RAIRO. INFORMATIQUE THÉORIQUE

I. MARGARIA
M. ZACCHI

Right and left invertibility in $\lambda - \beta$-calculus

RAIRO. Informatique théorique, tome 17, nº 1 (1983), p. 71-88
<http://www.numdam.org/item?id=ITA_1983__17_1_71_0>

R.A.I.R.O. Informatique théorique/Theoretical Informatics
(vol. 17, n° 1, 1983, p. 71 à 88).

RIGHT AND LEFT INVERTIBILITY IN λ-β-CALCULUS (*)

by I. MARGARIA and M. ZACCHI [1]

Communicated by J. F. PERROT

Abstract. — *A characterization of λ-terms having left and/or right inverses in λ-β-calculus is given and the sets of all and only λ-terms left/right invertible are constructed. The above results are obtained using the concept of Böhm tree, so this study is further used to characterize the λ-terms left/right invertible in the graph model* $\mathbf{P}_\omega$.

Résumé. — *Dans ce papier on va caractériser les λ-termes invertibles à droite et/ou à gauche, en donnant les règles pour construire les deux ensembles constitués respectivement par tous les λ-termes ayant un inverse droite ou gauche. Puisque ces résultés ont été obtenus par la notion d'arbre de Böhm on peut utiliser cette étude au fin de caractériser les λ-termes invertibles à droite ou à gauche dans le modèle* $\mathbf{P}_\omega$.

0. INTRODUCTION

Aim of this paper is the characterization of λ-terms having left and/or right inverses in λ-β-calculus. The semigroup Λ of λ-β-(η)-terms, having the combinator $\mathbf{I} \equiv \lambda x . x$ as identity element and the operation $\circ$ defined by $X \circ Y = \mathbf{B} X Y$ (where $\mathbf{B} \equiv \lambda xyz . x(yz)$) as composition, has been studied with respect to the left/right invertibility problem in [2], [4], [7, p. 167-168], [8], [9].

In particular in the λ-β-calculus the set of normal forms having at least one left or right inverse has been characterized in [4]. The same paper shows that the combinator **I** is the only normal form having both left and right inverse.

The present paper tries to give the final solution to the invertibility problem in λ-β-calculus showing the necessary and sufficient conditions under which an arbitrary λ-term possesses a left (right) inverse and characterizing the set of terms for which there exists only one left (right) inverse; for the

(*) Received in December 1981, revised in June 1982.
[1] Istituto di Scienze dell'Informazione, Università degli Studi di Torino, Corso M. D'Azeglio n° 42, Torino.

other left (right) invertible terms an infinite number of inverses is proved to exist. The basic definitions which the paper relies on are those of direct approximation [11], of Böhm tree [1] and of partial order relation $\sqsubseteq$ on the set of λ-$\underline{\Omega}$-terms, as stated in [10]. Using these notions it is possible to carry on Λ the relation $\sqsubseteq$ defining a λ-term X less or equal to a λ-term Y ($X \sqsubseteq Y$) if and only if its direct approximation $\Phi(X)$ is less or equal to the direct approximation $\Phi(Y)$ of Y ($\Phi(X) \sqsubseteq \Phi(Y)$) and to associate with a λ-term X the approximation set as the set of λ-$\underline{\Omega}$-terms $\Phi(X')$ such that X is β-convertible to X'. Firstly we notice that every left (right) inverse of a λ-term X is a left (right) inverse of all λ-terms Y such that $X \sqsubseteq Y$. Then in order to characterize the set of terms having left inverse, an operation, called terminal extension, is introduced on the set of Böhm trees. Roughly speaking a terminal extension of a Böhm tree A is a Böhm tree A' obtained from A modifying a terminal node of A either introducing in its label the abstraction of a new variable or pushing the head variable down of a level and substituting it by a bound variable. So we can prove that a λ-term X has a left inverse if and only if there exists in the approximation set of X at least one approximation which can be obtained from **I** applying a sequence of terminal extensions. Moreover it results that every term left invertible, different from **I**, possesses an infinite number of non-convertible left inverses.

The problem of the right invertibility is approached in a similar way. The operation of adding a son with label $\underline{\Omega}$ to the root of a Böhm tree A to obtain a Böhm tree A' is called initial extension. This allows to assert that a λ-term X has right inverse iff there exists at least one approximation of X which can be obtained from **I** applying a sequence of initial extensions. Obviously, as a corollary, it results that **I** is the only λ-term both left and right invertible. Furthermore we can prove that the number of right inverses for a right invertible term X is either one or infinite depending on the form of the term itself.

Finally we notice that the above results about invertibility can be carried on the graph model $\mathbf{P}_\omega$ [1, p. 467] and we show that the two functions which map an element of $\mathbf{P}_\omega$ into the set of all its right or left inverses, respectively, are not monotonic, i. e. it is possible to find a left (right) inverse of an element X of $\mathbf{P}_\omega$ which is not a left (right) inverse of an element Y, whereas $X \underset{\sim}{\sqsubseteq} Y$ ($\underset{\sim}{\sqsubseteq}$ is the usual order relation on $\mathbf{P}_\omega$).

1. NOTATIONS AND DEFINITIONS

In the sequel we will use the following notions and conventions:

i) λ-calculus means λ-β-calculus, normal form λ-β-normal form, $\geqslant$, $=$, $\equiv$

denote β-reducibility, α-β-convertibility and modulo α identity, respectively; moreover Λ represents the set of λ-terms;

ii) the word combinator will refer to closed λ-terms, i. e. terms without free variables; the combinators will be indicated by uppercase, boldface characters, for example $\mathbf{B} \equiv \lambda xyz.x(yz)$, $\mathbf{I} \equiv \lambda x.x$, etc.;

iii) we indicate by means of the ordered sequences of λ-terms

$$\langle X_0, X_1, \ldots, X_k \rangle$$

the λ-terms $\lambda z.zX_0X_1 \ldots X_k$ where z does not occur free in any X_i, $0 \leqslant i \leqslant k$ (Church n-tuple) [6];

iv) $\mathbf{C}[\ ]$ denotes a context, i. e. a λ-term where one subterm is missing; $\mathbf{C}[X]$ then denotes the result of filling the missing subterm with X (for a more formal definition see [11]);

v) $X[x := Y]$ indicates the λ-term obtained from a λ-term X by substituting in it the λ-term Y to every free occurrence (if any) of the variable x.

