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ANALYSIS OF AN ALGORITHM TO CONSTRUCT FIBONACCI PARTITIONS (*)

by Helmut PRODINGER [1]

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Abstract. — *A Fibonacci partition of* $\{1, 2, \ldots, n\}$ *is a partition such that* i *and* $i+1$ *are never in the same block. It can be shown by an umbral argument that the corresponding number* C_n *equals* B_{n-1}, *which is the number of partitions of* $\{1, \ldots, n-1\}$. *There exists an algorithm which constructs a unique Fibonacci partition of* $\{1, \ldots, n+1\}$ *if a partition of* $\{1, \ldots, n\}$ *is given. The interesting parameter of this algorithm is the number of companions of* $n+1$ *in the Fibonacci partitions.*

Using umbral methods, expressions of average and variance can be found, which can be evaluated asymptotically using de Bruijn's treatment of the Bell numbers.

Résumé. — *Une partition de Fibonacci de* $\{1, 2, \ldots, n\}$ *est une partition dans laquelle* i *et* $i+1$ *ne sont jamais dans le même bloc. On peut montrer par un argument de type ombral que le nombre* C_n *de partitions de Fibonacci de* $\{1, \ldots, n\}$ *est égal au nombre* B_{n-1} *de partitions de* $\{1, \ldots, n-1\}$. *Il existe un algorithme qui construit, à partir d'une partition de* $\{1, \ldots, n\}$, *une unique partition de Fibonacci de* $\{1, \ldots, n+1\}$. *Le paramètre intéressant de cet algorithme est le nombre de compagnons de* $n+1$ *dans les partitions de Fibonacci.*

A l'aide de méthodes ombrales, on peut trouver des expressions de moyenne et de variance qui peuvent être évaluées asymptotiquement en utilisant le traitement de de Bruijn des nombres de Bell.

1. INTRODUCTION

It is a classical result that the number of subsets A of $\bar{n}=\{1, 2, \ldots, n\}$ such that $i \in A$ and $i+1 \in A$ is impossible for all i equals the $n+1-st$ Fibonacci number F_{n+1} (see [2]). ($F_{n+2}=F_{n+1}+F_n$, $F_0=F_1=1$.) Let us call such a set A a Fibonacci set. One may also think of sequences of a's and b's of length n without consecutive a's. Regarding this result it is natural to define a Fibonacci partition of $\bar{n}$ to be a partition such that no pair i, $i+1$ lies in the

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same block (see [3]). For the readers convenience, let us review some basic facts about partitions (see [5]): A partition of $\bar{n}$ is a family of disjoint nonempty subsets (called blocks) whose union is $\bar{n}$. For instance, 124 | 3 | 5 is a partition of $\bar{5}$ consisting of 3 blocks. The number of partitions of $\bar{n}$ is the n-th Bell number B_n; they satisfy $B_{n+1}=\sum_{0\leq k\leq n}\binom{n}{k}B_k$, $B_0=1$; furthermore $\sum_{n\geq 0} B_n x^n/n!=\exp(e^x-1)$.

Let C_n be the number of Fibonacci partitions of $\bar{n}$. In [3] it was proved by umbral methods that $C_n=B_{n-1}$. This result will come out in the sequel as as corollary. On the other hand, a bijection between the set of partitions of $\bar{n}$ and the set of Fibonacci partitions of $\overline{n+1}$ was constructed in [3] by means of the following

ALGORITHM INPUT: partition of $\bar{n}$
OUTPUT: Fibonacci partition of $\overline{n+1}$

A1: $n+1$ is adjoined to the given partition in a new class

A2: Do step A3 for all blocks.

A3: If the block is a Fibonacci subset, do nothing.
Otherwise, repeatedly find i, $i+1$, elements of this subset, i maximal, and put i into the class of $n+1$.

