $X^\omega \setminus F = U^\delta$ for appropriately chosen subsets W', $U \subseteq X^*$. The above derived properties of the δ-limit allow us to assume $e \in W'$ and $W' \cap U = \emptyset$, for $(W' \cap U)^\delta \subseteq W'^\delta \cap U^\delta = \emptyset$. Now, we add to the language W' for each $w' \in W'$ all its successors with respect to "$\sqsubseteq$" up to the time we meet a word in U, i. e. we define

$$W := \{w : w \in X^* \text{ and there is a } w' \in W' \text{ such that no } u' \in U \text{ satisfies } w' \sqsubseteq u' \sqsubseteq w\}.$$

One easily observes that $X^* \setminus W$ is constructed in the same manner, only interchanging the roles of W' and U. Moreover, if $w \sqsubset u$ (or $u \sqsubset w$) for $w \in W$ and $u \in X^* \setminus W$, then there are $w' \in W'$ and $u' \in U$ such that $w' \sqsubseteq w \sqsubset u' \sqsubseteq u$ (or $u' \sqsubseteq u \sqsubset w' \sqsubseteq w$ resp.). Consequently, if $A(\beta) \cap W$ and $A(\beta) \setminus W$ are both infinite, so are $A(\beta) \cap W'$ and $A(\beta) \cap U$. The latter case is impossible, for $W'^\delta \cap U^\delta = \emptyset$. Thus W is a (σ, δ)-subset of X^*. Finally, $W^\delta = W'^\delta$ follows from $W' \subseteq W$, $U \subseteq X^* \setminus W$ and $W^\delta \cap (X^* \setminus W)^\delta = \emptyset$. $\square$

Remark 1: First, we will emphasize that, though W^δ is an F_σ-set provided W is a (σ, δ)-subset, the converse need not be true. So $(X^2)^*$ is not a (σ, δ)-subset, but $((X^2)^*)^\delta = X^\omega$ is an F_σ-set.

Remark 2: The construction in the above proof immediately shows that W is a regular (recursive) language if only W' and U are regular (recursive) languages.

We add some properties of (σ, δ)-subsets.

PROPOSITION 13 [St2]: *Let U be a (σ, δ)-subset of X^*. Then*

$$(U \cap W)^\delta = U^\delta \cap W^\delta \quad \textit{for all } W \subseteq X^*.$$

Proof: The inclusion "$\subseteq$" is obvious. Let $\beta \in U^\delta \cap W^\delta$. Then $\beta \in U^\delta$ implies that $A(\beta) \setminus U$ is finite. If $\beta \in W^\delta$ then $A(\beta) \cap W$ is infinite. Therefore $A(\beta) \cap U \cap W$ is infinite too, since $(A(\beta) \cap W) \setminus U$ is finite. □

PROPOSITION 14 [St2]: *The class of all* (σ, δ)*-subsets of* X^* *is a Boolean algebra.*

Proof: Closure under complementation is shown above. Let U, W be (σ, δ)-subsets of X^*. Consider $(W \cap U)^\delta$ and

$$(X^* \setminus (W \cap U))^\delta = (X^* \setminus U)^\delta \cup (X^* \setminus W)^\delta.$$

Since U and W are (σ, δ)-subsets, we have $U^\delta \cap (X^* \setminus U)^\delta = \varnothing$ and $W^\delta \cap (X^* \setminus W)^\delta = \varnothing$. Hence $(U \cap W)^\delta \cap (X^* \setminus (W \cap U))^\delta = \varnothing$, and $W \cap U$ is a (σ, δ)-subset. □

We return to the consideration of sequential mappings. If V is a (σ, δ)-subset of Y^*, proposition 13 and equation (5) imply $(\varphi(A(\beta)) \cap V)^\delta = \{\bar{\varphi}(\beta)\} \cap V^\delta$, i.e. $\varphi(A(\beta)) \cap V$ is infinite, iff $\beta \in \bar{\varphi}^{-1}(V^\delta)$.

In view of equation (10) this proves.

LEMMA 15: *If* V *is a* (σ, δ)*-subset of* Y^* *then*

$$\bar{\varphi}^{-1}(V^\delta) = \varphi^{-1}(V)^\delta \cap \operatorname{dom}(\bar{\varphi}).$$

Since $\operatorname{ls} W = A(W)^\delta$ and $A(W)$ is a particular kind of (σ, δ)-subset, lemma 15 yields a connection between $\bar{\varphi}^{-1}(\operatorname{ls} V)$ and $\operatorname{ls} \varphi^{-1}(A(V))$.

PROPOSITION 16: *For every* $V \subseteq Y^*$ *we have*

$$\bar{\varphi}^{-1}(\operatorname{ls} V) = \operatorname{ls} \varphi^{-1}(A(V)) \cap \operatorname{dom}(\bar{\varphi}).$$

Proof: Lemma 15 and equation (1 *a*) imply

$$\bar{\varphi}^{-1}(\operatorname{ls} V) = \varphi^{-1}(A(V))^\delta \cap \operatorname{dom}(\bar{\varphi}).$$

Since φ is a sequential mapping,

$$\varphi^{-1}(A(V)) = A(\varphi^{-1}(A(V))) \cap \operatorname{dom}(\varphi)$$

and

$$\operatorname{dom}(\varphi)^\delta \supseteq \operatorname{dom}(\bar{\varphi}).$$

Then equation (1 *a*) and proposition 13 yield

$$(A(\varphi^{-1}(A(V))) \cap \operatorname{dom}(\varphi))^\delta = \operatorname{ls} \varphi^{-1}(A(V)) \cap \operatorname{dom}(\varphi)^\delta,$$

and the assertion follows. □

With corollary 4 we get immediately an analogue to property 9 for closed sets.