As the concept of approximation of a λ-term [11] and the related one of Böhm tree [1, p. 211] are very useful for this study, we summarize here the principal definitions and conventions about them.

A λ-term X has *head normal form* if it has the form $\lambda x_1 x_2 \ldots x_m . yX_1X_2 \ldots X_n$ where:

— $x_1, x_2, \ldots, x_m$ are variables and $m \geqslant 0$;
— $X_1, X_2, \ldots, X_n$ are λ-terms and $n \geqslant 0$;
— y is a variable, free or bound (as usual it will be called the head variable of X).

The *direct approximation* $\Phi(X)$ of a λ-term X is defined as follows:

$\Phi(X) = \lambda x_1 \ldots x_m . y\Phi(X_1)\Phi(X_2) \ldots \Phi(X_n)$ if $X = \lambda x_1 \ldots x_m . yX_1X_2 \ldots X_n$;
$\Phi(X) = \underline{\Omega}$, where $\underline{\Omega}$ is an extra constant, if X has not a head normal form.

The set $\Phi(\Lambda)$ will be indicated by $\mathscr{N}$ (set of λ-$\underline{\Omega}$-terms). Inside $\mathscr{N}$ the following partial order relation $\sqsubseteq$ is defined [10]: for any M, N of $\mathscr{N}$ $M \sqsubseteq N$ iff either

i) $M \equiv \underline{\Omega}$; or

ii) $M \equiv \lambda x_1 x_2 \ldots x_n . x_j M_0 \ldots M_k$
$N \equiv \lambda x_1 x_2 \ldots x_n . x_j N_0 \ldots N_k$

and $M_i \sqsubseteq N_i$ for any i $(0 \leqslant i \leqslant k)$.

Given a λ-term X we call *approximation set* of $X : \mathscr{A}(X)$ the subset of $\mathscr{N}$ so defined:

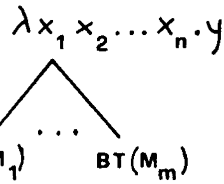
$$\mathscr{A}(X) = \{ M \in \mathscr{N} \mid M \sqsubseteq \Phi(X) \}.$$

The partial order relation $\sqsubseteq$ can be carried on Λ as follows: for any X, Y of Λ, $X \sqsubseteq Y$ iff $\Phi(X) \sqsubseteq \Phi(Y)$.

We can visualize every element M of $\mathcal{N}$ by means of a suitable tree: the Böhm tree (B. T.) of M. Given an element M of $\mathcal{N}$, the B. T. of M: $BT(M)$ is the labelled tree so defined:

i) if $M \equiv \Omega$ $\quad BT(M) \equiv \ .\Omega$

ii) if $M \equiv \lambda x_1 x_2 \ldots x_n . y M_1 \ldots M_m$ $\quad BT(M) \equiv$



We will refer to $\mathscr{B}$ as to the set of the B. T. of the elements of $\mathcal{N}$. The nodes of a B. T. will be indicated by strings of natural numbers (included the empty string ε, labelling the root) in the usual way, so that β denotes a successor of α if and only if α is a prefix of β: $\beta = \alpha\gamma$ for some γ. Let A be a B. T. and α be a node with label $\lambda x_1 \ldots x_n . y$, in the sequel we will use the following conventions [*see* 1, p. 218] :

i) A_α indicates the subtree of A having as root the node α;

ii) $\bar{\alpha}$ indicates the path from the root to the node α;

iii) $b(\alpha)$ indicates the vector of the bound variables occurring in the label of α, i. e. $b(\alpha) = x_1 x_2 \ldots x_n$;

iv) $b(\bar{\alpha})$ indicates the vector of the bound variables occurring in the labels of the nodes of the path $\bar{\alpha}$, inductively defined as follows:

— $b(\bar{\varepsilon}) = b(\varepsilon)$

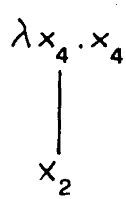
— $b(\alpha' \langle k \rangle) = b(\alpha')b(k)$.

By way of example, for the B. T. A of figure 1, if we choose as node α the node $\langle 1\ 0 \rangle$, we have:

$$b(\alpha) = x_4$$

$$b(\bar{\alpha}) = x_0 x_1 x_2 x_3 x_4$$

$A_\alpha \equiv$



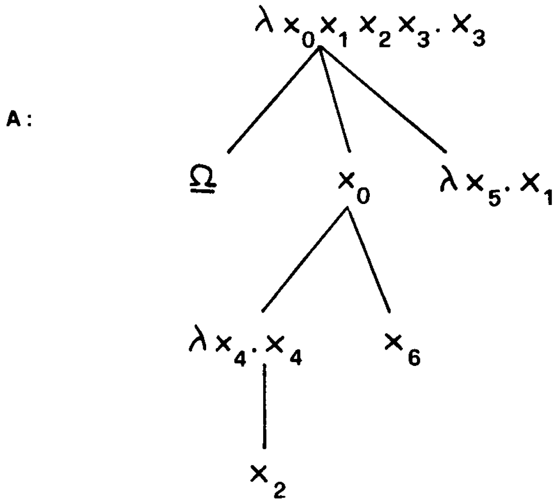


Figure 1. — A Böhm tree A.

By streching the Böhm tree definition, in the sequel sometimes we will refer to the B. T. of an element X of Λ: $BT(X)$, as to the B. T. of its direct approximation.

Obviously any B. T. A of $\mathscr{B}$ will define one and only one term of $\mathscr{N}$: M_A such that $BT(M_A)=A$ (for example for the B. T. A of figure 1

$$M_A \equiv \lambda x_0 x_1 x_2 x_3 . x_3 \underline{\Omega}(x_0(\lambda x_4 . x_4 x_2) x_6) \lambda x_5 . x_1);$$

hence the order relation $\sqsubseteq$ on $\mathscr{N}$ can be carried on $\mathscr{B}$: $A \sqsubseteq B$ iff $M_A \sqsubseteq M_B$.