It is not complicated to prove that the algorithm constructs a bijection, because one can describe the inverse mapping. To illustrate the algorithm, we start with the partition 1235 | 467 | 89 of $\bar{9}$:

1235 | 467 | 89 → 1235 | 467 | 89 | 10 → 135 | 467 | 89 | 210
→ 135 | 47 | 89 | 2610 → 135 | 47 | 9 | 26810.

The last partition is a Fibonacci partition of $\overline{10}$.

The aim of this note is to analyse this algorithm. The interesting parameter is the number k of elements which are shifted into the class of $n+1$ in step A3. It is easily seen that $0\leq k\leq [n/2]$ holds. Let $C_{n+1,k}$ be this number. We give exact formulas and asymptotic estimates for the average

$$M_f(n)=\sum_{k\geq 0} k\,C_{n+1,k}/C_{n+1}$$

and also for the variance $V_f(n)$. As a showcase, we consider also the corresponding values for the unrestricted case, i. e. $B_{n+1,k}$, its average $M(n)$ and the variance $V(n)$.

We point out that Ph. Flajolet (private communication) has mentioned, that the identities $C_n=B_{n-1}$ (and more general ones, see [3]) would probably lead to new continued fractions.

2. ANALYSIS

As already mentioned in the introduction, we consider first the numbers $B_{n+1,k}$ (i.e. the numbers of partitions of $\overline{n+1}$ such that k elements $\in \bar{n}$ are in the same block as $n+1$).

THEOREM 1:
$$\sum_{k \geqq 1} k B_{n+1,k} = n B_n . \tag{1}$$

Proof: It is quite obvious that
$$B_{n+1,k} = \binom{n}{k} B_{n-k} \tag{2}$$
holds. Let L be the linear functional, defined on the vector space of all polynomials over $\mathbb{R}$, sending each polynomial $(x)_n := x(x-1) \ldots (x-n+1)$ to 1. It is Rota's principal result [5] that $B_n = L(x^n)$ holds. Now
$$\sum_{k \geqq 1} k B_{n+1,k} = L \sum_{k \geqq 1} k \binom{n}{k} x^{n-k} = nL \sum_{k \geqq 1} \binom{n-1}{k-1} x^{n-1-(k-1)}$$
$$= nL(1+x)^{n-1} = nLx^n = nB_n ,$$
since Rota has shown that for all polynomials p
$$L(xp(x-1)) = L(p(x)) \tag{3}$$
holds.

Now we turn to the $C_{n+1,k}$'s. Let us recall the following facts from [5] and [3]: To compute B_n, the set of all functions $f: \bar{n} \to \bar{x}$ must be divided with respect to the kernels:
$$x^n = \sum_{\pi : \text{partition}} (x)_{N(\pi)} ,$$
where $N(\pi)$ is the number of blocks of the partition π. An application of the functional L to (4) yields $L(x^n) = B_n$. To compute C_n, the set of functions $f: \bar{n} \to \bar{x}$ with the property "$f(i) = f(i+1)$ impossible" is considered. The number of these functions is easy to determine: For $f(1)$ there are x possibilities, for $f(2)$ there are $x-1$ possibilities, for $f(3)$ there are $x-1$ possibilities and so on. Thus there are $x(x-1)^{n-1}$ such functions; the set of these functions is divided with respect to the kernels
$$x(x-1)^{n-1} = \sum_{\substack{\pi : \text{Fibonacci} \\ \text{partition}}} (x)_{N(\pi)} . \tag{5}$$
Thus we get as a corollary:

COROLLARY 2: $C_n = B_{n-1}$.

Proof: We apply the functional L on both sides of (5) to get
$$C_n = L(x(x-1)^{n-1}) = L(x^{n-1}) = B_{n-1} .$$

LEMMA 3:
$$C_{n+1,k} = L\left(\binom{n-k-1}{k-1} x^k (x-1)^{n-2k} + \binom{n-k-1}{k} x^{k+1} (x-1)^{n-2k-1}\right).$$

Proof: We have to compute the number of all partial functions f: $\bar{n} \to \bar{x}$ with "$f(i)=f(i+1)$ impossible" such that f is undefined for exactly k values which are not adjacent (i.e. the set of these k values is a Fibonacci subset). Furthermore, $f(n)$ must be defined. The application of L yields then the desired number $C_{n+1,k}$.