COROLLARY 17: *Let* φ *be a sequential mapping. The inverse* $\bar{\varphi}^{-1}$ *preserves closed sets iff* $\operatorname{dom}(\bar{\varphi})$ *is closed.*

In order to derive the same proposition for open sets, we prove a statement similar to proposition 16.

PROPOSITION 18: *For every* $V \subseteq Y^*$ *we have*

$$\bar{\varphi}^{-1}(V \cdot Y^\omega) = \varphi^{-1}(V \cdot Y^*) \cdot X^\omega \cap \operatorname{dom}(\bar{\varphi}).$$

Proof: Inserting the (σ, δ)-subset $V \cdot Y^*$ instead of V into the identity of lemma 15 yields

$$\bar{\varphi}^{-1}(V \cdot Y^\omega) = \varphi^{-1}(V \cdot Y^*)^\delta \cap \operatorname{dom}(\bar{\varphi}).$$

In view of $W^\delta \subseteq W \cdot X^\omega$ we have $\varphi^{-1}(V \cdot Y^*)^\delta \subseteq \varphi^{-1}(V \cdot Y^*) \cdot X^\omega$. Hence, it suffices to show

$$\varphi^{-1}(V \cdot Y^*) \cdot X^\omega \cap \operatorname{dom}(\bar{\varphi}) \subseteq \varphi^{-1}(V \cdot Y^*)^\delta.$$

If $\beta \in \varphi^{-1}(V \cdot Y^*) \cdot X^\omega \cap \operatorname{dom}(\bar{\varphi})$ then there is a $w \in A(\beta)$ such that $\varphi(w) \in V \cdot Y^*$ and, moreover, $\varphi(A(\beta))$ is infinite. Thus, $\varphi(A(\beta) \cap w \cdot X^*)$ is an infinite subset of $\varphi(w) \cdot Y^* \subseteq V \cdot Y^*$, which implies $\beta \in \varphi^{-1}(V \cdot Y^*)^\delta$. □

COROLLARY 19: *Let* φ *be a sequential mapping. The inverse* $\bar{\varphi}^{-1}$ *preserves open sets iff* $\operatorname{dom}(\bar{\varphi})$ *is open.*

We conclude this section by giving a necessary and sufficient condition for the validity of Equation (II).

THEOREM 20: *It holds for all* $V \subseteq Y^*$

$$\bar{\varphi}^{-1}(\operatorname{ls} V) = \operatorname{ls} \varphi^{-1}(A(V)) \tag{II}$$

if and only if $\operatorname{dom}(\bar{\varphi}) = \operatorname{ls} \operatorname{dom}(\varphi)$.

Proof: The only if part is easily verified by insertion of $V := Y^*$ into (II).

Conversely, in view of $\varphi^{-1}(A(V)) \subseteq \operatorname{dom}(\varphi)$, we have

$$\operatorname{ls} \varphi^{-1}(A(V)) \subseteq \operatorname{ls} \operatorname{dom}(\varphi) = \operatorname{dom}(\bar{\varphi}).$$

Hence, the assertion follows from proposition 16. □

4. EQUATION (I) AND CONTINUITY

In this section we investigate conditions under which the equation (I) holds. To this end we quote a theorem of [St1] (*cf.* also [LS, BN]) which shows the validity of the equation under a strong hypothesis.

THEOREM 21 [St1]: *Let* φ *be a fully defined* ($\mathrm{dom}(\varphi)=X^*$) *sequential mapping and let also* $\mathrm{dom}(\bar{\varphi})=X^\omega$. *Then*

$$\bar{\varphi}(\mathrm{ls}\, W)=\mathrm{ls}\,\varphi(A(W)) \tag{I}$$

for all subsets $W \subseteqq X^*$.

Therefore, our main technique in deriving sufficient conditions is to extend a sequential mapping φ in such a way that the extension ψ satisfies $\mathrm{dom}(\bar{\psi})=X^\omega$, i. e. is a continuous mapping from X^ω to Y^ω.

To this end we study in more detail the relations between extensions and restrictions of sequential mappings. First we introduce the following notion.

DEFINITION: A sequential mapping φ is called totally unbounded provided for every infinite subset $U \subseteqq \mathrm{dom}(\varphi)$ the image $\varphi(U)$ is also infinite.

PROPERTY 22: *Let* ψ *be an extension of a sequential mapping* $\varphi: X^* \to Y^*$ (*or short:* $\psi \supseteqq \varphi$). *Then the following assertions are true:*

$$\bar{\psi} \supseteqq \bar{\varphi}$$

if dom $(\bar{\psi})=X^\omega$ *then* $\bar{\psi}: X^\omega \to Y^\omega$ *is continuous,*
and φ *is totally unbounded if* ψ *is totally unbounded.*

Proof: The first and third assertion are immediate, and the second one follows from theorem 10. □

Remark: In particular, as it was noted in corollary 6.22 of [LS] (*see* also lemma 8 of [BN]), it follows from theorem 21 that if ψ and $\bar{\psi}$ are fully defined, then ψ is totally unbounded. As a first approximation to equation (I) we mention the following easily verified inequality [being a counterpart to equation (12)]:

$$\bar{\varphi}(\mathrm{ls}\, W) \subseteqq \mathrm{ls}\,\varphi(A(W)). \tag{13}$$

Next we prove our first extension result.

THEOREM 23: *If* $\mathrm{dom}(\bar{\varphi})=\mathrm{ls}\ \mathrm{dom}(\varphi)$ *then there is a fully defined extension* $\psi \supseteqq \varphi$ *such that* $\mathrm{dom}(\bar{\psi})=X^\omega$.