2. RIGHT AND LEFT INVERTIBILITY

Aim of this section is to study the conditions under which an arbitrary λ-term X has right and/or left inverses. In the sequel we use the following notations:

i) $X_R(X_L)$ denotes a right (left) inverse of a λ-term X, i. e.,:

$$\mathbf{B}XX_R = \mathbf{I} \qquad (\mathbf{B}X_L X = \mathbf{I}).$$

ii) $\mathscr{I}_R(X)$ $(\mathscr{I}_L(X))$ denotes the set of all the right (left) inverses of a λ-term X.

THEOREM 1: Let X, Y be two λ-terms of Λ for which $X \sqsubseteq Y$, then:

$$i)\ \mathscr{I}_R(X) \subseteq \mathscr{I}_R(Y)$$
$$ii)\ \mathscr{I}_L(X) \subseteq \mathscr{I}_L(Y).$$

Proof: *i*) The assertion is trivially true for $\mathscr{I}_R(X)$ empty.

If it is not true, we prove that any right inverse X_R of X is also a right inverse for Y. By definition we have:

$X(X_R y) \geqslant y$ for any variable y not free in X and X_R.

Since Lévy has proved (th. 5.8, p. 105 of [10]) that if $X \sqsubseteq Y$ then $\mathbf{C}[X] \sqsubseteq \mathbf{C}[Y]$ for any context $\mathbf{C}[\ ]$, if we choose as context $[\ ](X_R y)$ it will be:

$$y \leqslant X(X_R y) \sqsubseteq Y(X_R y) \quad \text{hence} \quad Y(X_R y) \geqslant y.$$

ii) The proof is analogous to the preceding one if we choose as context $X_L([\ ]y)$.

2.1. Left Invertibility

DEFINITION 2.1.1: Let A, A' be two Böhm trees and α a terminal node of A with label $\lambda x_{i_1} x_{i_2} \ldots x_{i_h}.x_t$. We say that A' is a *terminal extension of A in* α if A' results from A by one of the following substitutions:

1) the label of the node α in A is replaced in A' by the label

$$\lambda x_{i_1} x_{i_2} \ldots x_{i_h} x_{i_{h+1}}.x_t \quad \text{(terminal extension of type 1);}$$

2) the subtree A_α is replaced in A' by a subtree A'_α such that:

a) the label of α is $\lambda x_{i_1} \ldots x_{i_h}.x_j$, where x_j is a bound variable distinct from x_t;

b) α has m sons with $m \geqslant 1$. Each of these sons are terminal nodes, one and only one of them has label x_t, whereas the remaining $m-1$ have label $\underline{\Omega}$ (*see* fig. 2) (terminal extension of type 2).

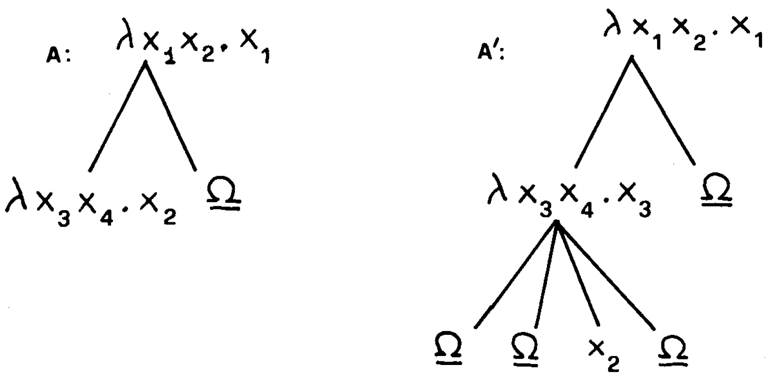


Figure 2. — A terminal extension of type 2.

With every terminal extension e of type 2, we associate the triple

$$\tau(e) = \langle x_j, m, k \rangle,$$

where x_j and m are respectively the name of the head variable and the number of sons of the node α in A' and k indicates that the only son of α in A' with label different from $\underline{\Omega}$ is the k-th.

DEFINITION 2.1.2: Let A, A' be two Böhm trees.

We say that *A' is a terminal extension of A* ($A \underset{t-\text{ext}}{\longrightarrow} A'$) if it is a terminal extension of A in some terminal node.

DEFINITION 2.1.3: We call *Left Invertible Term Generator Set* the subset $\mathscr{L} \subset \mathscr{N}$ inductively defined as follows:

i) $\mathbf{I} \in \mathscr{L}$

ii) $N \in \mathscr{L}$ and $BT(N) \underset{t-\text{ext}}{\longrightarrow} BT(N') \Rightarrow N' \in \mathscr{L}$.

DEFINITION 2.1.4: Let N be an element of $\mathscr{L}$. We call *history of N* : $\mathscr{H}(N)$ a sequence of elements of $\mathscr{L}$: $\langle N^0, N^1, \ldots, N^h \rangle$ such that $N^0 \equiv \mathbf{I}$, $N^h \equiv N$ and for any i, $0 \leqslant i \leqslant h-1$, $BT(N^i) \underset{t-\text{ext}}{\longrightarrow} BT(N^{i+1})$.

LEMMA 2.1.1: Every element N of $\mathscr{L}$ has one and only one history: $\mathscr{H}(N)$.

Proof: Obvious from definition 2.1.1 and definition 2.1.3.

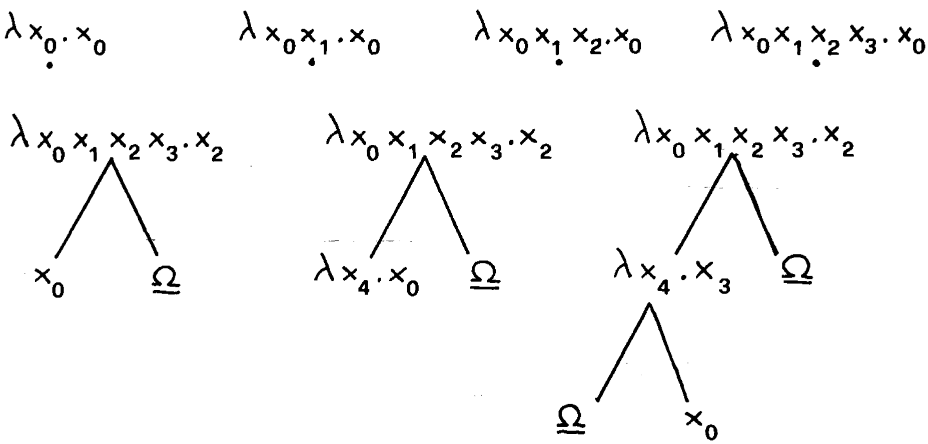


Figure 3. — Böhm trees of the history of the λ-$\underline{\Omega}$-term $\lambda x_0 x_1 x_2 x_3 . x_2(\lambda x_4 . x_3 \underline{\Omega} x_0)\underline{\Omega}$.

