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NÉRON MODELS FROM THE RIGID ANALYTIC VIEWPOINT

by Siegfried BOSCH (*)

The theory of Néron models has proved as an important tool for the treatment of abelian varieties over number fields or over fields with a discrete non-archimedean valuation. For example, the uniformization of abelian varieties as suggested by RAYNAUD, is based on the existence of Néron models. Recently, through Faltings' proof of Mordell's conjecture, these models have been brought to attention again. Néron models were originally constructed by NÉRON [N] in 1963. It was then indicated by RAYNAUD [R] in 1966 how to carry out the construction within the framework of modern algebraic geometry. Furthermore, GROTHENDIECK [SGA 7] showed the existence of Néron models with semi-abelian reduction.

In [BL 2], one finds an approach to the uniformization of abelian varieties in terms of rigid analysis, avoiding the use of Néron models. In fact, the existence of Néron models with semi-abelian reduction in an easy consequence of the results in [BL 2]. This will be explained in the present lecture ; for more details (see [BL 3]). The main idea behind the construction centers around the fact that a flat scheme over a valuation ring is uniquely characterized by its generic fibre and by its formal completion along the special fibre. Under certain finiteness conditions, the latter can be viewed as a rigid analytic variety. It is for this reason that the application of rigid analysis is possible.

1. Rigid analysis and formal algebraic geometry.

Let k be a complete non-archimedean field with a discrete valuation and let K be a dense subfield of k. The valuation ring of k is denoted by k°, and the residue field by $\tilde{k}$; similarly for K. We will consider schemes over k° and K, and their formal completions (in terms of formal schemes over k°) or their analytifications (in terms of rigid analytic varieties), respectively.

Let X be a formal scheme l. t. f. t. (locally of topologically finite type) over k°. One associates with X a formal analytic variety X^{f-an}, ([BL 1], sect. 1), and a (rigid) analytic variety X^{an}. Namely, if X is affine, say $X = \operatorname{Specf} R$ with R t. f. t. (of topologically finite type) over k°, then $R_k := R \otimes_{k^\circ} k$ is a k-affinoid algebra, and

$$X^{f-an} := \operatorname{Spf} R_k \ , \quad X^{an} := \operatorname{Sp} R_k \ .$$

Both varieties are essentially the same ; their "points" consist of the maximal

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ideals in R_k . However, on the affinoid variety $\mathrm{Sp}\, R_k$, we consider the usual Grothendieck topology, as characterized in ([BGR], 9.1.4/2), whereas on the formal affinoid variety $\mathrm{Spf}\, R_k$ we restrict ourselves to the topology generated by all formal subdomaine ([BL 1], sect. 1). Using coverings, the definitions of X^{f-an} and X^{an} extend to the general case. Formal analytic varieties admit reductions, derived from the canonical reductions of their formal open affinoid parts. The reduction $\tilde{X}^{f-an}$ of X^{f-an} is related to the special fibre X_s of X by a finite surjective map $\tilde{X}^{f-an} \longrightarrow X_s$. If $X = \mathrm{Specf}\, R$ is as before, this map corresponds to the homomorphism

$$R \otimes_{k^\circ} \tilde{k} \longrightarrow \tilde{R}_k ,$$

where

$$\tilde{R}_k := \{f \in R_k \; ; \; |f|_{\sup} \leq 1\}/\{f \in R_k \; ; \; |f|_{\sup} < 1\} .$$

There is a converse procedure which associates a formal scheme Y^{f-sch} (not necessarily) l. t. f. t. over k°) to each formal analytic variety Y . Namely, let $\mathcal{O}$ be the sheaf of analytic functions on Y and denote by $\mathcal{O}^\circ$ the subsheaf of functions of sup-norm ≤ 1 . Then Y^{f-sch} is constructed by glueing the affine formal schemes $\mathrm{Specf}\; \mathcal{O}^\circ(U)$ where U varies over all formal open affinoid subvarieties of Y .

LEMMA 1.1. - <u>The mappings</u> $X \mapsto X^{f-an}$ <u>and</u> $Y \mapsto Y^{f-sch}$ <u>are functorial. They set up an equivalence between</u>

(a) <u>all formal schemes</u> X <u>which are l. t. f. t. and flat over</u> k° , <u>and have a reduced special fibre</u> X_s , <u>and</u>

(b) <u>all distinguished formal analytic varieties</u> Y . (<u>A formal analytic variety</u> Y <u>is called distinguished if, for each affinoid formal open subvariety</u> $U \subset Y$, <u>there exists a closed immersion</u> $U \hookrightarrow \underset{\sim}{B}^n$ <u>into some unit ball</u> $\underset{\sim}{B}^n$ <u>such that each</u> $f \in \mathcal{O}(U)$ <u>extends to a function</u> $g \in \mathcal{O}(\underset{\sim}{B}^n)$ <u>with</u> $|f|_{\sup} = |g|_{\sup}$ (<u>cf.</u> [BL 1], <u>sect.</u> 1 <u>and</u> [BGR], 6.4.3).