Proof: We define $\psi: X^* \to Y^*$ by induction as follows:

$$\psi(e) := \begin{cases} \varphi(e), & \text{if } e \in \operatorname{dom}(\varphi) \\ e, & \text{otherwise,} \end{cases}$$

and

$$\psi(wx) := \begin{cases} \varphi(wx), & \text{if } wx \in \operatorname{dom}(\varphi) \\ \psi(w), & \text{if } wx \in A(\operatorname{dom}(\varphi)) \setminus \operatorname{dom}(\varphi) \\ \psi(w)\,y, & \text{if } wx \notin A(\operatorname{dom}(\varphi)), \end{cases}$$

where $w \in X^*$, $x \in X$ and y is a fixed letter in Y.

Clearly ψ is a fully defined sequential mapping extending φ. Hence, $\beta \in \operatorname{dom}(\bar{\varphi})$ implies $\bar{\psi}(\beta) = \bar{\varphi}(\beta)$. If $\beta \notin \operatorname{dom}(\bar{\varphi}) = \operatorname{ls}\operatorname{dom}(\varphi)$ then there is a $w \in A(\beta)$ not in $A(\operatorname{dom}(\varphi))$. Consequently, we have a longest $u \in A(\beta)$ contained in $\{e\} \cup A(\operatorname{dom}(\varphi))$. In that case, we have

$$\psi(A(\beta)) \supseteq \{\psi(u), \psi(u)y, \psi(u)yy, \ldots\},$$

and $\bar{\psi}(\beta)$ is also defined. $\square$

Three remarks are in order here

Remark 1: By the construction of ψ we have

$$\psi(A(W)) \setminus \{e\} = \varphi(A(W)) \setminus \{e\} \tag{14}$$

for every $W \subseteq A(\operatorname{dom}(\varphi))$.

Remark 2: One easily verifies that by dropping the factor y in the third line of the definition of $\psi(wx)$ we obtain a fully defined sequential mapping ψ extending φ which satisfies $\bar{\psi} = \bar{\varphi}$.

Remark 3: The extension result of theorem 23 is in particular applicable to the totally unbounded sequential mappings satisfying $\operatorname{dom}(\varphi) = A(\operatorname{dom}(\varphi))$ (which readily implies $\operatorname{dom}(\bar{\varphi}) = \operatorname{ls}\operatorname{dom}(\varphi)$) investigated in section V of [BN].

Now we obtain the condition guaranteeing the validity of equation (I).

THEOREM 24: *It holds* $\operatorname{dom}(\bar{\varphi}) = \operatorname{ls}\operatorname{dom}(\varphi)$ *iff* φ *is totally unbounded and*

$$\bar{\varphi}(\operatorname{ls} W) = \operatorname{ls}\varphi(A(W)) \tag{I}$$

for all $W \subseteq X^*$.

Proof: First, suppose $\operatorname{dom}(\bar{\varphi}) \neq \operatorname{ls}\operatorname{dom}(\varphi)$, i.e. $\operatorname{dom}(\bar{\varphi}) \subset \operatorname{ls}\operatorname{dom}(\varphi)$, and consider an arbitrary $\beta \in \operatorname{ls}\operatorname{dom}(\varphi) \setminus \operatorname{dom}(\bar{\varphi})$. According to properties 1 and

2 there is an infinite subset $U \subseteqq \mathrm{dom}(\varphi)$ such that $\{\beta\} = \mathrm{ls}\, U$. Since $\bar{\varphi}(\beta)$ is not defined, $\bar{\varphi}(\mathrm{ls}\, U) = \varnothing$. Now, if $\varphi(U)$ is finite then φ is not totally unbounded; and if $\varphi(U)$ is infinite we have $\bar{\varphi}(\mathrm{ls}\, U) = \varnothing \neq \mathrm{ls}\, \varphi(A(U))$.

In the case $\mathrm{dom}(\bar{\varphi}) = \mathrm{ls}\, \mathrm{dom}(\varphi)$ we apply theorem 23 and consider the extension $\psi \supseteqq \varphi$ defined there. Since $\mathrm{dom}(\bar{\psi}) = X^{\omega}$, ψ is totally unbounded. Hence, φ is also totally unbounded, by property 22. In order to prove equation (I) we start from the identity

$$\bar{\varphi}(\mathrm{ls}\, W) = \bar{\psi}(\mathrm{ls}\, W \cap \mathrm{dom}(\bar{\varphi}))$$

implied by $\psi \supseteqq \varphi$. Utilizing the hypothesis and equation (5) we obtain

$$\mathrm{ls}\, W \cap \mathrm{dom}(\bar{\varphi}) = \mathrm{ls}(A(W) \cap A(\mathrm{dom}(\varphi))).$$

Now, we can apply theorem 21 which yields

$$\bar{\varphi}(\mathrm{ls}\, W) = \mathrm{ls}\, \psi(A(W) \cap A(\mathrm{dom}(\varphi))),$$

and the assertion follows from the above equation (14). □

The following example shows that the condition "totally unbounded" and "equation (I)" in theorem 24 are likewise independent.

Example 3: Let $U := \{0\}^* \cdot \{1\}$ and let φ_1, φ_2 be defined via

$$\mathrm{dom}(\varphi_1) := U, \qquad \mathrm{dom}(\varphi_2) := X^*$$

$$\varphi_1(u) := u \quad \text{if} \quad u \in \mathrm{dom}(\varphi_1) = U,$$

$$\varphi_2(u) := e \quad \text{if} \quad u \in \mathrm{dom}(\varphi_2) = X^*.$$

Moreover $\bar{\varphi}_1 = \bar{\varphi}_2 = \varnothing$ (the empty mapping).