DEFINITION 2.1.5: Let N be an element of $\mathscr{L}$. We say that *N is a term non-homogeneous for the variable x_t* if in its history $\mathscr{H}(N)$ there are at least two terminal extensions e, e' of type 2 with $\tau(e) = \langle x_t, m, k \rangle$ and $\tau(e') = \langle x_t, m', k' \rangle$ such that $m \neq m'$ and/or $k \neq k'$.

Figure 4(a) shows the Böhm tree of a term non-homogeneous for the variable x_1, whereas it is homogeneous for the variable x_2; instead the term whose Böhm tree is in figure 4(b) is homogeneous for each variable occurring as head variable; in such a case we say that the term is *homogeneous.*

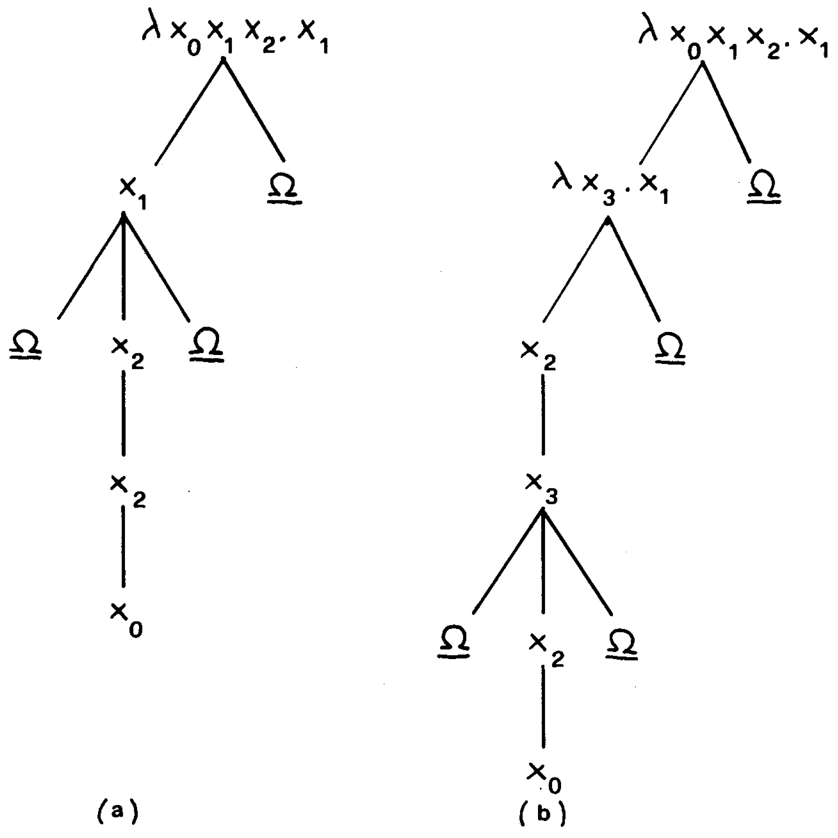


Figure 4. — Böhm trees of a non-homogeneous (a) and of a homogeneous λ-$\underline{\Omega}$-term (b).

From lemma 3 of [3] it follows lemma 2.1.2 which has been rewritten and proved (in a simpler way) using the notation of the present work.

LEMMA 2.1.2: Let N be a λ-$\underline{\Omega}$ term of $\mathscr{L}$, non-homogeneous for a set of variables $\{x_{l_1}, x_{l_2}, \ldots, x_{l_k}\}$. We state that there is a normal combinator

$\mathbf{C}_{[m]}\mathbf{I}$ such that the term $N'[x_{l_i} := \mathbf{C}_{[m]}\mathbf{I}]$, where N' is obtained from N by eliminating the abstraction of x_{l_i}, is non-homogeneous for the set

$$\{ x_{l_1}, x_{l_2}, \ldots, x_{l_{i-1}}, x_{l_{i+1}}, \ldots, x_{l_k} \}.$$

Proof: Let $e_1, e_2, \ldots, e_n$ be the terminal extensions of type 2, occurring in $\mathscr{H}(N)$ such that the first element of $\tau(e_j)$ is $l_i (1 \leqslant j \leqslant n)$, i. e.:

$$\tau(e_1) = \langle x_{l_i}, m_1, k_1 \rangle$$
$$\tau(e_2) = \langle x_{l_i}, m_2, k_2 \rangle$$
$$\ldots$$
$$\tau(e_n) = \langle x_{l_i}, m_n, k_n \rangle .$$

Let $m = \max(m_1, m_2, \ldots, m_n)$. It is easy to prove that the normal combinator $\mathbf{C}_{[m]}\mathbf{I} \equiv \lambda t_0 t_1 \ldots t_m . t_m t_0 t_1 \ldots t_{m-1}$ satisfies the thesis, because it substitutes the different occurrences of x_{l_i} by different variables.

LEMMA 2.1.3: Every λ-term X of Λ, whose direct approximation is in $\mathscr{L}$, has at least a left inverse.

Proof: Firstly we prove that every λ-term X, whose direct approximation is a homogeneous element of $\mathscr{L}$ has a left inverse. From definition 2.1.3 it follows that there is one and only one terminal node of $BT(X)$ having label different from $\underline{\Omega}$; let such a node be α and let $b(\bar{\alpha}) = x_0 x_1 \ldots x_n$, $n \geqslant 0$. We assert that there are n suitable λ-terms $\Psi_1, \Psi_2, \ldots, \Psi_n$ such that the sequence $\langle \Psi_1, \Psi_2, \ldots, \Psi_n \rangle$ is a left inverse for X. We prove this assertion by induction on the number h of elements of $\mathscr{H}(\Phi(X))$.

$h = 1$. $\quad X = \mathbf{I} \quad X_L = \lambda z . z \equiv \mathbf{I}.$

$h + 1$.

