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FRONTIERS OF INFINITE TREES (*)

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Communiqué par Maurice Nivat

Abstract. — *The frontier of an infinite tree is a generalized (infinite) word called an* arrangement. *An equation in arrangements has an initial solution which is the frontier of its solution in the domain of infinite trees. Certain systems of equations can be solved explicitely.*

INTRODUCTION

The theory of languages deals with finite words and sets of finite words. *Infinite words* as sequences of elements of some finite alphabet X indexed by $\mathcal{N}$ have been considered in many papers (in particular McNaughton [5] and Nivat [6]).

It is clear that the reversal of an infinite word is not defined: it should be a sequence indexed by the set of negative integers. And the concatenation of finite words is extended to infinite words in such a way that $uv = u$ if u is infinite and v finite or infinite, which is somewhat unnatural.

We shall consider a more general concept of infinite word, called an *arrangement*. An arrangement of elements of X is a family of elements of X indexed by some linearly ordered set. (It is convenient to keep the term "infinite word" for arrangements indexed by $\mathcal{N}$.) The reversal of an arrangement is now defined, and the concatenation of arrangements does not satisfy the above property.

We shall consider equations (and systems of equations) in arrangements. For example the equation $u = aub$ (where u is a variable ranging over arrangements and a, b are symbols from X) has the solution $u = a^{\omega} b^{-\omega}$. It has many other solutions (of the general form $a^{\omega} A b^{-\omega}$ where A is an arbitrary arrangement) but $a^{\omega} b^{-\omega}$ is clearly "the simplest one". For every system of equations, we shall define a "simplest" solution, which is in some sense "generated" by the system considered as a context-free grammar.

To do this rigorously, we shall use derivation trees. Let us recall that the *frontier* of a finite tree is the finite string of the labels of its terminal nodes considered from left to right. A word is generated by a context-free grammar if and only if it is the frontier of one of its derivation trees.

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The same idea will be used for our equations. Each equation has one infinite derivation tree. Its "simplest" solution is the frontier of this derivation tree (the frontier can be naturally defined as the arrangement of the labels of the terminal nodes, considered from left to right). We characterize it as *an (the) initial object in the category of all solutions* of the given equation. Equations in categories are more generally considered by Lehmann [4]. Some other results are proved:

1) every countable arrangement is the frontier of an infinite tree;

2) to have the same frontier (up to an isomorphism) is an equivalence relation on infinite trees which is a "global" property, not expressible in terms of the finite approximations of the considered trees (technically which is *not algebraic* in the sense of Courcelle and Nivat [2]);

3) certain systems of equations (called *quasi-rational*) can be solved explicitely by means of *regular expressions with parameters.*

1. ARRANGEMENTS

1.1. Let X be a finite alphabet. An *arrangement* is a triple $A = \langle \mid A \mid, \pi, h \rangle$ consisting of:

1) a set $\mid A \mid$;

2) a linear order π on $\mid A \mid$;

3) a mapping $h : \mid A \mid \to X$.

Let $\mathscr{A}(X)$ be the class of all arrangements and $\mathscr{A}_f(X)$ [resp. $\mathscr{A}_\omega(X)$] the class of *finite* (resp. *countable*) arrangements i. e. such that $\mid A \mid$ is finite [resp. countable].

If $X = \{ a \}$, an arrangement A is simply a *linearly ordered set* denoted by $A = \langle \mid A \mid, \pi \rangle$. By convention, "let $A = \langle \mid A \mid, \pi \rangle$ be an arrangement..." means that X is singleton.

Let ε be the unique arrangement A such that $\mid A \mid = \emptyset$. Words on X will be identified with finite arrangements: to the word $u = a_1 a_2 \ldots a_n$ (where $a_i \in X$) will correspond the arrangement $\langle \{ 1, 2, \ldots, n \}, \leqq, h \rangle$ where $h(i) = a_i$ for $i = 1, 2, \ldots, n$. The notation X^* will be used instead of $\mathscr{A}_f(X)$. An arrangement of the form $\langle \mathscr{N}, \leqq, h \rangle$ corresponds to an infinite word in the sense of [6]. Let X^ω be the set of such arrangements and $X^\infty = X^\omega \cup X^*$. Finally, for $a \in X$, let us define

$$a^\omega = \langle \mathscr{N}, \leqq, h \rangle, a^{\mathbf{Z}} = \langle \mathbf{Z}, \leqq, h \rangle$$

and $a^{-\omega} = \langle \{ -n / n \in \mathscr{N} \}, \leqq, h \rangle$ with $h(i) = a$ for all i.

In order to compare arrangements, we define morphisms.

Let $A = \langle | A |, \pi, h \rangle$ and $A' = \langle | A' |, \pi', h' \rangle$ belong to $\mathscr{A}(X)$. A *morphism* $\alpha : A \to A'$ is a mapping $\alpha : | A | \to | A' |$

1) which is one-to-one and order-preserving, i. e. (since π and π' are linear), $\alpha(x)\, \pi'\, \alpha(y) \Leftrightarrow x\, \pi\, y$ for all $x, y \in | A |$;

2) such that $h(x) = h'(\alpha(x))$ for $x \in | A |$.

If α is onto, it is called an *isomorphism* and we write $A \simeq A'$.

It would be pleasant to identify isomorphic arrangements but this will not be always possible since there may exist *several* isomorphisms: $A \to A'$. For A and A' in X^{∞} there exists at most one isomorphism: $A \to A'$, hence $A \simeq A'$ can be replaced by $A = A'$ without ambiguity. We will often do so for arbitrary arrangements when no difficulty arises. Let us finally point out that there may exist morphisms $\alpha : A \to B$ and $\beta : B \to A$ although A and B are not isomorphic. Just take $A = a\,(ab^{\mathbf{Z}}\, ab^{\mathbf{Z}}\, a)^{\omega}$ and $B = ab^{\mathbf{Z}}\, a\,(ab^{\mathbf{Z}}\, ab^{\mathbf{Z}}\, a)^{\omega}$ (this depends on definitions and notations given below; thanks to B. Lang for this example).

1.2. We will define the concatenation of arrangements, which will generalize the concatenation of words for finite arrangements.

This will be done in terms of another operation called the *substitution* which we now define.

Let $A = \langle | A |, \pi, h \rangle \in \mathscr{A}(X)$ and $\sigma = | A | \to \mathscr{A}(X)$ be a partial mapping. Let us extend σ into a total mapping $\bar{\sigma}$ by taking for $\bar{\sigma}(x)$ the one-element arrangement $h(x)$ [i. e. $\langle \{1\}, \pi, k \rangle$ where $k(1) = h(x)$] when $\sigma(x)$ is undefined and $\sigma(x)$ otherwise. Let $\bar{\sigma}(x) = \langle | \bar{\sigma}(x) |, \pi_x, h_x \rangle$ for $x \in | A |$.