The mapping φ_1 is totally unbounded, but for

$$\varnothing = \bar{\varphi}_1(\mathrm{ls}\, U) \neq \mathrm{ls}\, \varphi_1(A(U)) = \mathrm{ls}\, U.$$

equation (I) does not hold. On the other hand $\mathrm{ls}\, \varphi_2(W) = \varnothing$ for all $W \subseteqq X^*$. So φ_2 satisfies equation (I), but φ_2 is not totally unbounded. □

In order to get rid of the bycondition "φ is totally unbounded", which guarantees that there is no influence onto $\mathrm{ls}\, \varphi(A(W))$ from outside $A(\mathrm{dom}(\bar{\varphi})) = A(\mathrm{ls}\, \mathrm{dom}(\varphi))$, we have to confine our equation (I) to the essential part $A(\mathrm{dom}(\bar{\varphi}))$. Therefore, we consider the following concept of restriction of sequential mappings.

Let $U \subseteqq X^*$ and define the restriction φ_U of φ to U via

$$\mathrm{dom}(\varphi_U) := \mathrm{dom}(\varphi) \cap U$$

and

$$\varphi_U(w) := \begin{cases} \varphi(w), & \text{if } w \in \operatorname{dom}(\varphi) \cap U \\ \text{undefined}, & \text{otherwise.} \end{cases}$$

Clearly, $\varphi_U(W) = \varphi(W \cap U)$.

The domain $\operatorname{dom}(\bar{\varphi}_U)$ satisfies the following equation.

$$\operatorname{dom}(\bar{\varphi}_U) = \operatorname{dom}(\bar{\varphi}) \cap (\operatorname{dom}(\varphi) \cap U)^\delta. \tag{15}$$

Proof: The inclusion "$\subseteqq$" is obvious.

Suppose $\beta \in \operatorname{dom}(\bar{\varphi}) \cap (\operatorname{dom}(\varphi) \cap U)^\delta$. Then $\varphi(A(\beta))$ is infinite, and $A(\beta)$ contains an infinite subset of dom $(\varphi) \cap U$. Now, the assertion follows with property 5. □

Consider the case $U := \mathbf{U}_\varphi(Y^*)$. Equations (6) and (7) show that

$$\operatorname{dom}(\bar{\varphi}_U) = \operatorname{dom}(\bar{\varphi}) = U^\delta.$$

Hence, $\bar{\varphi}_U = \bar{\varphi}$.

This example and the fully defined mapping ψ of remark 2 (after theorem 23) above show that the set $\{\psi : \bar{\psi} = \bar{\varphi}\}$ contains as well mappings having large $(\operatorname{dom}(\psi) = X^*)$ as mappings having small $(\operatorname{dom}(\bar{\psi}) = \operatorname{dom}(\psi)^\delta)$ domains.

Since $\operatorname{dom}(\bar{\varphi}) \subseteqq \operatorname{dom}(\varphi)^\delta$, we obtain with proposition 13 in the special case when $\operatorname{dom}(\varphi)$ or U is a (σ, δ)-subset of X^* the following corollary to equation (15).

COROLLARY 25: *If one of the sets* $\operatorname{dom}(\varphi)$ *or* U *is a* (σ, δ)-*subset of* X^* *then*

$$\operatorname{dom}(\bar{\varphi}_U) = \operatorname{dom}(\bar{\varphi}) \cap U^\delta.$$

The next corollary follows immediately.

COROLLARY 26:

$$\operatorname{dom}(\bar{\varphi}_U) = \operatorname{dom}(\bar{\varphi}) \text{ whenever } A(\operatorname{dom}(\bar{\varphi})) \subseteqq U.$$

In the sequel the case $U := A(\operatorname{dom}(\bar{\varphi}))$ will be of importance. Therefore we introduce the following notation

$$\varphi_A := \varphi_{A(\operatorname{dom}(\bar{\varphi}))}.$$

By corollary 26 this canonical restriction φ_A of φ has the following properties

$$\bar{\varphi} = \bar{\varphi}_A. \tag{16}$$

PROPERTY 27: *The domain* $\operatorname{dom}(\bar{\varphi})$ *is closed if and only if* $\operatorname{dom}(\bar{\varphi}) = \operatorname{ls}\text{-}\operatorname{dom}(\varphi_A)$.

Proof: The if-part is immediate from corollary 4. To prove the converse, we mention that equation (16) readily implies $\operatorname{dom}(\varphi) \subseteqq \operatorname{ls}\operatorname{dom}(\varphi_A)$. Now, let $\operatorname{dom}(\bar{\varphi})$ be closed, i. e. $\operatorname{ls} A(\operatorname{dom}(\bar{\varphi})) = \operatorname{dom}(\bar{\varphi})$. Since $\operatorname{dom}(\varphi_A) \subseteqq A\text{-}(\operatorname{dom}(\bar{\varphi}))$, the assertion follows. □

Inserting φ_A instead of φ into theorem 23 and taking notice of the preceding considerations we obtain another sufficient condition for the extendability of a sequential mapping $\bar{\varphi}$ to a continuous mapping.

LEMMA 28: *If* $\operatorname{dom}(\bar{\varphi})$ *is closed, then there is a fully defined extension* ψ *of* φ_A *such that* $\operatorname{dom}(\bar{\psi}) = X^\omega$.

Remark 1: Now theorem 24, in particular, shows that if $\operatorname{dom}(\bar{\varphi})$ is closed the canonical restriction φ_A is totally unbounded and $\bar{\varphi}(X^\omega)$ is also closed. The converse, however, need not be true. We give an example.

Example 4: Let

$$Y := X := \{0, 1\}, \qquad \operatorname{dom}(\varphi) := \{0\}^* \cdot \{1\} \cdot X^*$$

and

$$\varphi(0^n 1 w) := \begin{cases} 0^{|w|+n}, & \text{if } n \text{ even}, \\ 1^{|w|+n}, & \text{if } n \text{ odd}. \end{cases}$$

Then φ is totally unbounded, $\bar{\varphi}(X^\omega) = \{0\}^\omega \cup \{1\}^\omega$ is closed, and $\operatorname{dom}(\bar{\varphi}) = X^\omega \setminus \{0\}^\omega$ is open but not closed.