First we compute in how many ways these k places can be distributed. This is the number of configurations

$$i_0+i_1+\ldots+i_k=n-k, \qquad i_0\geq 0, \qquad i_s\geq 1 \tag{6}$$

or, equivalently, the sum of the numbers of configurations in (7) and (8):

$$i_1+\ldots+i_k=n-k, \qquad i_s\geq 1, \tag{7}$$

$$i_0+\ldots+i_k=n-k, \qquad i_s\geq 1. \tag{8}$$

Let a configuration as in (7) be fixed. The number of functions f with this configuration is then

$$x(x-1)^{i_1-1}x(x-1)^{i_2-1}\ldots x(x-1)^{i_k-1}=x^k(x-1)^{n-2k}$$

The coefficient (i.e. the number of configurations in (7) can be computed as follows: ($[x^n]g$ is the coefficient of x^n in g)

$$\sum_{i_1+\ldots+i_k=n-k;\ i_s\geqq 1} 1 = [x^{n-k}](x+x^2+\ldots)^k=[x^{n-k}]\frac{x^k}{(1-x)^k}$$

$$=[x^{n-2k}](1-x)^{-k}=\binom{-k}{n-2k}(-1)^{n-2k}=\binom{n-k-1}{k-1}$$

Similarly, the number of functions for a configuration as in (8) is

$$x(x-1)^{i_0-1}\ldots x(x-1)^{i_k-1}=x^{k+1}(x-1)^{n-2k-1}$$

The coefficient is

$$\sum_{i_0+\ldots+i_k=n-k;\ i_s\geqq 1} 1 = [x^{n-k}]\frac{x^{k+1}}{(1-x)^{k+1}}=[x^{n-2k-1}](1-x)^{-k-1}$$

$$=\binom{-k-1}{n-2k-1}(-1)^{n-2k-1}=\binom{n-k-1}{k},$$

which finishes the proof.

THEOREM 3: $\sum_{k\geqq 1} kC_{n+1,k}=(n-1)B_{n-1}-(n-2)B_{n-2}+\ldots+(-1)^n B_1$.

Proof: From Lemma 2 we have

$$\sum_{k\geqq 1} kC_{n+1,k}=L\Bigg(\sum_{k\geqq 1} k\binom{n-k-1}{k-1}x^k(x-1)^{n-2k}$$

$$+\sum_{k\geqq 1} k\binom{n-k-1}{k}x^{k+1}(x-1)^{n-2k-1}\Bigg). \tag{9}$$

We compute the two polynomials in (9): For this let $y=x/(x-1)^2$ and $\alpha=\sqrt{1+4y}=(x+1)/(x-1)$. We refer to an identity in [4; p. 76] and give only the key steps.

$$\sum_{k\geqq 1} k\binom{n-1-k}{k-1}x^k(x-1)^{n-2k}=\sum_{k\geqq 0}\binom{n-2-k}{k}(k+1)x^{k+1}(x-1)^{n-2-2k}$$
$$=x(x-1)^{n-2}\frac{d}{dy}y\frac{1}{\alpha}\left[\left(\frac{1+\alpha}{2}\right)^{n-1}-\left(\frac{1-\alpha}{2}\right)^{n-1}\right]$$
$$=\frac{x}{(x+1)^3}(x^{n+1}+(n-1)x^n+nx^{n-1}+n(-1)^n x^2+(n-1)x(-1)^n+(-1)^n) \quad (10)$$

$$\sum_{k\geqq 1} k\binom{n-1-k}{k}x^{k+1}(x-1)^{n-1-2k}=x(x-1)^{n-1}\sum_{k\geqq 1} k\binom{n-1-k}{k}y^k$$
$$=x^2(x-1)^{n-3}\frac{d}{dy}\sum_{k\geqq 0}\binom{n-1-k}{k}y^k$$
$$=x^2(x-1)^{n-3}\frac{d}{dy}\frac{1}{\alpha}\left[\left(\frac{1+\alpha}{2}\right)^{n}-\left(\frac{1-\alpha}{2}\right)^{n}\right]$$
$$=\frac{x^2}{(x+1)^3}((n-2)x^n-(n-2)(-1)^n+nx^{n-1}-nx(-1)^n) \quad (11)$$