Then $\sigma(A) = A' = \langle | A' |, \pi', h' \rangle$ with:

1) $| A' | = \{ (x, y) / x \in | A |$ and $y \in | \bar{\sigma}(x) | \}$;

2) $(x, y)\, \pi'\, (x', y')$ if and only if either $x \neq x'$ and $x\, \pi\, x'$ or $x = x'$ and $y\, \pi_x\, y'$;

3) $h'((x, y)) = h_x(y)$.

Note that $\sigma(A)$ does not depend on h if σ is total.

Furthermore, if $\alpha : A_1 \to A_2$ is a morphism, $\sigma_i : | A_i | \to \mathscr{A}(X)$ is a substitution for $i = 1, 2$, $\delta_x : \sigma_1(x) \to \sigma_2(\alpha(x))$ is a morphism for all x such that $\sigma_1(x)$ is defined, one can define a canonical morphism $\gamma : \sigma_1(A_1) \to \sigma_2(A_2)$ by taking:

$\gamma((x, 1)) = (\alpha(x), 1)$ if $\sigma_1(x)$ is not defined,

$\gamma((x, y)) = (\alpha(x), \delta_x(y))$ if $\sigma_1(x)$ is defined and $y \in | \sigma_1(x) |$.

As an example, for $A \in \mathscr{A}(X)$, let $A^{\omega} = \sigma(\langle \mathscr{N}, \leqq \rangle)$ and $A^{-\omega} = \sigma(\langle \mathscr{N}, \leqq' \rangle)$ where $\sigma(i) = A$ for all $i \in \mathscr{N}$ and $i \leqq' j$ iff $j \leqq i$ for $i, j \in \mathscr{N}$.

The *concatenation* of A and $A' \in \mathscr{A}(X)$ is defined by $AA' = \sigma(\langle \{0, 1\}, \leqq \rangle)$ where, $\sigma(0) = A$ and $\sigma(1) = A'$. In particular $a^{-\omega} a^{\omega} \simeq a^{\mathbf{Z}}$.

1.3. PROPOSITION: *The concatenation is associative. Namely, there exists a canonical isomorphism:* $A_1 (A_2 A_3) \to (A_1 A_2) A_3$ *for* $A_1, A_2, A_3 \in \mathscr{A}(X)$. Hence we will write $A_1 A_2 A_3$ for $A_1 (A_2 A_3)$.

For arrangements in X^*, this concatenation coincides with the usual one. But it does not with the concatenation on X^∞ described in [6] (Recall that $uv = u$ if $u \in X^\omega$, $v \in X^\infty$.)

1.4. *Examples:*

$aa^{\omega} = a^{\omega}$, i.e. $aa^{\omega} \simeq a^{\omega}$

$a^{-\omega} a = a^{-\omega}$,

$a^{\omega} a \neq a^{\omega}$,

$a^{\omega} a^{\omega} \neq a^{\omega}$.

1.5. We now focus our attention on *countable arrangements* and represent them in the form $\langle L, \leqq_l, h \rangle$ where $L \subset \{0, 1\}^*$ and $\leqq_l$ is the lexicographic ordering on L.

Let us recall some definitions.

Let $u, v \in \{0, 1\}^*$; then $u \prec v$ iff $v = uw$ for some $w \in \{0, 1\}^*$ and $u \leqq_l v$ if and only if:

(1) either $u \prec v$;

(2) or $u = w\,0\,u'$, $v = w\,1\,v'$ for some $w, u', v' \in \{0, 1\}^*$.

A language $L \subset \{0, 1\}^*$ is *prefix-free* if $u \prec v$ implies $u = v$ for all $u, v \in L$.

A *complete* language is a maximal element of the set of prefix-free languages ordered by inclusion.

1.6. LEMMA [7]: *A prefix-free language* $L \subset \{0, 1\}^*$ *is complete if for all* $u, v \in \{0, 1\}^*$:

$u\,0\,v \in L \;\Rightarrow\; u\,1\,w \in L$ *for some* $w \in \{0, 1\}^*$

and

$u\,1\,v \in L \;\Rightarrow\; u\,0\,w \in L$ *for some* $w \in \{0, 1\}^*$.

1.7. THEOREM: *Every $A \in \mathscr{A}_\omega(X)$ is of the form $\langle K, \leqq_l, k \rangle$ for some complete language $K \subset \{0, 1\}^*$.*

Proof: The case of a finite arrangement A is trivial. For an infinite A, the proof will be done in two steps: we first define a prefix-free language L such that $A \simeq \langle L, \leqq_l, l \rangle$; secondly, we transform L into a complete language K such that $\langle L, \leqq_l, l \rangle \simeq \langle K, \leqq_l, k \rangle$.

Let us take $A = \langle \mathscr{N}, \pi, h \rangle$ for some linear order π on $\mathscr{N}$ and $h : \mathscr{N} \to X$.

We define a sequence $(u_n)_{n \in \mathscr{N}}$ of words on $\{0, 1\}$ such that, for all n:

1) $L_n = \{u_0, u_1, \ldots, u_n\}$ is prefix-free;

2) for all $m \neq n$, $u_m \neq u_n$ and $u_m \leqq_l u_n$ iff $m \, \pi \, n$;

3) there exists $v \in \{0, 1\}^*$ such that $L_n \cup \{v\}$ is prefix-free and $v \leqq_l u_i$ for all $i = 0, \ldots, n$;

4) there exists $w \in \{0, 1\}^*$ such that $L_n \cup \{w\}$ is prefix-free and $u_i \leqq_l w$ for all $i = 0, \ldots, n$;

5) for all $i, j \in \{0, 1, \ldots n\}$ such that $i \neq j$ and $i \, \pi \, j$, there exists $z_{i,j} \in \{0, 1\}^*$ such that $L_n \cup \{z_{i,j}\}$ is prefix-free and $u_i \leqq_l z_{i,j} \leqq_l u_j$.