Moreover, it is impossible to extend $\bar{\varphi}$ to a continuous mapping defined on the whole space X^ω. □

Remark 2: In lemma 28 we have dealt only with the extension of mappings $\bar{\varphi}$ having a closed domain. We could follow this line farther by utilizing the oscillation approach indicated by theorem 1 (§ 35, I) of [Ku], but this would lead beyond the scope of this paper.

Instead, we conclude this section by returning to the restricted version of equation (I).

THEOREM 29: *Let* φ *be a sequential mapping. Then the following three conditions are equivalent:*

1. $\operatorname{dom}(\bar{\varphi})$ *is closed.*

2. *For all infinite subsets* U *of* $A(\operatorname{dom}(\bar{\varphi}))$ *the image* $\varphi(A(U))$ *is also infinite.*

3. *For all subsets* $W \subseteqq A\ (\mathrm{dom}(\bar{\varphi}))$ *we have*

$$\bar{\varphi}(\mathrm{ls}\, W) = \mathrm{ls}\, \varphi(A(W)). \tag{I}$$

Before proceeding to the proof, we mention that condition 2 is a stronger version of the statement, that φ_A be totally unbounded. Moreover, we have to derive an auxiliary property.

PROPERTY 30: *It holds* $w \in A(\mathrm{dom}(\bar{\varphi}))$ *iff there is a*

$$u \in \mathbf{U}_\varphi\,(Y^*) \cap A(\mathrm{dom}(\bar{\varphi}))$$

such that $w \sqsubseteq u$ *and* $|\varphi(u)| \geqq |w|$.

Proof: Clearly, the condition is sufficient. Now, let $w \in A(\mathrm{dom}(\bar{\varphi}))$. Then there is a $\beta \in \mathrm{dom}(\bar{\varphi})$ such that $w \sqsubset \beta$. Since $\varphi(A(\beta))$ is infinite, there is a $u \in A(\beta)$ such that $w \sqsubseteq u$ and $|\varphi(u)| \geqq |w|$. □

Proof of theorem 29: First we show that 1 implies 2 and 3. If $\mathrm{dom}(\bar{\varphi})$ is closed, in view of equation (16) and property 27 we have $\mathrm{dom}(\bar{\varphi}) = \mathrm{dom}(\bar{\varphi}_A) = \mathrm{ls}\,\mathrm{dom}(\varphi_A)$, and according to theorem 24 this yields $\bar{\varphi}(\mathrm{ls}\, W) = \mathrm{ls}\, \varphi_A(A(W))$. Let $W \subseteqq A\ (\mathrm{dom}(\bar{\varphi}))$. Then, by definition of φ_A, we have $\varphi(A(W)) = \varphi_A(A(W))$, what proves 3. If moreover $W \subseteqq A\ (\mathrm{dom}(\bar{\varphi}))$ is infinite, then $\varnothing \neq \mathrm{ls}\, W \subseteqq \mathrm{dom}(\bar{\varphi})$.

Thus $\varnothing \neq \bar{\varphi}(\mathrm{ls}\, W) = \mathrm{ls}\, \varphi(A(W))$, and 2 follows from property 2.

Now, let condition 2 be satisfied, and let $A(\beta) \subseteqq A(\mathrm{dom}(\bar{\varphi}))$. Then $\varphi(A(\beta))$ is infinite, what proves $\beta \in \mathrm{dom}(\bar{\varphi})$. Hence, $\mathrm{dom}\ (\bar{\varphi})$ is closed.

Finally, let condition 3 be satisfied, and let $A(\beta) \subseteqq A(\mathrm{dom}(\bar{\varphi}))$. According to property 30, there is an infinite subset $\{u_i : i \in N\}$ of $\mathbf{U}_\varphi\,(Y^*) \cap A(\beta)$ such that $|\varphi(u_i)| > i$. Consequently, $\{\beta\} = \mathrm{ls}\,\{u_j : j \in M\}$ for some $M \subseteqq N$. Inserting $W := \{u_j : j \in M\}$ into condition 3 yields $\{\bar{\varphi}(\beta)\} \supseteqq \mathrm{ls}\, \varphi(W) \neq \varnothing$, since $\varphi(W)$ is infinite. Thus $\bar{\varphi}(\beta)$ is defined, i. e. $\mathrm{dom}(\bar{\varphi})$ is closed. □

5. *agsm*-MAPPINGS

A special type of sequential mappings, defined by generalized sequential machines is widely investigated in connection with language theory. In the book [Sa] a generalized sequential machine (*gsm*) has been combined with a deterministic finite automaton (*dfa*) in order to obtain a type of transducer capable of accepting inputs, a so called accepting generalized sequential machine (*agsm*). Since any *gsm* defines a (fully defined) sequential mapping the mappings defined by *agsms*, as restrictions of *gsm*-mappings, constitute an important class of sequential mappings. In [Au], lemma 1, it has been mentioned, that for languages the power of *agsm*-mappings (or their inverses) coincides with the power of *gsm*-mappings (their inverses resp.) combined with intersection of regular languages.

Though in the case of ω-languages the behaviour of totally unbounded *gsm*-mappings has been investigated in [LS], only few is known for *agsm*-mappings (including the case of arbitrary *gsm*-mappings). It is the aim of this section to throw more light on the behaviour of *agsm*-mappings on ω-languages. In particular, we prove a statement analogous to lemma 1 of [Au]. We start with some necessary definitions and considerations concerning regular ω-languages. For all necessary background in language theory: and finite automata we refer to a standard book, e. g. [Sa].