The sum of (10) and (11) is

$$\frac{x}{(x+1)^3}((n-1)x^{n+1}+(2n-1)x^n+nx^{n-1}+x(-1)^n+(-1)^n)$$
$$=\frac{x}{(x+1)^2}((n-1)x^n+nx^{n-1}+(-1)^n)$$
$$=\sum_{k=1}^{n-1}(-1)^{n+1-k}kx^k. \quad (12)$$

The result now follows by an application of the linear functional L to (12).

THEOREM 4: $M(n)=\dfrac{nB_n}{B_{n+1}}=\log n+0(\log\log n)$ as $n\to\infty$.

Proof: From [1; pp. 107] we infer

$$\log B_n-\log B_{n-1}=\log n-\frac{1}{2}\log\left(1+\frac{1}{u_n}\right)+\frac{1}{2}\log\left(1+\frac{1}{u_{n-1}}\right)$$
$$+\log(1+0(e^{-u_n}))+\log(1+0(e^{-u_{n-1}}))+e^{u_n}-e^{u_{n-1}}$$
$$-(n+1)\log u_n+n\log u_{n-1}-\frac{1}{2}(u_n-u_{n-1}). \quad (13)$$

u_n, u_{n-1} are given as the unique positive numbers for which hold:

$$u_n e^{u_n} = n+1 \tag{14}$$

$$u_{n-1} e^{u_{n-1}} = n \tag{15}$$

Dividing (14) and (15) yields

$$\frac{u_n}{u_{n-1}} e^{u_n - u_{n-1}} = 1 + \frac{1}{n},$$

thus

$$\log u_n - \log u_{n-1} + u_n - u_{n-1} = \log\left(1+\frac{1}{n}\right) = 0\left(\frac{1}{n}\right).$$

Now $u_n \sim \log(n+1)$ ([1; pp. 25]), thus

$$\log u_n - \log u_{n-1} = \log \frac{u_n}{u_{n-1}} = 0\left(\frac{\log(n+1)}{\log n}\right)$$

$$= 0\left(\log\left(1+\frac{\log\left(1+\frac{1}{n}\right)}{\log n}\right)\right) = 0\left(\frac{\log\left(1+\frac{1}{n}\right)}{\log n}\right) = 0\left(\frac{1}{n \log n}\right),$$

hence

$$u_n - u_{n-1} = 0\left(\frac{1}{n\log n}\right) + 0\left(\frac{1}{n}\right) = 0\left(\frac{1}{n}\right),$$

$$\log(1+0(e^{-u_n})) = \log\left(1+0\left(\frac{1}{n}\right)\right) = 0\left(\frac{1}{n}\right).$$

So (13) yields

$$\log B_n - \log B_{n-1} = \log n + 0\left(\frac{1}{u_n}\right) + 0\left(\frac{1}{u_{n-1}}\right) + 0\left(\frac{1}{n}\right)$$

$$+ \frac{n+1}{u_n} - \frac{n}{u_{n-1}} - n(\log u_n - \log u_{n-1}) - \log u_n + 0\left(\frac{1}{n}\right)$$

$$= \log n + 0\left(\frac{1}{\log n}\right) + \frac{n(u_{n-1}-u_n)}{u_n u_{n-1}} + \frac{1}{u_n} + n0\left(\frac{1}{n\log n}\right)$$

$$- \log\log n + 0\left(\frac{\log\log n}{\log n}\right)$$

$$= \log n - \log\log n + 0\left(\frac{1}{\log^2 n}\right) + 0\left(\frac{1}{\log n}\right) + 0\left(\frac{\log\log n}{\log n}\right)$$

$$= \log n - \log\log n + 0\left(\frac{\log\log n}{\log n}\right). \tag{16}$$

Now

$$\log \frac{nB_n}{B_{n+1}} = \log n + \log B_n - \log B_{n+1}$$
$$= \log \log n + 0\left(\frac{\log \log n}{\log n}\right).$$

and the result follows by exponentiation.