Assuming this, we will take $L = \bigcup_{n \in \mathscr{N}} L_n$ and $l : L \to X$ defined by $l(u_n) = h(n)$. We construct u_n inductively:

• $u_0 = 10$. Note that 3) and 4) hold, the other conditions being trivially satisfied;

•• having defined u_n we now define u_{n+1} in the following way:

(i) if $(n+1) \, \pi \, i$ for all $i = 0, 1, \ldots, n$, let v be a word defined by 3) for L_n Then we take $u_{n+1} = v\,10$;

(ii) if $i \, \pi \, (n+1)$ for $i = 0, 1, \ldots, n$, let w be defined by 4) for L_n. Then we take $u_{n+1} = w\,10$;

(iii) if $i \, \pi \, (n+1) \, \pi \, j$ for some $i, j \in \{0, \ldots, n\}$ and

$$\{k/0 \leqq k \leqq n, i \, \pi \, k \, \pi \, j\} = \{i, j\}$$

then we take $u_{n+1} = z_{i,j}\,10$ where $z_{i,j}$ is defined by 5) for L_n.

Conditions 3 to 5 express that there is room left for other elements to the left and to the right of what has already been constructed and between any two elements. These possibilities are used in cases (i), (ii) and (iii) of the definition of L_{n+1}. The factors "10" are used to preserve conditions 3 to 5.

The rest of the proof follows from:

1.8. LEMMA: *Let* $L \subset \{0, 1\}^*$ *be prefix-free. There exists a complete language* $K \in \{0, 1\}^*$ *such that* $\langle L, \leqq_l \rangle \simeq \langle K, \leqq_l \rangle$.

Proof: Let $L \subset \{0, 1\}^*$ be prefix-free.

We define a mapping $\varphi : L \to \{0, 1\}^*$, we will take $K = \varphi(L)$, and φ will be the required isomorphism $\varphi : \langle L, \leqq_l \rangle \to \langle K, \leqq_l \rangle$.

A prefix u of some word in L is *critical* if:

$$u0\{0, 1\}^* \cap L = \emptyset \quad \not\Leftrightarrow \quad u1\{0, 1\}^* \cap L = \emptyset.$$

Note that L is complete if and only if it has no critical prefix by lemma 1.6.

Let $x_1 \ldots x_k \in L$ with $x_1, \ldots, x_k \in \{0, 1\}$.

Then $\varphi(x_1 \ldots x_k) = y_1 \ldots y_k$,
where

$$\begin{aligned} y_i &= x_i \quad \text{if } x_1 \ldots x_{i-1} \text{ is not critical} \\ &= \varepsilon \quad \text{if } x_1 \ldots x_{i-1} \text{ is critical.} \end{aligned}$$

CLAIM 1 : *Let* $u, u' \in L$. *If* $\varphi(u) \prec \varphi(u')$ *then* $u = u'$.

Assume by contradiction that $u \neq u'$. Then $u = v\,0\,w$ and $u' = v\,1\,w'$ for some $v, w, w' \in \{0, 1\}^*$ (or vice versa); but v is *not* critical hence $\varphi(u) = x\,0\,y$ and $\varphi(u') = x\,1\,y'$ for some $x, y, y' \in \{0, 1\}^*$ and we cannot have $\varphi(u) \prec \varphi(u')$.

Q. E. D.

CLAIM 2 : *Let* $u, u' \in L$. *If* $u \leqq_l u'$ *then* $\varphi(u) \leqq_l \varphi(u')$.

We have $u = v\,0\,w$ and $u' = v\,1\,w'$ for some $v, w, w' \in \{0, 1\}^*$ hence v is *not* critical and $\varphi(u) = x\,0\,y$, $\varphi(u') = x\,1\,y'$ for some $x, y, y' \in \{0, 1\}^*$.

Q. E. D.

Claim 1 shows that K is prefix-free and that φ is one-to-one, claim 2 that φ preserves $\leqq_l$. We need only show that K is complete.

Let $\varphi(u) = v\,0\,w$ for some $u \in L$, $v, w \in \{0, 1\}^*$. Then, by the definition of φ, we can write:

$$u = x_1 \ldots x_k \quad \text{with} \quad x_l = 0 \quad \text{for some } 1 \leqq l \leqq k,$$

$$\left.\begin{aligned} v &= y_1 \ldots y_{l-1} \\ w &= y_{l+1} \ldots y_k \end{aligned}\right\} \text{ as in the definition of } \varphi,$$

$x_1 \ldots x_{l-1}$ is not critical.

Hence there exists $u' = x_1 \dots x_{l-1} 1 z \in L$ for some $z \in \{0, 1\}^*$ and clearly,

$$\varphi(u') = y_1 \dots y_{l-1} 1 w' \quad \text{for some } w' \in \{0, 1\}^*$$
$$= v 1 w'.$$

Lemma 1.6 shows then that K is complete.

If we represent prefix-free languages by trees, the transformation of L into K corresponds to the suppression of nodes with only one son.

As an example, if

$L = \{0, 1010, 100010, 100011\}$

then $K = \{0, 100, 101, 11\}$.

2. THE FONTIER OF AN INFINITE TREE

2.1. Let $F = \{\star\} \cup X$. The symbol $\star$ will be given the arity 2 and each $x \in X$ the arity 0.

We recall from [1] the definition of $M^\infty(F)$, the set of infinite (and finite) trees on F by adapting it to the special set F that we are considering.

A *tree* T is a partial mapping: $\{0, 1\}^* \to X \cup \{\star\}$ such that the set of its *nodes*, $\text{Dom}(T) = \{u \in \{0, 1\}^* / T(u) \text{ is defined}\}$ satisfies the following properties:

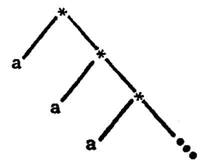
1) $u \prec v \in \text{Dom}(T) \Rightarrow u \in \text{Dom}(T)$;

2) $T(u) \in X$ if and only if u is maximal in $\text{Dom}(T)$ w. r. t. $\prec$.

Let $T_\# = \{u \in \{0, 1\}^* / T(u) \in X\}$, the set of *terminal nodes*. Then $T_\#$ is prefix-free. It is complete if and only if T is *locally finite* [1] i. e. if each node is prefix of some terminal node. We let $M(F)$ denote the set of finite trees (with a finite set of nodes) and $M^{\text{loc}}(F)$, the set of locally finite trees.

A binary operation on trees is defined by $T = T_0 \star T_1$ if $T(\varepsilon) = \star$ and $T(iu) = T_i(u)$ for $u \in \{0, 1\}^*$ and $i = 0, 1$.

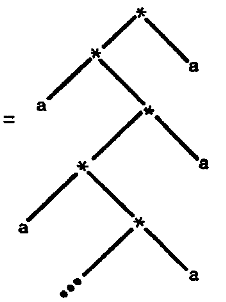
2.2. *Examples:*

$T_0 =$ 

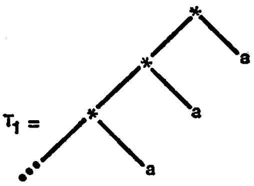
corresponds to the mapping T_0 such that $T_0(1^n 0) = a$ and $T_0(1^n) = \star$ for $n \geq 0$, otherwise undefined.