DEFINITION: An *agsm* is a 7-tuple $\mathfrak{m}=(X, Y, Z, f, g, z_0, Z')$, where X and Y are the input and output alphabets resp.,

Z is a finite set of states,

$z_0 \in Z$ the initial state,

$Z' \subseteq Z$ the set of final (accepting) states,

$f: Z \times X \to Z$ the next state function, and

$g: Z \times X \to Y^*$ the output function.

As usual f and g may be extended to $Z \times X^*$ via

$$f(z, e)=z, \qquad f(z, w . v)=f(f(z, w), v),$$

and

$$g(z, e)=e, \qquad g(z, w \cdot v)=g(z, w) \cdot g(f(z, w), v).$$

An *agsm* $\mathfrak{m}$ defines a mapping $\varphi: X^* \to Y^*$ in the following way:

$$\operatorname{dom}(\varphi)=\{ w: f(z_0, w) \in Z' \}$$

and

$$\varphi(w)=g(z_0, w) \qquad \text{if} \quad w \in \operatorname{dom}(\varphi).$$

Clearly, an *agsm*-mapping is a sequential mapping. A *gsm* is an *agsm* satisfying $Z=Z'$, i. e. the domain of its mapping is X^*. From the considerations one easily observes, that we can split every *agsm*

$$\mathfrak{m}=(X, Y, Z, f, g, z_0, Z')$$

into a *gsm*

$$\mathfrak{m}_0 = (X, Y, Z, f, g, z_0, Z)$$

and a *dfa*

$$\mathfrak{a} = (X, Z, f, z_0, Z')$$

accepting the language $\mathrm{dom}(\varphi)$. As usual we denote by $T(\mathfrak{a}) := \{ w : f(z_0, w) \in Z' \}$ the language accepted by $\mathfrak{a}$, and we call a language $L \subseteq X^*$ regular iff it is accepted by some *dfa* (or equivalently by some nondeterministic finite automaton).

Vice versa to every *gsm* $\mathfrak{m} = (X, Y, Z, f, g, z_0, Z)$ and every *dfa* $\mathfrak{a} = (X, S, h, s_0, S')$ by the usual product construction one obtains an *agsm*

$$\mathfrak{m} \times \mathfrak{a} = (X, Y, Z \times S, f, g, (z_0, s_0), Z \times S')$$

such that $\mathfrak{m} \times \mathfrak{a}$ defines a mapping φ satisfying

$$\mathrm{dom}(\varphi) = \mathrm{T}(\mathfrak{a})$$

and

$$\varphi(w) = g(z_0, w) \qquad \text{for} \quad w \in \mathrm{dom}(\varphi).$$

Similar to the language case, in the case of ω-languages a subset $F \subseteq X^\omega$ is called regular iff it is accepted by some nondeterministic finite automaton, or equivalently, iff there are regular languages W_i, $U_i \subseteq X^*$ such that (*cf.* [Bü])

$$F = \bigcup_{i=1}^{n} W_i . U_i^\omega .$$

In particular, for every regular language $W \subseteq X^*$ its δ-limit W^δ is a regular ω-language, but not every regular ω-language $F \subseteq X^\omega$ can be represented in the form $F = W^\delta$.

The following properties hold (*cf.* [La, LS]).

PROPERTY 31 [La]: *An ω-language F is simultaneously regular and a G_δ-set iff there is a regular language W such that $F = W^\delta$.*

PROPERTY 32 [Bü, La]: *The family of regular ω-languages is closed under Boolean operations, and the family of regular G_δ-sets is closed under union and intersection but not under complementation.*

Now, we are going to investigate the inverse $\bar{\varphi}^{-1}$ of an *agsm*-mapping. To this end we return to the upper quasiinverse $\mathbf{U}_\varphi$ of φ and prove that $\mathbf{U}_\varphi\,(Y^*)$ is regular. [In fact, we could even prove that $\mathbf{U}_\varphi\,(V)$ is regular provided only that V is regular, but for the sake of simplicity and, since in the sequel we do not need the more general result, we confine to the former case.]

PROPOSITION 33: *If φ is an agsm-mapping, then $\mathbf{U}_\varphi\,(Y^*)$ is a regular language.*

Proof: Let $\mathfrak{m}=(X, Y, Z, f, g, z_0, Z)$ be an *agsm* defining φ. We define an automaton $\mathfrak{a}$ accepting $\mathbf{U}_\varphi\,(Y^*)$ as follows

$$\mathfrak{a}=(X, S, h, s_0, S')$$

where

$$S=Z\times\{0, 1\}$$

$$s_0=\begin{cases}(z_0, 0), & \text{if} \quad e\in\mathbf{U}_\varphi\,(y^*)\\ (z_0, 1), & \text{otherwise}\end{cases}$$

$$S'=Z'\times\{1\},$$

and

$$h((z, a), x):=\begin{cases}(f(z, x), 0), & \text{if} \quad (z\in Z' \text{ or } a=0) \quad \text{and} \quad g(z, x)=e\\ (f(z, x), 1), & \text{if} \quad a=1 \quad \text{or} \quad g(z, x)\neq e.\end{cases}$$

Informally, the construction splits Z into two sets $Z\times\{0\}$ and $Z\times\{1\}$ according to whether the state $f(z, x)$ is reached from the latest accepting state $z'\in Z'$ via an e-labelled path or not. By this much explanation utilizing equation (6) it is readily seen that

$$\mathbf{U}_\varphi\,(Y^*)=\{w: h(s_0, w)\in S'\}. \quad \square$$

Equation (7) and property 31 yield the following.

COROLLARY 34: *If φ is an agsm-mapping, then* dom $(\bar{\varphi})$ *is a regular ω-language.*

If ψ is a sequential mapping extending φ, then

$$\bar{\varphi}^{-1}(F)=\bar{\psi}^{-1}(F)\cap \mathrm{dom}(\bar{\varphi}) \tag{17}$$

holds for every $F\subseteq Y^\omega$. In particular, if ψ is a *gsm*-mapping extending an *agsm*-mapping φ corollary 34 and equation (17) imply a statement analogous to lemma 1 in [Au].