Given $\mathscr{H}(\Phi(X)) = \langle N^0, N^1, \ldots, N^h, N^{h+1} \rangle$, let X^i, $0 \leqslant i \leqslant h$, be a λ-term such that $\Phi(X^i) = N^i$, let $X^{h+1} = X$ and $A^i = BT(N^i)$. We distinguish two cases either A^{h+1} extends A^h by a terminal extension of type 1 or A^{h+1} extends A^h by a terminal extension of type 2. In the first case we say that a left inverse for X can be obtained by adding to the left inverse of X^h (existing by induction hypothesis) a generic λ-term Ψ_n, i. e.

if $\quad X_L^h = \langle \Psi_1^h, \Psi_2^h, \ldots, \Psi_{n-1}^h \rangle$

X_L^{h+1} will be $\langle \Psi_1^h, \Psi_2^h, \ldots, \Psi_{n-1}^h, \Psi_n \rangle$.

In fact it follows from the definitions of $\mathscr{L}$ and of terminal extension of type 1 that:

$$(X^{h+1} y) = (X^h y)[y := \lambda x_n . y]$$

and by induction hypothesis:

$$(X^{h+1}y)\Psi_1^h\Psi_2^h \ldots \Psi_{n-1}^h \geqslant \lambda x_n . y$$

i. e.:

$$(\lambda x_n . y)\Psi_n \geqslant y.$$

In the second case, let $\langle x_j, m, k \rangle$ be the triple associated with the $(h+1)-$th terminal extension. If x_j occurs as head variable in some terminal extension preceding the $(h+1)$-th one, from homogeneity hypotesis it follows that the left inverse X_L^h (existing by induction hypothesis) is also a left inverse for X^{h+1}; otherwise we prove that a left inverse of X^{h+1} can be obtained by substituting in the left inverse X_L^h for the λ-term Ψ_j^h the normal combinator (selector)

$$\mathbf{U}_k^m = \lambda t_1 t_2 \ldots t_m . t_k, \quad \text{i. e.:}$$

$$X_L^{h+1} = \langle \Psi_1^h, \Psi_2^h, \ldots, \Psi_{j-1}^h, \mathbf{U}_k^m, \Psi_{j+1}^h, \ldots \rangle.$$

It follows from definitions of $\mathscr{L}$ and of terminal extension of type 2, that:

$$(X^{h+1}y) = (X^h y)\,[y := x_j X_1' X_2' \ldots X_{k-1}' y X_{k+1}' \ldots X_m']$$

where X_i' are unsolvable terms; then:

$$(X^{h+1}y)\Psi_1^h\Psi_2^h \ldots \Psi_{j-1}^h \mathbf{U}_k^m \Psi_{j+1}^h \ldots \geqslant \mathbf{U}_k^m X_1' X_2' \ldots X_{k-1}' y X_{k+1}' \ldots X_m' \geqslant y.$$

Now, let us suppose that X has a direct approximation non-homogeneous only for one variable x_i. From lemma 2.1.2 it follows that there exists an integer m such that the term $N'\ [x_i := \mathbf{C}_{[m]}\mathbf{I}]$, where N' is obtained from $\Phi(X)$ by eliminating the abstraction of x_i, is homogeneous. Let X' be a λ-term of Λ such that $\Phi(X') = N'\ [x_i := \mathbf{C}_{[m]}\mathbf{I}]$ and let X_L' be its left inverse, existing for the first part of this lemma: $X_L' = \langle \Psi_1', \Psi_2', \ldots, \Psi_n' \rangle$. We maintain that the sequence $X_L = \langle \Psi_1', \Psi_2', \ldots, \Psi_{i-1}', \mathbf{C}_{[m]}\mathbf{I}, \Psi_i', \Psi_{i+1}', \ldots, \Psi_n' \rangle$ is a left inverse for X. In fact:

$$(X'y)\Psi_1'\Psi_2' \ldots \Psi_{i-1}' = (Xy)\Psi_1'\Psi_2' \ldots \Psi_{i-1}'(\mathbf{C}_{[m]}\mathbf{I})$$

$$(Xy)\Psi_1'\Psi_2' \ldots \Psi_{i-1}'(\mathbf{C}_{[m]}\mathbf{I})\Psi_i' \ldots \Psi_n' = (X'y)\Psi_1'\Psi_2' \ldots \Psi_{i-1}'\Psi_i' \ldots \Psi_n' \geqslant y.$$

The proof can be generalized in a obvious way to the case of terms non-homogeneous for more than one variable.

LEMMA 2.1.4: Every λ-term of Λ, distinct from $\mathbf{I}$ and having the direct approximation in $\mathscr{L}$, has an infinite number of non convertible left inverses.

Proof: Let X be a λ-term satisfying the hypothesis of this lemma. If some of the λ-terms of the not empty sequence X_L, obtained by the construction of lemma 2.1.3, are arbitrary we can obtain an infinite number of left inverses choosing them in infinite ways.

Instead if each Ψ_i has been substituted by a suitable combinator, we can obtain an infinite number of left inverses as follows. Let U_k^m be a selector occurring in X_L (from proof of lemma 2.1.3 it is clear that in X_L we have surely some selectors), i. e.:

$$X_L = \langle \Psi_1, \Psi_2, \ldots, \Psi_{i-1}, U_k^m, \Psi_{i+1}, \ldots, \Psi_k \rangle.$$

It is easy to prove that

$$X_L' = \langle \Psi_1, \Psi_2, \ldots, \Psi_{i-1}, U_k^{m+n}, \Psi_{i+1}, \ldots, \Psi_h, \Phi_1, \ldots, \Phi_n \rangle$$

where $\Phi_i (1 \leqslant i \leqslant n)$ are generic λ-terms, is another left inverse for X, non-convertible to X_L:

$$X_L'(Xy) \geqslant (Xy)\Psi_1\Psi_2 \ldots \Psi_{i-1}U_k^{m+n}\Psi_{i+1} \ldots \Psi_n\Phi_1\Phi_2 \ldots \Phi_n \geqslant$$
$$\geqslant (\lambda t_1 t_2 \ldots t_n . y)\Phi_1\Phi_2 \ldots \Phi_n \geqslant y.$$

DEFINITION 2.1.6: *A λ-term X of Λ is of type Σ* if the set $\mathscr{A}(X) \cap \mathscr{L}$ is not empty.