Other examples of trees are:

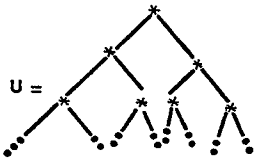
T_2 =



T_1 =



U =



The trees T_1, T_2, T_3 are locally finite, but U is not.

The *frontier* of $T \in M^\infty(F)$ is the countable arrangement
$\Phi(T) = \langle T_\#, \leqq_l, T \rangle$
(the third element of this triple is in fact the restriction of T to $T_\#$).

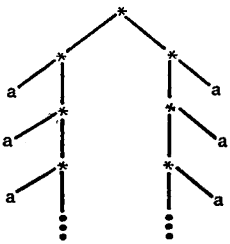
Hence $\Phi(T_0) = a^\omega$, $\Phi(T_1) = a^{-\omega}$, $\Phi(T_2) = a^\omega a^{-\omega}$, $\Phi(U) = \varepsilon$.

For a finite T, $\Phi(T) \in X^*$ and coincides with the usual frontier defined fot instance in [8]. An immediate consequence of theorem (1.7) is

2.3. THEOREM:
$\mathscr{A}_\omega(X) = \{ \Phi(T)/T \in M^\infty(F) \} = \{ \varepsilon \} \cup \{ \Phi(T)/T \in M^{\mathrm{loc}}(F) \}$.

2.4. PROPOSITION: $\Phi(T \star T') = \Phi(T)\,\Phi(T')$.

2.5. *Example:* Let $T_3 = T_0 \star T_1$ (where T_0, T_1 are from 2.2).
This tree looks like:



Clearly $\Phi(T_3) = \Phi(T_2) = a^\omega a^{-\omega}$.

3. SOLVING EQUATIONS IN $\mathscr{A}$ (X)

3.1. We want to solve in $\mathscr{A}(X)$ systems of equations of the form

$$\Sigma \begin{cases} u_1 = t_1 \\ \vdots \\ u_k = t_k \end{cases}$$

the u_i' s are variables of arity 0, $V = \{u_1, \ldots, u_k\}$, $t_i \in (X \cup V)^*$ and $t_i \notin V$ for $i = 1, \ldots, k$.

Such a system is a context-free grammar with the special property that right handsides of equations are monomials rather than polynomials.

It is known that the language generated by a context-free grammar is the set of frontiers of the trees generated by a regular tree-grammar. A similar result is used here.

Given Σ, we define a system $\bar{\Sigma}$ of the form $\langle u_i = \bar{t}_i, 1 \leqq i \leqq k \rangle$ such that $\bar{t}_i \in M(F \cup V)$ and $\Phi(\bar{t}_i) = t_i$ for $i = 1, \ldots, k$.

This new system has a unique solution $\langle T_1, \ldots, T_k \rangle$ in $M^\infty(F)$ and, by (2.4), $\langle \Phi(T_1), \ldots, \Phi(T_k) \rangle$ is a solution of Σ in $\mathscr{A}(X)$. We shall characterize it by an initiality property and show that it does not depend on the choice of $\bar{\Sigma}$.

3.2. *Examples* [*see* (2.2) for T_0, T_1, T_2 and U]:

Σ	$\bar{\Sigma}$	Solution of $\bar{\Sigma}$	Solution of Σ
$u_0 = au_0$	$u_0 = a \star u_0$	T_0	$a^\omega = \Phi(T_0)$
$u_1 = u_1 a$	$u_1 = u_1 \star a$	T_1	$a^{-\omega} = \Phi(T_1)$
$u_2 = au_2 a$	$u_2 = a \star (u_2 \star a)$	T_2	$a^\omega a^{-\omega} = \Phi(T_2)$
$u = uu$	$u = u \star u$	U	$\varepsilon = \Phi(U)$

Any solution in $\mathscr{A}_\omega(X)$ of $w = waw$ is a dense countable linearly ordered set without least and greatest element, hence is isomorphic to the order type of rational numbers.

3.3. Notations: If u_i is a variable of Σ, we let $T(\bar{\Sigma}, u_i)$ denote the component of the solution of $\bar{\Sigma}$ in $M^\infty(F)$ associated with u_i, and $A(\Sigma, u_i)$ denote $\Phi(T(\bar{\Sigma}, u_i))$.

Let $\vec{T}(\bar{\Sigma}) = \langle T(\bar{\Sigma}, u_1), \ldots, T(\bar{\Sigma}, u_k) \rangle$ and

$\vec{A}(\Sigma) = \langle A(\Sigma, u_i), \ldots, A(\Sigma, u_k) \rangle$.

3.4. *An open decision problem*

(P): Given Σ and Σ', can we decide whether $A(\Sigma, u_1)$ is isomorphic to $A(\Sigma', u'_1)$?

In the special case where $X = \{ a \}$, this problem is equivalent to the following one:

(P'): Given two prefix-free rational languages L and L' on $\{ 0, 1 \}$, can we decide whether $\langle L, \leqq_l \rangle$ and $\langle L', \leqq_l \rangle$ are isomorphic order types?

In fact these two problems are equivalent. Let us show that (P) reduces to (P'). If $X = \{ a_1, a_2, \ldots, a_n \}$ we can "code" a_1 by $a^{-\omega} a^{\omega}$, a_2 by $a^{-\omega} a^{-\omega} a^{\omega}, \ldots, a_n$ by $(a^{-\omega})^n a^{\omega}$. Formally, if $A(\Sigma, u_1) = \langle | A |, \pi, h \rangle$ we define a substitution σ such that

$$\sigma(x) = (a^{-\omega})^i a^{\omega} \quad \text{if} \quad x \in |A| \quad \text{and} \quad h(x) = a_i$$

and similarly, a substitution σ' associated with $A(\Sigma', u'_1)$.

One easily constructs Σ_1 and Σ'_1 such that

$$\sigma(A(\Sigma, u_1)) = A(\Sigma_1, u_1),$$

$$\sigma'(A(\Sigma', u'_1)) = A(\Sigma'_1, u'_1),$$

and one shows that:

$$A(\Sigma, u_1) \simeq A(\Sigma', u'_1) \quad \text{iff} \quad A(\Sigma_1, u_1) \simeq A(\Sigma'_1, u'_1).$$

3.5. *Quasi-rational systems*

A system $\Sigma = \langle u_i = t_i, 1 \leqq i \leqq k \rangle$ is *quasi-rational* if it is a quasi-rational context-free grammar i. e. if for all u_i and $t \in (X \cup V)^*$ such that $u_i \xrightarrow[\Sigma]{*} t$ then u_i occurs at most once in t.