COROLLARY 35: *The sets of operations* $\{\bar{\varphi}^{-1}:\varphi$ *an agsm-mapping*$\}$ *and* $\{\bar{\psi}^{-1}:\psi$ *a gsm-mapping*$\}\cup\{\cap E: E$ *a regular* G_δ*-set*$\}$ *have exactly the same power.*

But we can derive a stronger connection between the inverses of related *agsm*- and *gsm*-mappings, more exactly, between $\bar{\varphi}^{-1}$ and ψ^{-1}. To this end consider the following property of *gsm*-mappings.

Every *gsm*-mapping ψ has the property that

$$|\psi(wx)|-|\psi(w)|\leqq m \tag{18}$$

for some $m\in N$ and arbitrary $w\in X^*$, $x\in X$. Hence, if $w\in \mathbf{U}_\psi(v)$ then $|\psi(w)|-|v|\leqq m$. This observation shows, that for every *gsm*-mapping ψ there is an $m\in N$ such that

$$\mathbf{U}_\psi(V)\subseteqq \psi^{-1}(V\cdot A(Y^m)) \tag{19}$$

for arbitrary $V\subseteqq Y^*$.

These considerations yield the following.

LEMMA 36: *Let* ψ *be a gsm-mapping. Then there is an* $m\in N$ *such that for all* $V\subseteqq Y^*$ *the equation*

$$\bar{\psi}^{-1}(V^\delta)=\psi^{-1}(V\cdot A(Y^m))^\delta\cap \mathbf{U}_\psi(Y^*)^\delta=(\psi^{-1}(V\cdot A(Y^m))\cap \mathbf{U}_\psi(Y^*))^\delta$$

holds true.

Proof: It is readily seen that $V^\delta=(V\cdot A(X^m))^\delta$.

Thus, equations (6) and (12) prove the first inclusion « $\supseteq$ » the second « $\supseteq$ » being obvious.

On the other hand, by theorem 8, $\bar{\psi}^{-1}(V^\delta)=\mathbf{U}_\psi(V)^\delta$, and due to equation (19) $\mathbf{U}_\psi(V)\subseteqq\psi^{-1}(V\cdot A(Y^m))\cap\mathbf{U}_\psi(Y^*)$. □

Utilizing the above mentioned splitting of an *agsm* into a *gsm* and a *dfa* we obtain for *agsm*-mappings the following strengthening of equation (12).

THEOREM 37: *If* φ *is an agsm-mapping, then there are a gsm-mapping* ψ *extending* φ *and an* $m\in N$ *such that*

$$\bar{\varphi}^{-1}(V^\delta)=\psi^{-1}(V\cdot A(Y^m))^\delta\cap\mathbf{U}_\varphi(Y^*)^\delta$$

for arbitrary $V\subseteqq Y^*$.

Proof: Since $\psi\supseteqq\varphi$, we have $\bar{\varphi}^{-1}(V^\delta)=\bar{\psi}^{-1}(V^\delta)\cap\operatorname{dom}(\bar{\varphi})$, and the result follows from the previous lemma. □

Our theorem 37 may be used to show that inverse *agsm*-mappings preserve several classes of ω-languages. We shall return to this point in the next section.

We conclude this section by reconsidering the extension results of theorem 23 and lemma 28 in the case of *agsm*-mappings. To this end let us note the following properties of *agsm*-mappings.

PROPERTY 38: *Let* φ *be an agsm-mapping. Then* $A(\operatorname{dom}(\bar{\varphi}))$ *is a regular language and the restriction* φ_A *of* φ *to* $A(\operatorname{dom}(\bar{\varphi}))$ *is also an agsm-mapping.*

Proof: The first assertion follows from the obvious fact that $A(F)$ is a regular language whenever F is a regular ω-language, and the second one is readily verified from the first one. □

THEOREM 39: *If* φ *is an agsm-mapping and* $\operatorname{dom}(\bar{\varphi}) = \operatorname{ls} \operatorname{dom}(\varphi)$, *then there is a totally unbounded gsm-mapping* ψ *such that* $\varphi \subseteq \psi$.

Proof: We start from an arbitrary *gsm*-mapping ψ' extending φ and define

$$\psi(w) := \psi'(w) \qquad \text{if} \quad w \in A(\operatorname{dom}(\varphi)) \cup \{e\},$$

and

$$\psi(wx) := \psi(w) \cdot y \qquad \text{if} \quad wx \notin A(\operatorname{dom}(\varphi)), \qquad x \in X,$$

where y is a fixed letter in Y. Clearly, ψ is a *gsm*-mapping. Now the rest of the proof is the same as for theorem 23. □

In the same way as lemma 28 was derived from theorem 23, we obtain from theorem 39 and property 38 the following.

LEMMA 40: *If* φ *is an agsm-mapping and* $\operatorname{dom}(\bar{\varphi})$ *is closed, then there is a gsm-mapping* ψ *extending* φ_A *such that* $\operatorname{dom}(\bar{\psi}) = X^\omega$.

6. APPLICATIONS TO FAMILIES OF ω-LANGUAGES

In this section we investigate the closure of several families of ω-languages under inverses *agsm*-mappings and related operations. It will turn out that the closure properties of the families of ω-languages are already implied by several closure properties of the underlying families of languages. Here we consider families of ω-languages defined via the δ-limit in the following way:

$$\Delta\mathscr{L} := \{ L^\delta : L \in \mathscr{L} \},$$

where $\mathscr{L} \subseteqq \{ W : W \subseteqq X^* \text{ and } X \text{ finite and } X \subseteqq \Sigma \}$ is a family of languages [Σ — being an infinite (universal) alphabet]. Similar investigations have been carried out in [SW 1].