Remark 1: For any Böhm tree $BT(X)$ of a λ-term X of type Σ (shortly B. T. of type Σ), there is at least a terminal node σ, such that:

i) the first component of the vector $b(\overline{\sigma})$ occurs as head variable only in the label of σ;

ii) every head variable in the label of a not terminal node of the path $\overline{\sigma}$, is bound.

The Böhm tree of figure 5 is of type Σ, because the terminal nodes $\langle 2 \rangle$ and $\langle 1\ 1 \rangle$ satisfy the conditions of the remark 1.

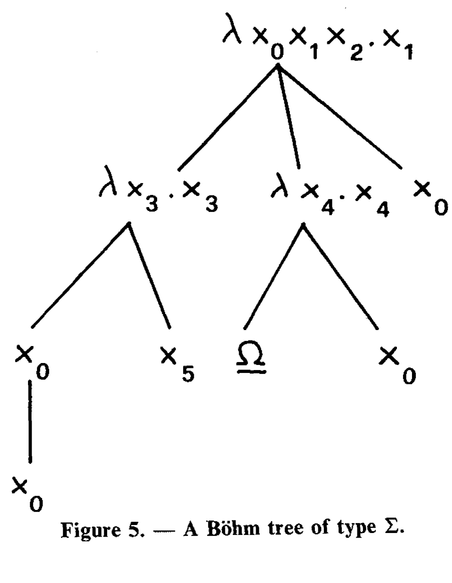


Figure 5. — A Böhm tree of type Σ.

THEOREM 2.1.1: A λ-term X has at least a left inverse if and only if it is of type Σ.

Proof: If X is of type Σ, there is at least an approximation $N' \sqsubseteq \Phi(X)$ belonging to $\mathscr{L}$, so for theorem 2.1 and lemma 2.1.3 X has at least a left inverse.

Now, let us suppose, *per absurdum*, that the λ-term X not of type Σ has a left inverse. If X is not of type Σ one of the conditions of remark 1 is not satisfied.

If for any path $\bar{\sigma}$ of $BT(X)$ the condition i) of remark 1 does not hold, then in (Xy) the free variable y, if it occurs, always occurs applied to a positive number of arguments, which cannot be eliminated using only β-reductions. Instead if for any path for which condition i) of remark 1 holds, there is some non-terminal node whose label has as head variable a free variable, then there is no λ-term Y such that in $Y(Xy)$ this free variable can be erased to obtain y.

2.2. Right Invertibility

DEFINITION 2.2.1: Let A, A' be two B. T., different from $\underline{\Omega}$. We say that A' *is an initial extension of* A ($A \underset{i-\text{ext}}{\longrightarrow} A'$) if A' results from A by adding to its root a son with label $\underline{\Omega}$ (*see* fig. 6)

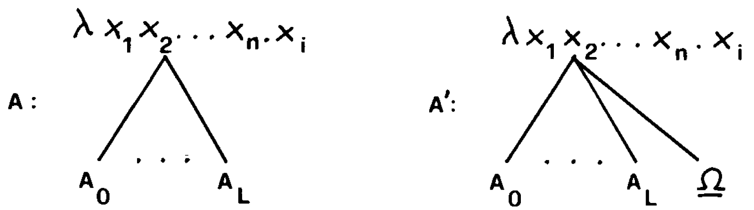


Figure 6. — Two Böhm trees A and A′ such that A $\underset{i-\text{ext}}{\longrightarrow}$ A′.

DEFINITION 2.2.2: We call *Right Invertible Term Generator Set* the subset $\mathscr{R} \subset \mathscr{N}$ inductively defined as follows:

i) $\mathbf{I} \in \mathscr{R}$

ii) $N \in \mathscr{R}$ and $BT(N) \underset{i-\text{ext}}{\longrightarrow} BT(N') \Rightarrow N' \in \mathscr{R}$.

LEMMA 2.2.1: Every λ-term X of Λ, whose direct approximation $\Phi(X)$ is in $\mathscr{R}$ has one and only one right inverse X_R.

Proof: " One " part. Let X be $\lambda x . x X_1 X_2 \ldots X_h$, with $X_i (1 \leqslant i \leqslant h)$ unsolvable; we take as X_R the λ-term $\mathbf{U}_1^{h+1} \equiv \lambda x_0 x_1 \ldots x_h . x_0$. It's trivial that $X(X_R\, y) \geqslant y$, so X_R is a right inverse for X.

" Only one " part. Let us suppose, *per absurdum*, that $X = \lambda z . z X_1 \ldots X_h$, with $X_i (1 \leqslant i \leqslant h)$ unsolvable, has a right inverse $X'_R = \lambda x_0 x_1 \ldots x_n . x_j Y_1 Y_2 \ldots Y_t$ distinct from $X_R = \lambda x_0 x_1 \ldots x_h . x_0$.

Since, from the definition of right inverse,

$$(X'_R y) X_1 \ldots X_h \geqslant y$$

we must have $n \leqslant h$, otherwise we cannot eliminate the $n-h$ initial abstractions.

Since from theorem 2.1.1 it follows that X'_R is of type Σ, the head variable x_j of X'_R must be exactly x_0 if $t=0$, different from x_0 and bound if $t \neq 0$. In the first case we must have $n=h$, otherwise y remains applied to a positive number of λ-terms, which cannot be eliminated to give y, hence $X'_R = X_R$, contrary to the hypothesis. In the second case, we should have, for some X_j unsolvable:

$$X_j Y'_1 \ldots Y'_t X_{n+1} \ldots X_h \geqslant y,$$

where:

$$Y'_i = Y_i [x_0 := y, x_1 := X_1, \ldots, x_n := X_n] \quad \text{for} \quad 1 \leqslant i \leqslant t$$

and this is an *absurdum*.

DEFINITION 2.2.3: We say that a *λ-term X of Λ is of type* Ξ if the set $\mathscr{A}(X) \cap \mathscr{R}$ is not empty.

Example: The λ-terms whose B. T. is shown in figure 7 are of type Ξ, because they have as approximation the λ-$\underline{\Omega}$-term $\lambda x_0 . x_0 \underline{\Omega}\underline{\Omega}\underline{\Omega}$.