The system Σ is *preordered* if there exists a preorder θ on $\{ 1, \ldots, k \}$ such that for all i and j:

(i) if u_j occurs in t_i then $j \,\theta\, i$ (i. e. j is less than i w. r. t. θ),

(ii) if u_j and u_l have distinct occurrences in t_i (and possibly $j = l$) $i \,\theta\, j$ and $i \,\theta\, l$ do not hold together.

3.6. PROPOSITION: *A system Σ is quasi-rational if and only if it is preordered.*

Proof: Only if. Let Σ be quasi-rational. Take $j \theta i$ if $i = j$ or $u_i \xrightarrow[\Sigma]{*} t$ for some $t \in (X \cup V)^*$ with at least one occurrence of u_j. Then θ is clearly a preorder, (i) is satisfied; if u_j and u_l have distinct occurrences in t_i and if $i \theta j$ and $i \theta l$ then one can find some $t \in (X \cup V)^*$ such that: $u_i \rightarrow t_i \xrightarrow{*} t$ and u_i occurs twice in t.

Hence Σ is not quasi-rational.

If. Let Σ be preordered by θ. By induction on the length of a derivation $u_i \xrightarrow{*} t$, one shows that if u_j and u_l have distinct occurrences in t then $i \theta j$ and $i \theta l$ do not hold together. In particular, this implies that Σ is quasi-rational. □

3.7. In order to solve explicitly quasi-rational systems, we define a certain kind of *regular expressions* (*r. e.*) involving exponentiation to ω and $-\omega$. Similar expressions are used in [5].

They are obtained by a finite number of applications of the following rules:

(i) every $a \in X$ is a r. e., and ε is a r. e.,

(ii) RR' is a r. e.,

(iii) $(R)^\omega$ is a r. e., and

(iv) $(R)^{-\omega}$ is a r. e., if R and R' are r. e. We also define *regular expressions with parameters* $U_1, \ldots, U_k$ by adding following rule:

(v) U_i is a r. e. for $i = 1, \ldots, k$.

By definitions 1.2 and proposition 1.3, every r. e. R has a well defined value in $\mathscr{A}_\omega (X)$. A r. e. with parameters $U_1, \ldots, U_k$ has a value in $\mathscr{A} (X)$ for every assignment of values in $\mathscr{A} (X)$ to the parameters. (The same symbol will denote a regular expression and its value.)

An arrangement A is *quasi-rational* if $A = A (\Sigma, u_i)$ for some quasi-rational system Σ.

3.8. THEOREM: *An arrangement is quasi-rational if and only if it is defined by some regular expression.*

Proof: The "if" part is proved by induction on the structure of regular expressions:

(i) if $R = a \in X \cup \{ \varepsilon \}$, then $R = A (\Sigma, u_1)$ where $\Sigma = \langle u_1 = a \rangle$;

(ii) if $R = R_1 R_2$, one constructs $\Sigma_1 = \langle u_1 = t_1, \ldots, u_k = t_k \rangle$; $\Sigma_2 = \langle u_{k+1} = t_{k+1}, \ldots, u_l = t_l \rangle$ such that $R_1 = A (\Sigma_1, u_1)$, $R_2 = A (\Sigma_2, u_{k+1})$ then $R = A (\Sigma, u_{l+1})$ where $\Sigma = \Sigma_1 \cup \Sigma_2 \cup \langle u_{l+1} = u_1 u_{k+1} \rangle$;

(iii) if $R = R_1^\omega$, where $R_1 = A (\Sigma, u_1)$ then $R = A (\Sigma, u_{k+1})$ where $\Sigma = \Sigma_1 \cup \langle u_{k+1} = u_1 u_{k+1} \rangle$ (easy lemma left to the reader) and similarly:

(iv) if $R = R_1^{-\omega}$, then $R = A (\Sigma, u_{k+1})$ where $\Sigma = \Sigma_1 \cup \langle u_{k+1} = u_{k+1} u_1 \rangle$.

One easily checks at each step of this construction that one gets preordered systems.

Let Σ be preordered by θ and $A = A(\Sigma, u_1)$.

For each $i \in \{1, 2, \ldots, k\}$ one can find $t'_i \in (X \cup V)^*$ such that:

(1) $t_i \xrightarrow{*} t'_i$;

(2) if u_j occurs in t'_i then $j \, \theta \, i$;

(3) if u_j occurs in t'_i and $i \, \theta \, j$ then $i = j$;

(4) u_i has at most one occurrence in t'_i.

Let $\Sigma' = \langle u_1 = t'_1, \ldots, u_k = t'_k \rangle$; then $\vec{A}(\Sigma) = \vec{A}(\Sigma')$.

We now define R_i such that $R_i = A(\Sigma', u_i)$ for $i = 1, \ldots, k$.

We define R_i in terms of the R_j's such that $j \, \theta \, i$ and $i \neq j$.

(α) If t'_i contains no occurrence of u_i, then by (2) t'_i contains only occurrences of u_j such that $j \, \theta \, i$ and $i \neq j$. Then we take $R_i = t'_i [R_j/u_j]$, the substitution of R_j to each occurrence of u_j in t'_i for all j.

(β) If t'_i contains an occurrence of u_i, then we obtain three cases:

(β 1) $t'_i = s u_i s'$;

(β 2) $t'_i = u_i s'$;

(β 3) $t'_i = s u_i$;

where s and s' only contain occurrences of u_j such that $j \, \theta \, i$ and $j \neq i$. Let $S = s' [R_j/u_j]$ and $S' = s [R_j/u_j]$ [same notation as in case (α)]. We then take $R_i = S^{\omega} S'^{-\omega}$ in the first case, $R_i = S'^{-\omega}$ in the second and $R_i = S^{\omega}$ in the third. $\square$

3.9. *Example:* Let us define $R_1 = A(\Sigma, u_1)$ where

$$\Sigma \begin{cases} u_1 = a u_2 u_3, \\ u_2 = u_3 b u_1, \\ u_3 = u_4 u_3 b, \\ u_4 = a u_5, \\ u_5 = ab; \end{cases}$$

$R_5 = ab$,

$R_4 = aab$,

$R_3 = (aab)^{\omega} b^{-\omega}$.

The first equation is transformed into

$u_1 = a u_3 b u_1 u_3$,

$R_1 = (a(aab)^{\omega} b^{-\omega} b)^{\omega} ((aab)^{\omega} b^{-\omega})^{-\omega} = (a(aab)^{\omega} b^{-\omega})^{\omega} ((aab)^{\omega} b^{-\omega})^{-\omega}$,

$R_2 = (aab)^{\omega} b^{-\omega} (a(aab)^{\omega} b^{-\omega})^{\omega} ((aab)^{\omega} b^{-\omega})^{-\omega}$.