As a first simple closure property, from equation (13) it follows that $\Delta\mathscr{L}$ is closed under union whenever $\mathscr{L}$ is closed under union.

Lemma 36 implies the next closure property.

PROPERTY 41: If $\mathscr{L}$ is closed under $\cdot A(Y^m)$ ($Y \subseteqq \Sigma$ finite and $m \in N$ arbitrary), inverse *gsm*-mapping and intersection with regular languages, then $\Delta\mathscr{L}$ is closed under inverse *gsm*-mapping.

Now, in view of corollary 35, we are interested in an operation for languages which implies that $\Delta\mathscr{L}$ is closed under intersection with regular G_δ-sets.

To this end we introduce the continuation $L > U$ of a language L to a language U:

$$w > U := \operatorname{Min} U \cap w \cdot X^* \qquad (w \in X^*, U \subseteqq X^*),$$

and

$$L > U := \bigcup_{w \in L} w > \mathbf{U}.$$

The term continuation becomes clear from the first line of the definition, where $w > U$ denotes the set of all smallest with respect to « $\sqsubseteq$ » words in U which have w as an initial word.

This operation looks somehow artificially, but the following informal consideration shows that most classes of languages defined by (deterministic) accepting devices are closed under continuation to regular languages:

Take the device ϑ accepting L and let it be conveyed during the accepting process by a *dfa* $\mathfrak{a}$ and a controlling unit having an active and a dormant state.

Whenever ϑ accepts an initial part of the input word, the controlling unit switches to "active" and remains there until $\mathfrak{a}$ reaches a final state. After leaving this final state the controlling unit becomes dormant. An input word w is accepted if and only if w is accepted by $\mathfrak{a}$ and the controlling unit is active.

Moreover, for the continuation we have

$$(L > U)^\delta = L^\delta \cap U^\delta. \tag{20}$$

Before proceeding to the proof of equation (20) we mention the following easily verified properties of the continuation.

PROPERTY 42: *Let* $w \sqsubseteq u$, $w \in L$, *and* $u \in U$. *Then there is a* $u' \in L > U$ *such that* $w \sqsubseteq u' \sqsubseteq u$.

PROPERTY 43: *Let* $u \sqsubset u'$, *and* $u, u' \in L > U$. *Then there is a* $w \in L$ *such that* $u \sqsubseteq w \sqsubseteq u'$.

Proof of equation (20): By definition $L > U \subseteq U$, hence $(L > U)^\delta \subseteq U^\delta$. If, moreover, $(L > U) \cap A(\beta)$ is infinite, property 43 shows that also $L \cap A(\beta)$ is infinite, which finishes the proof of $(L > U)^\delta \subseteq L^\delta \cap U^\delta$.

Now, let $L \cap A(\beta)$ and $U \cap A(\beta)$ both be infinite. Then for infinitely many $w \in L$ we have $w \sqsubseteq u \sqsubseteq \beta$ for some $u \in U$; and property 42 implies $(L > U) \cap A(\beta)$ is infinite. Hence $L^\delta \cap U^\delta \subseteq (L > U)^\delta$, and the assertion is proved. □

As an immediate consequence of equation (20) we obtain that the family $\Delta\mathscr{L}$ is closed under intersection with regular G_δ-sets whenever the family $\mathscr{L}$ is closed under continuation to regular languages. Applying now theorem 37 yields the following closure property for inverse *agsm*-appings.

LEMMA 44: *If* $\mathscr{L}$ *is closed under* $\cdot A(Y^m)$ ($Y \subseteq \Sigma$ *finite and* $m \in N$ *arbitrary*), *inverse gsm-mapping and continuation to regular languages, then* $\Delta\mathscr{L}$ *is closed under inverse agsm-mapping.*

Finally, we consider the special case $\mathscr{L} = \mathscr{REC}$ where $\mathscr{REC}$ is the family of recursive languages. The family $\Delta\mathscr{REC}$ has been investigated in detail in [CG], [SW 2] and [St 4] where closure under union and intersection and the identity $\Delta\mathscr{REC} = \Delta\mathscr{RE}$ ($\mathscr{RE}$ being the family of recursively enumerable languages) were shown. In [SW 2] and [St 4], moreover, it has been shown that $\Delta\mathscr{REC}$ is closed under $\bar{\psi}^{-1}$ when ψ is a totally unbounded recursive (as a function) sequential mapping. We shall extend this result to arbitrary partial recursive sequential mappings (or processes, as they were called in [Sc 2]), which are interesting in connection with complexity questions for infinite sequences [Sc 1,2].

To this end we quote lemma 6.3 from [Sc 1].

LEMMA 45: *If* φ *is a partial recursive function being a sequential mapping. Then there is a (fully defined) recursive sequential mapping* ψ *such that* $\bar{\varphi} = \bar{\psi}$.

Remark: It should be mentioned that the extension ψ of remark 2 after theorem 23 need not be a recursive function if φ is a partial recursive function.

We get our result.

LEMMA 46: *$\Delta\mathscr{R}\mathscr{E}\mathscr{C}$ is closed under* $\bar{\varphi}^{-1}$, when the sequential mapping φ is also a partial recursive function.

Proof: In view of lemma 45 we may assume φ to be a recursive function. Since $w \in \mathbf{U}_\varphi(V)$ iff there is a $v \in V$ such that $v \sqsubseteq \varphi(w)$ and for all $u \sqsubset w$ we have $|v| > |\varphi(u)|$, one easily verifies that $\mathbf{U}_\varphi(V)$ is a recursive language whenever V is recursive. Now, an application of theorem 8 ends the proof. □

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