THEOREM 5: $M_f(n) = \sum_k k C_{n+1,k}/C_{n+1} \sim \log n$.

Proof: By Corollary 2 and Theorem 4 we have

$$M_f(n) = \frac{(n-1)B_{n-1} - (n-2)B_{n-2} + \ldots + (-1)^n B_1}{B_n}.$$

An argument similar to the Leibniz convergence criterion for alternating series yields the bounds

$$\frac{(n-1)B_{n-1}}{B_n} - \frac{(n-2)B_{n-2}}{B_n} \leqslant M_f(n) \leqslant \frac{(n-1)B_{n-1}}{B_n}.$$

Now $(n-1)B_{n-1}/B_n \sim \log n$ and

$$\frac{(n-2)B_{n-2}}{B_n} = \frac{(n-2)B_{n-2}}{B_{n-1}} \cdot \frac{(n-1)B_{n-1}}{B_n} \cdot \frac{1}{n-1} = 0\left(\frac{\log^2 n}{n}\right),$$

yielding the asymptotic equivalent of $M_f(n)$.

At a first look this asymptotic result may be surprising, since in the ordinary case k varies between 0 and n, and in the Fibonacci case only between 0 and $[n/2]$, but it is extremely improbable that a value $\geqslant n/2$ appears, so that the big values of k do not influence the average too much.

THEOREM 6: $V(n) = \dfrac{nB_n + n(n-1)B_{n-1}}{B_{n+1}} - M^2(n) = 0(\log n \,.\, \log \log n)$.

Proof:

$$\sum_k k^2 B_{n+1,k} = \sum_k k(k-1) B_{n+1,k} + \sum_k k B_{n+1,k}$$
$$= n(n-1) \sum_k \binom{n-2}{k-2} B_{n-k} + nB_n$$
$$= n(n-1) L \sum_k \binom{n-2}{k-2} x^{n-k} + nB_n$$
$$= n(n-1) L(1+x)^{n-2} + nB_n = n(n-1) L x^{n-1} + nB_n,$$

which gives the exact expression for $V(n)$. The asymptotic estimate is obtained as follows:

$$V(n)=\frac{nB_n}{B_{n+1}}+\frac{(n-1)B_{n-1}}{B_n}\cdot\frac{nB_n}{B_{n+1}}-M^2(n)$$
$$=\log n+0(\log\log n)+\log^2 n+0(\log n\,.\,\log\log n)$$
$$-\log^2 n+0(\log n\,.\,\log\log n).$$

THEOREM 7: $V_f(n)=\sum_k k(k-1)C_{n+1,k}/C_{n+1}+M_f(n)-(M_f(n))^2$

$$=0(\log n\,.\,\log\log n).$$

Proof (Sketch): A similar, but much longer computation as in Theorem 3 shows that

$$\sum_{k\geq 1} k(k-1)\binom{n-k-1}{k-1}x^k(x-1)^{n-2k}+\sum_{k\geq 1} k(k-1)\binom{n-k-1}{k}x^{k+1}(x-1)^{n-2k-1}$$
$$=\sum_{k=2}^{n-2}(n-1-k)k(k-1)(-1)^{n-k}x^k.$$

Hence we get, by applying L to the last expression,

$$V_f(n)=\frac{1}{B_n}\left[\sum_{k=2}^{n-2}(n-1-k)k(k-1)(-1)^{n-k}B_k\right]+M_f(n)-(M_f(n))^2.$$

The asymptotic estimation is as in Theorem 6.

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