By using regular expressions with parameters, we can also define *all* solutions of a given quasi-rational system Σ. In order to do so we state without proof the following:

3.10 LEMMA: *Let A, $A' \in \mathscr{A}(X)$. An arrangement U satisfies the equation $U = AUA'$ (resp. $U = UA'$)* (resp. $U = AU$) *if and only if $U = A^{\omega} BA'^{-\omega}$ (resp. $U = BA'^{-\omega}$) (resp. $U = A^{\omega} B$) for some $B \in \mathscr{A}(X)$.*

Given Σ as in the second part of the proof of 3.8, let U_i be a parameter for all $i \in \{1, \ldots, k\}$ such that t_i' contains an occurrence of u_i let P be this set of parameters.

We define regular expressions $\overline{R}_i$ with parameters in P in the same way as before except in case (β) where we take:

$\overline{R}_i = S^{\omega} U_i S'^{-\omega}$ in subcase (β 1),

$\overline{R}_i = U_i S'^{-\omega}$ in subcase (β 2),

$\overline{R}_i = S^{\omega} U_i$ in subcase (β 3).

We obtain with lemma 3.10 and the above notations:

3.11. THEOREM: *The class of solutions of Σ is exactly the class of values of $\langle \overline{R}_1, \ldots, \overline{R}_k \rangle$ where parameters range over $\mathscr{A}(X)$.*

3.12. *Example:* Let Σ be defined in example 3.9:

$R_5 = ab$,
$R_4 = aab$,
$R_3 = (aab)^{\omega} U_3 b^{-\omega}$,
$R_1 = (a(aab)^{\omega} U_3 b^{-\omega})^{\omega} U_1 ((aab)^{\omega} U_3 b^{-\omega})^{-\omega}$,

R_2 is left to the reader.

Hence, the general solution of Σ depends on two parameters, U_1 and U_3.

We conclude this section by another characterization of quasi-rational systems.

Let $T \in M^{\infty}(F)$. An *infinite branch* of T is a word $w \in \{0, 1\}^{\omega}$ such that $w[n]$ (i. e. the finite prefix of w of length n) belongs to Dom (T) for all $n \in \mathscr{N}$.

A tree is *sparse* (in French: *éparpillé*) if it has at most countably many infinite branches.

3.13 PROPOSITION: *A system Σ is quasi-rational if and only if each $T(\overline{\Sigma}, u_i)$ is sparse.*

Proof: Assume that Σ is not quasi-rational. Then $u_i \xrightarrow{*} t$ with two occurrences of u_i in t. Then $T(\Sigma, u_i)$ contains as a subgraph the tree $U = U \star U$ of 2.2.

Its set of infinite branches is $\{0, 1\}^\omega$ which is not countable.

Conversely, let Σ be preordered. One shows that $T(\bar{\Sigma}, u_i)$ is sparse by the same induction as for the construction of the R_i's in the proof of 3.8 and the remark that if T is the solution in $M^\infty(F)$ of an equation

$$T = t[T/v_0, T_1/v_1, \ldots, T_n/v_n],$$

such that

$$\begin{cases} T_1, \ldots, T_k \text{ are sparse,} \\ t \in M(F \cup \{v_0, \ldots, v_n\}),\ t \neq v_0 \text{ and } v_0 \text{ has one occurrence in } t, \end{cases}$$

then T is sparse. □

This proposition shows in particular that the arrangement A such that $A \simeq A\,a\,A$ (see examples 3.2) is *not* quasi-rational.

4. INITIALITY OF $\vec{A}(\Sigma)$

4.1. Let $\Sigma = \langle u_i = w_i;\ 1 \leqq i \leqq k \rangle$ be a system of equations over $\mathscr{A}(X)$ where $w_i \in (X \cup V)^*$ for all i (and $V = \{u_1, u_2, \ldots, u_k\}$). Let $\bar{\Sigma} = \langle u_i = t_i;\ 1 \leqq i \leqq k \rangle$ be a system over $M^\infty(F)$ where $t_i \in M(F \cup V)$ is choosen in such a way that $\Phi(t_i) = u_i$. We shall see that all possible choices for $\bar{\Sigma}$ yield the same result.

We redefine the concept of a solution of Σ in $\mathscr{A}(X)$, more precisely than we have done yet:

A *solution* of Σ is a k-tuple $\langle (A_i, \alpha_i);\ 1 \leqq i \leqq k \rangle$ of pairs consisting of an arrangement A_i and an isomorphism $\alpha_i : A_i \to w_i[\vec{A}]$ where $w_i[\vec{A}]$ is the substitution of A_i to the parameter u_i in the regular expression w_i, or equivalently, $w_i[\vec{A}] = \sigma(w_i)$ where σ is the substitution such that $\sigma(x) = A_i$ if $w_i(x) = u_i$ (considering now w_i as a finite arrangement on $X \cup V$).

A *morphism* of solutions of Σ, denoted

$$\vec{\beta} : \langle (A_i, \alpha_i);\ 1 \leqq i \leqq k \rangle \to \langle (A'_i, \alpha'_i);\ 1 \leqq i \leqq k \rangle$$

is a k-tuple of morphisms $\vec{\beta} = \langle \beta_i;\ 1 \leqq i \leqq k \rangle$ where, for each i, β_i is a morphism $: A_i \to A'_i$ and the following diagram commutes:

$$\begin{array}{ccc} A_i & \xrightarrow{\alpha_i} & w_i[\vec{A}] \\ \downarrow{\scriptstyle \beta_i} & & \downarrow{\scriptstyle w_i[\vec{\beta}]} \\ A_i & \xrightarrow{\alpha'_i} & w_i[\vec{A'}] \end{array}$$

The morphism $w_i[\vec{\beta}]$ is canonically defined : $w_i[\vec{A}] \to w_i[\vec{A'}]$ as γ in 1.2 with $A_1 = A_2 = w_i$, π the identity on w_i, $\delta_x = \beta_i$ if $w_i(x) = u_i$. We redefine it in this context: using the fact that $w_i = \Phi(t_i)$, we get for $w_i[\vec{\beta}]$ the following definition with

$t_{i\#}^{(j)} = \{ z \in \mathrm{Dom}(t_i)/t_i(z) = u_j \}$

and $t_{i\#}^{(X)} = \{ z \in \mathrm{Dom}(t_i)/t_i(z) \in X \}$:

$$w_i[\vec{\beta}]((z, 1)) = (z, 1) \quad \text{if} \quad z \in t_{i\#}^{(X)}, \tag{4.1.1}$$

$$w_i[\vec{\beta}]((z, z')) = (z, \beta_j(z')) \quad \text{if} \quad z \in t_{i\#}^{(j)} \text{ (whence } z' \in |A_j|). \tag{4.1.2}$$

4.2. Let now $\langle T_1, T_2, \ldots, T_k \rangle$ be the unique solution of $\bar{\Sigma}$ in $M^{\infty}(F)$. We shall define morphisms $\gamma_1, \gamma_2, \ldots, \gamma_k$ such that $\langle (\Phi(T_i), \gamma_i); 1 \leqq i \leqq k \rangle$ is a solution of Σ in the sense of 4.1.