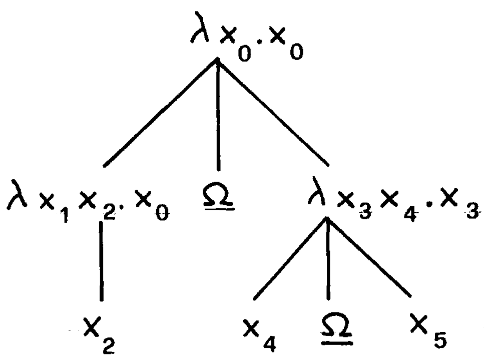


Figure 7. — A Böhm tree of λ-terms of type Ξ.

Remark 2: If X is of type Ξ, it has the form $\lambda x . x X_1 \ldots X_h$.

THEOREM 2.2.1: A λ-term X has at least a righ inverse if and only if it is of type Ξ.

Proof: If X is a λ-term of type Ξ, $\mathscr{I}_R(X)$ is not empty from lemma 2.2.1 and theorem 2.1. Now let us suppose X not of type Ξ; then X can have in its head more than one abstraction: $X=\lambda x_0 x_1 \ldots x_n . x_i X_1 \ldots X_h$ and/or X can have as head variable a free variable $X=\lambda x_0 x_1 \ldots x_n . y X_1 \ldots X_h$. In the first case the $n+1$ initial abstractions cannot be eliminated using β-reductions; in the second case the free variable cannot be erased.

COROLLARY: The only λ-term having left and right inverse is the combinator **I**.

DEFINITION 2.2.4: We say that a λ-term X is Ω-like if either:

i) X is unsolvable, or
ii) X is solvable and its head variable is free.

We say X not Ω-like on the contrary.

LEMMA 2.2.2: Let X be a λ-term:

i) if X is Ω-like, for any λ-term Y, the application $(X\,Y)$ is also an Ω-like term.

ii) if X is Ω-like, for any variable y different from the head variable of X, if any, there are no h λ-terms $Y_1, Y_2, \ldots, Y_h$ such that:

$$X\,Y_1 Y_2 \ldots Y_h \geqslant y$$

iii) if X is not Ω-like there are h λ-terms $Y_1, Y_2, \ldots, Y_h$ such that:

$$X\,Y_1 Y_2 \ldots Y_h \geqslant \mathbf{I}.$$

Proof: Both assertions *i*) and *ii*) are trivially true for X unsolvable. Let us suppose X solvable with head variable free: $X=\lambda x_1 x_2 \ldots x_k . a X_1 \ldots X_h$, then the head variable a cannot be eliminated using only β-reductions, so $(X\,Y)$ is solvable with head variable a, moreover it is impossible to reduce X to a free variable y different from a.

To prove assertion *iii*), let us suppose $X=\lambda x_1 x_2 \ldots x_k . x_j X_1 \ldots X_s$, with x_j bound. If we choose $h=k$, $Y_i=\Psi_i$, where Ψ_i is a generic λ-term, for $1 \leqslant i < j$ and $j < i \leqslant h$, and $Y_j=\mathbf{U}_{s+1}^{s+1}$, where $\mathbf{U}_{s+1}^{s+1} \equiv \lambda x_0 x_1 \ldots x_s . x_s$, it is trivially true that $X\,Y_1 \ldots Y_h \geqslant \mathbf{I}$.

THEOREM 2.2.2: Let X be a λ-term of type Ξ: $X=\lambda z . z X_1 \ldots X_h$. If every X_i is Ω-like, then X has one and only one right inverse, else X has an infinite number of right inverses.

Proof: Let us suppose $X=\lambda z . z X_1 \ldots X_h$ with $X_i (1 \leqslant i \leqslant h)$ Ω-like. We must prove that X has only the right inverse given in the proof of lemma 2.2.1: $X_R = \lambda x_0 \ldots x_h . x_0$.

The existence of another inverse X'_R should cause an *absurdum*, in fact being $X'_R = \lambda x_0 x_1 \ldots x_n . x_j Y_1 \ldots Y_h$ of type Σ because of theorem 2.1.1, its head variable must be bound and different from x_0 (*see* proof of lemma 2.2.1), then we should have, for some X_i Ω-like and some $Z_1, Z_2, \ldots, Z_k$:

$$X_i Z_1 \ldots Z_k \geqslant y$$

and this is an *absurdum* because of lemma 2.2.2, case *ii*).

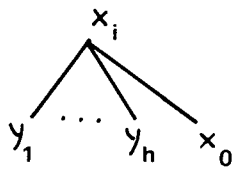
Now let us suppose that at least one λ-term X_i is not Ω-like. For lemma 2.2.2, case *iii*), there exist h λ-terms $Y_1, Y_2, \ldots, Y_h$ such that:

$$X_i Y_1 Y_2 \ldots Y_h \geqslant \mathbf{I}.$$

Let $\mathscr{S}(X)$ be the set inductively defined as follows:

i) $X_R = \lambda x_0 \ldots x_h . x_0$ is in $\mathscr{S}(X)$

ii) if Y is in $\mathscr{S}(X)$ and Y' is a term obtained by substituting in the $BT(Y)$ to the terminal node x_0 the subtree:



;

then Y' is in $\mathscr{S}(X)$.

It is obvious that $\mathscr{S}(X)$ has an infinite number of elements, which are all right inverses of X.

3. LEFT AND RIGHT INVERTIBILITY IN THE GRAPH MODEL $\mathbf{P}_\omega$

H. Barendregt [1, p. 496-500], reformulating in terms of Böhm trees the Hyland's characterization of the equality in the graph model $\mathbf{P}_\omega$, has shown that

$$\mathbf{P}_\omega \models X = Y \Leftrightarrow BT(X) = BT(Y).$$

So we can say that the above results about invertibility on $\mathscr{N}$ (or $\mathscr{B}$) can be carried on $\mathbf{P}_\omega$. Now let f and g be the following functions

$$f : \mathbf{P}_\omega \to 2^{\mathbf{P}_\omega} \qquad f(X) = \mathscr{I}_L(X)$$
$$g : \mathbf{P}_\omega \to 2^{\mathbf{P}_\omega} \qquad g(X) = \mathscr{I}_R(X),$$

since both $\mathbf{P}_\omega$ and $2^{\mathbf{P}_\omega}$ are complete lattices [1, p. 19], it is of some interest to investigate whether f and g are monotonic functions, i. e.

whether $\mathscr{I}_L(X) \subseteq \mathscr{I}_L(Y)$ whenever $X \sqsubseteq Y$

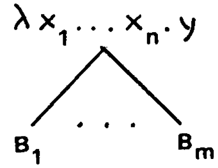
and whether $\mathscr{I}_R(X) \subseteq \mathscr{I}_R(Y)$ whenever $X \sqsubseteq Y$,

being $\sqsubseteq$ the order relation on $\mathbf{P}_\omega$. H. Barendregt [1, p. 228-240, 496-500] has shown that

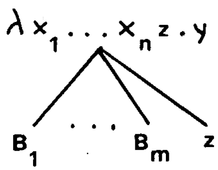
$$\mathbf{P}_\omega \models X \sqsubseteq Y \Leftrightarrow BT(X)\eta_{\sqsubseteq}BT(Y),$$

where $\eta_{\sqsubseteq}$ is the order relation defined as it follows.