From the equation

$$T_{i\#} = t_{i\#}^{(1)} T_{1\#} \cup \ldots \cup t_{i\#}^{(k)} T_{k\#} \cup t_{i\#}^{(X)}, \tag{4.2.1}$$

there exists a bijection

$$\gamma_i : \quad T_{i\#} \to t_{1\#}^{(1)} \times T_{1\#} \cup \ldots \cup t_{i\#}^{(k)} \times T_{k\#} \cup t_{i\#}^{(X)} \times \{1\} \tag{4.2.3}$$

such that

$$\begin{aligned} \gamma_i(z) &= (z, 1) \quad \text{if} \quad z \in t_{i\#}^{(X)} \\ &= (z_1, z_2) \quad \text{if} \quad z = z_1 z_2 \quad \text{for some } z_1 \in t_{i\#}^{(j)} \quad \text{and} \quad z_2 \in T_{j\#}. \end{aligned}$$

It follows from the definitions and the properties of trees (see for instance [1]) that γ_i is a bijection. Its codomain is exactly $|\sigma_i(\Phi(t_i))|$ where $\sigma_i : |\Phi(t_i)| \to \mathscr{A}(X)$ is the substitution such that $\sigma_i(z) = \Phi(T_j)$ if $t_i(z) = u_j$, for $z \in t_{i\#}$. Hence, γ_i is an isomorphism of $\Phi(T_i)$ onto $\sigma_i(\Phi(t_i)) = w_i[\Phi(T_1)/u_1, \ldots, \Phi(T_k)/u_k]$.

We can now state and prove:

4.3. THEOREM: $\langle (\Phi(T_i), \gamma_i); 1 \leqq i \leqq k \rangle$ *is initial in the category of solutions of* Σ.

Initial means that for every solution $\langle (D_i, \delta_i); 1 \leqq i \leqq k \rangle$ of Σ there exists one and only one morphism from $\langle (\Phi(T_i), \gamma_i); 1 \leqq i \leqq k \rangle$ to it.

Proof: Given a solution $\langle (D_1, \delta_1), \ldots, (D_k, \delta_k) \rangle$, assume first the existence of $(\beta_1, \ldots, \beta_k)$ such that:

$$w_i[\vec{\beta}]\gamma_i = \delta_i \beta_i \quad \text{for} \quad 1 \leqq i \leqq k. \tag{4.3.1}$$

Then $\beta_i : T_{i\#} \to |D_i|$ satisfies the following:

$$\text{if} \quad z \in t_{i\#}^{(X)} \quad \text{then} \quad \gamma_i(z) = (z, 1) \quad \text{and} \quad w_i[\vec{\beta}]((z, 1)) = (z, 1); \tag{4.3.2}$$

by (4.1.1) hence $\beta_i(z) = \delta_i^{-1}((z, 1))$ by (4.3.1),

$$\left.\begin{array}{l} \text{if } \quad z = z_1 z_2 \quad \text{with} \quad z_1 \in t_{i\#}^{(j)},\ z_2 \in T_{j\#}, \quad \text{then} \quad \gamma_i(z) = (z_1, z_2) \\ \text{and} \\ w_i[\vec{\beta}]((z_1, z_2)) = (z_1, \beta_j(z_2)) \end{array}\right\} \quad (4.3.3)$$

by 4.1.2, hence $\beta_i(z) = \delta_i^{-1}((z_1, \beta_j(z_2)))$.

These properties insure in fact the existence and unicity of $\vec{\beta}$.

Since γ_i is bijective, every $z \in T_{i\#}$ either belongs to $t_{i\#}^{(X)}$ and $\beta_i(z)$ is defined by (4.3.2) in a unique way, or is of the form $z = z_1 z_2$ with $z_1 \in t_{i\#}^{(j)}$ and $z_2 \in T_{j\#}$ in a unique way *and* z_2 *is shorter than* z (since $t_i \notin V$) and $\beta_i(z)$ is defined by (4.3.3) in a unique way if $\beta_j(z_2)$ is, which can be assumed since $|z_2| < |z|$ (in a proof by induction).

Hence we have shown the existence and unicity of $\vec{\beta}$. □

Since initial objects are isomorphic, $\langle (\Phi(T_i), \gamma_i); 1 \leqq i \leqq k \rangle$ does not depend on the precise choice of $\bar{\Sigma}$ which has been done to define $T_1, \ldots, T_k$.

More precisely, if $\Sigma^{(j)} = \langle u_i = t_i^{(j)}; 1 \leqq i \leqq k \rangle$, $j = 1, 2$ are two systems such that $t_i^{(j)} \in M(F \cup V) - V$ and $\Phi(t_i^{(1)}) = \Phi(t_i^{(2)})$ for $1 \leqq i \leqq k$ then:

4.4. COROLLARY: $\Phi(T(\Sigma^{(1)}, u_i)) \simeq \Phi(T(\Sigma^{(2)}, u_i))$ *for* $1 \leqq i \leqq k$.

5. THE ALGEBRAIC STRUCTURE OF $\mathscr{A}_\omega(X)/\simeq$

5.1. Let us try to find an algebraic presentation of $\mathscr{A}_\omega(X)/\simeq$, the set of isomorphism classes of countable arrangements.

We have shown that every countable arrangement is isomorphic to $\Phi(T)$ for some $T \in M^\infty(F)$ (theorem 2.3).

Let $\equiv$ be the equivalence relation on $M^\infty(F)$ defined by $T \equiv T'$ iff $\Phi(T) \simeq \Phi(T')$. It is a congruence by 2.4, i. e. $T_1 \star T_2 \equiv T_1' \star T_2'$ if $T_i \equiv T_i'$ for $i = 1, 2$.

It follows that $\mathscr{A}_\omega(X)/\simeq$ is isomorphic to $M^\infty(F)/\equiv$.

Equivalences on $M^\infty(F)$ have been investigated in [2] for a different purpose. We shall use the concepts of [2] and show a negative result, that $\equiv$ is not an algebraic congruence on $M^\infty(F)$. It will follow that $M^\infty(F)/\equiv$ and $\mathscr{A}_\omega(X)/\simeq$ are not "pleasantly" presented, as we would like them to be.

Let us recall that $M^\infty(F)$ is the set of maximal elements of an ordered set $M_\Omega^\infty(F)$ defined as follows: the alphabet is augmented with a new symbol Ω and $M_\Omega^\infty(F) = M^\infty(F \cup \{\Omega\})$. An order relation $\prec$ on $M_\Omega^\infty(F)$ is defined as follows: $T \prec T'$ iff $\mathrm{Dom}(T) \subset \mathrm{Dom}(T')$ and for all u in $\mathrm{Dom}(T)$, $T(u) = \Omega$ or $T(u) = T'(u)$.

Intuitively $T \prec T'$ if and only if T' is obtained from T by the substitution of arbitrary elements for certain occurrences of Ω in T. One can also think of T as an "initial subtree of T'" if $T \prec T'$.

Let $M_\Omega(F)$ be the set of trees of $M_\Omega^\infty(F)$ with a finite set of nodes. Letters t, t', s, s', t_1, ... will be reserved to elements of $M_\Omega(F)$.

Every increasing sequence (t_n) in $M_\Omega(F)$ has a least upper bound Sup (t_n) in $M_\Omega^\infty(F)$, and conversely, every element of $M_\Omega^\infty(F)$ is the least upper bound of an increasing sequence in $M_\Omega(F)$.

We shall show that the equivalence $\equiv$ does *not* satisfy the following *continuity property:*

If (t_n) and (t'_n) are increasing sequences in $M_\Omega(F)$ such that $t_n \equiv t'_n$ for all $n \in \mathcal{N}$, then Sup $(t_n) \equiv$ Sup (t'_n).

To do so we take an example.

5.2. *Example:*

Let s_n, s'_n and s''_n be the increasing sequences in $M_\Omega(F)$ defined by:

$$\begin{cases} s_0 = s'_0 = s''_0 = \Omega, \\ s_{n+1} = a \star s_n, \\ s'_{n+1} = s'_n \star a, \\ s''_{n+1} = a \star (s''_n \star a). \end{cases}$$

The sequences $s_n \star s''_n$ and $s''_n \star s'_n$ also are increasing and have least upper bounds $T_0 \star T_2$ and $T_2 \star T_1$ respectively (T_0, T_1 and T_2 are defined in 2.2).

We have clearly:

$$\Phi(s_n \star s''_n) = a^n \Omega a^n \Omega a^n = \Phi(s''_n \star s'_n),$$
$$\Phi(T_0 \star T_2) = a^\omega a^\omega a^{-\omega} \neq a^\omega a^{-\omega} a^{-\omega} = \Phi(T_2 \star T_1). \quad \square$$

5.3. We can also show that $\equiv$ is not an *algebraic congruence* on $M^\infty(F)$.

Using definitions of [2], a congruence on $M^\infty(F)$ is said *algebraic* if there exists a preorder R on $M_\Omega(F)$ satisfying the following

$$t \prec t' \Rightarrow t R t', \tag{5.3.1}$$

$$t_1 R t'_1 \text{ and } t_2 R t'_2 \Rightarrow (t_1 \star t_2) R (t'_1 \star t'_2) \tag{5.3.2}$$

for all increasing sequences (t_n) and (t'_n) in $M_\Omega(F)$ with least upper bounds in $M^\infty(F)$:

$$\text{Sup}(t_n) \equiv \text{Sup}(t'_n) \Leftrightarrow \forall n \exists m \text{ s.t. } t_n R t'_m \text{ and } \forall m \exists n \text{ s.t. } t'_m R t_n \tag{5.3.3}$$

5.4. PROPOSITION: *The congruence $\equiv$ on $M^\infty(F)$ is not algebraic.*

Proof: Assuming the existence of a preorder R on $M_\Omega(F)$ satisfying (5.3.1) to (5.3.3) we shall show that

$$\forall n, \exists m \text{ s.t. } (s_n \star s''_n) R (s''_m \star s'_m) \tag{5.4.1}$$

and

$$\forall m, \exists n \text{ s.t. } (s''_m \star s'_m) R (s_n \star s''_n). \tag{5.4.2}$$

But a contradiction will follow since $\mathrm{Sup}(s_n \star s''_n) \not\equiv \mathrm{Sup}(s''_m \star s'_m)$ as shown in 5.2.

In order to prove (5.4.1), let us fix n and consider $S^{(n)} = U^{(n)} \star U'^{(n)}$ where

$$\begin{cases} U^{(0)} = T_2; & U'^{(0)} = T_1, \\ U^{(n+1)} = a \star U^{(n)}; & U'^{(n+1)} = a \star (U'^{(n)} \star a). \end{cases}$$

It is clear that $\Phi(U^{(n)}) = a^n \Phi(T_2) = a^\omega a^{-\omega}$ and $\Phi(U'^{(n)}) = a^n a^{-\omega} a^n = a^n a^{-\omega}$.

Finally, $\Phi(S^{(n)}) = a^\omega a^{-\omega} a^{-\omega}$ hence $S^{(n)} \equiv T_2 \star T_1$.

Since $s_n \star s''_n \prec S^{(n)}$, we get from (5.3.3) the existence of m such that $(s_n \star s''_n) R (s''_m \star s'_m)$. And (5.4.1) is proved. One proves similarly (5.4.2) and a contradiction is obtained. □

The meaning of 5.2 and 5.4 is that the equivalence of two trees is a global property which cannot be deduced from a comparison of the finite approximations of the trees involved.

Another consequence is that $\mathscr{A}_\omega(X)/\simeq$ cannot be defined as a quotient of $M^\infty(F)$ in a usable way, and *a fortiori* is not the set of ideals of a quotient of $M_\Omega(F)$.

6. CONCLUSIONS

Many problems are left open.

1. Deciding whether $A_1(\Sigma) \simeq A_1(\Sigma')$ for systems Σ and Σ'.

2. Find a complete system of axioms and rules for the equality of regular expressions.

3. Extend regular expressions to represent the solutions of arbitrary systems.

4. What can be said of the set of arrangements $A(G)$ which are generated by a context-free grammar? of the equivalence relation on grammars defined by $A(G) \simeq A(G')$ i. e. for every $B \in A(G)$ there exists some $B' \in A(G')$ such that $B \simeq B'$ and *vice versa*?

This work being completed, I happened to know that equations in arrangements have already been considered by S. Heilbrunner [3]. In particular, he gives a solution to the third problem that I am leaving open.

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