DEFINITION 3.1: Let A be a B. T. and α one of its nodes having label $\lambda x_1 \ldots x_n . y$. The B. T. A' *is an* η-*expansion of* A *at* α if it results from A by replacing the subtree A_α, which has the form



by the subtree A'_α having the form



In the sequel if α is the root of A, we call the η-expansion at α *head* η-*expansion*, if α is a terminal node of A we call it *terminal* η-*expansion.*

DEFINITION 3.2: Let A, A' be two Böhm trees. A' *is a (possibly) infinite* η-*expansion of* A (shortly $A \leqslant_\eta A'$) if it results from A by the application of a (possibly infinite) sequence of η-expansions.

DEFINITION 3.3: Let A, A' be two Böhm trees. $A \eta_{\sqsubseteq} A'$ if there exists a Böhm tree B, which is a (possibly) infinite η-expansion of A, such that $B \sqsubseteq A'$, i. e. $A \leqslant_\eta B \sqsubseteq A'$.

In the sequel if $A \eta_{\sqsubseteq} B$, i. e. $A \leqslant_\eta A' \sqsubseteq B$ for some A', and no terminal (head) η-expansion is applied to A in order to obtain A', we say that B results from A without terminal (head) η-expansions.

LEMMA 3.1: Let X, Y be two λ-terms for which $BT(X)\eta_{\sqsubseteq}BT(Y)$ and let X be of type Ξ. Y is of type Ξ if and only if $BT(Y)$ results from $BT(X)$ without head η-expansions.

Proof: Obvious.

THEOREM 3.1: The function g is not monotonic.

Proof: Let X be a λ-term of type Ξ and let Y be a λ-term for which $BT(X)\eta_{\sqsubseteq}BT(Y)$. If $BT(Y)$ results from $BT(X)$ by some head η-expansion, from lemma 3.1 it follows that $\mathscr{I}_R(Y)$ is empty so $\mathscr{I}_R(X) \nsubseteq \mathscr{I}_R(Y)$, being $\mathscr{I}_R(X)$ not empty.

Notice that also in the case in which $BT(Y)$ results from $BT(X)$ without head η-expansions we can have $\mathscr{I}_R(X) \nsubseteq \mathscr{I}_R(Y)$. For example if

$$X=\lambda x_0 . x_0(\lambda x_1 . x_1) \quad \text{and} \quad Y=\lambda x_0 . x_0(\lambda x_1 x_2 . x_1 x_2),$$

we have that $X_R=\lambda t_0 t_1 . t_1 t_0$ is a right inverse for X but not for Y.

LEMMA 3.2: Let X, Y be two λ-terms for which $BT(X)\leqslant_\eta BT(Y)$ and let X be of type Σ. Y is of type Σ if and only if there exists $A\in\mathscr{A}(X)\cap\mathscr{L}$ such that $BT(Y)$ results from $BT(A)$ without terminal η-expansions.

Proof: Obvious.

THEOREM 3.2: The function f is not monotonic.

Proof: Obvious from lemma 3.2.

Notice that also in the case in which Y is of type Σ as X, we can have $\mathscr{I}_L(X) \nsubseteq \mathscr{I}_L(Y)$. For example if

$$X=\lambda x_0 x_1 x_2 . x_1(x_2 x_0) \quad \text{and} \quad Y=\lambda x_0 x_1 x_2 . x_1(\lambda x_3 . x_2 x_0 x_3)$$

we have that $X_L=\lambda z . z\mathbf{II}$ is a left inverse for X but not for Y.

REFERENCES

1. H. P. BARENDREGT, *The Lambda Calculus, its Sintax and Semantics*, North-Holland, Amsterdam, 1981.
2. J. BERGSTRA and J. W. KLOP, *Invertible Terms in the Lambda Calculus*, Theor., Comp. Sci., vol. 9, 1980, p. 27-38.
3. C. BÖHM, *Alcune proprietà delle forme* β-η*-normali nel* λ-*k calcolo*. Pubblicazioni dell'Istituto per le Applicazioni del Calcolo, n. 696, Roma, 1968.
4. C. BÖHM and M. DEZANI-CIANCAGLINI, *Combinatorial problems, combinator equations and normal forms*, Springer L. N. C. S., n° 14, 1974, p. 185-199.
5. A. CHURCH, *Combinatory logic as a semigroup* (abstract), Bull. Amer. Math. Soc., vol. 43, 1937, p. 333.

6. A. CHURCH, *The Calculi of Lambda Conversion*, Princeton University Press, Princeton, 1941.
7. H. B. CURRY and R. FEYS, *Combinatory Logic*, vol. 1, North-Holland, Amsterdam, 1958.
8. M. DEZANI-CIANCAGLINI, *Pattern-Matching Problems inside* λ-β-η *calculus*, Proceedings Informatica 76, Bled, 1976.
9. M. DEZANI-CIANCAGLINI, *Characterization of normal forms possessing inverse in the* λ-β-η *calculus*, Theor. Comput. Sci., vol. 2, 1976, p. 323-337.
10. J. J. LÉVY, *An algebraic interpretation of the* λ-β-*k*-*Calculus and an application of a labelled* λ-*Calculus*, Theor. Comput. Sci., vol. 2, 1976, p. 97-114.
11. C. P. WADSWORTH, *The relation between computational and denotational properties for Scott's* $\mathbf{D}_\infty$-*models of the lambda-calculus*, SIAM J. Comput., vol. 5, 1976, p. 488-521.