<u>Furthermore, if</u> X <u>is of type</u> (a) <u>and</u> $Y = X^{f-an}$ <u>is its counterpart of type</u> (b) <u>then the special fibre of</u> X <u>coincides canonically with the reduction of</u> Y .

We call a formal analytic variety Y smooth over k° if it is distinguished and has geometrically regular reduction $\tilde{Y}$. Equivalently, instead of assuming Y to be distinguished, we may require that the sup-norm of analytic functions on formal open subvarieties of Y assumes values only in $|k|$; this is clear by ([BGR], 6.4.3/1), since we are working over a complete field with a discrete valuation, which is stable by ([BGR], 3.6.2/1). If Y is smooth over k° , its associated analytic variety is geometrically regular. Furthermore, one can show that, under the equivalence set up in Lemma 1.1, formal schemes, smooth over k° , correspond bijectively to formal analytic varieties, smooth over k° .

Looking at the dense subfield $K \subset k$, let us consider a scheme $\mathfrak{X}$ which is l. f. t. (locally of finite type) over k^o . Then we set

$\hat{\mathfrak{X}}$:= formal completion of $\mathfrak{X}$ (a formal scheme l. t. f. t. over k^o),

$\bar{\mathfrak{X}} := \hat{\mathfrak{X}}^{f-an}$ = formal analytic variety associated to $\hat{\mathfrak{X}}$,

$\mathfrak{X}_\eta$:= generic fibre of $\mathfrak{X}$,

$\mathfrak{X}_\eta^{an}$:= analytification of $\mathfrak{X}_\eta$ (a rigid analytic variety over k).

One knows that $\bar{\mathfrak{X}}$ gives rise to an admissible open subvariety of $\mathfrak{X}_\eta^{an}$.

Example 1.2. - Let t be a uniformizing element in K^o . Consider an algebra R f. t. (of finite type) over K^o , and set

$R_K := R \otimes_{K^o} K$,

$\hat{R}$:= t-adic completion of R ,

$\bar{R} := \hat{R} \otimes_{K^o} K = \hat{R} \otimes_{k^o} k$.

Then, for $\mathfrak{X}$:= Spec R , one has $\hat{\mathfrak{X}}$ = Specf $\hat{R}$, $\bar{\mathfrak{X}}$ = Spf $\bar{R}$, and $\mathfrak{X}_\eta$ = Spec R_K . Choosing a closed immersion $\mathfrak{X} \hookrightarrow \underset{\sim}{A}^n_{K^o}$, one obtains closed immersions $\mathfrak{X}_\eta \hookrightarrow \underset{\sim}{A}^n_K$ and $\mathfrak{X}_\eta^{an} \hookrightarrow \underset{\sim}{A}^n_k$. In this situation, the formal analytic variety $\bar{\mathfrak{X}}$ corresponds to the "unit ball" in $\mathfrak{X}_\eta^{an}$; i. e., pointwise we have $\bar{\mathfrak{X}} = \mathfrak{X}_\eta^{an} \cap \mathfrak{X}\underset{\sim}{B}^n$.

PROPOSITION 1.3. - <u>Let</u> $\mathfrak{X}$, $\mathfrak{Y}$ <u>be schemes l. f. t. and flat over</u> K^o <u>and having reduced special fibres</u> $\mathfrak{X}_s$ <u>and</u> $\mathfrak{Y}_s$. <u>Let</u> $\varphi_\eta : \mathfrak{Y}_\eta \to \mathfrak{X}_\eta$ <u>be a</u> K-<u>morphism between the generic fibres. Then the following conditions are equivalents</u>

(i) φ_η <u>extends to a</u> K^o-<u>morphism</u> $\varphi : \mathfrak{Y} \to \mathfrak{X}$.

(ii) $\varphi_\eta^{an} : \mathfrak{Y}_\eta^{an} \to \mathfrak{X}_\eta^{an}$, <u>the analytification of</u> φ_η , <u>restricts to a morphism of formal analytic varieties</u> $\bar{\varphi} : \bar{\mathfrak{Y}} \to \bar{\mathfrak{X}}$.

<u>Furthermore, any extension</u> φ <u>of</u> φ_η <u>as in</u> (i) <u>is unique</u>.

We need an analogue of 1.3 which shows how to extend K-schemes to K^o-schemes. Let X_η be a scheme l. f. t. over K , and denote its analytification by X_η^{an} . Furthermore, let $\bar{X}$ be a distinguished formal analytic variety, together with an open immersion $\bar{X}^{an} \hookrightarrow X_\eta^{an}$ of its analytification $\bar{X}^{an}$ into X_η^{an} . More precisely, let us assume that, there is an open affine covering $\{U_i\}_{i\in I}$ of X_η as well as a formal open affinoid covering $\{\bar{U}_i\}_{i\in I}$ of $\bar{X}$ such that, for all $i \in I$, we have $\bar{U}_i \subset U_i^{an}$ and the image of $\mathcal{O}(U_i)$ is dense in $\mathcal{O}(\bar{U}_i)$. Then

PROPOSITION 1.4. - X_η <u>extends uniquely (up to canonical isomorphism) to a scheme</u> $\mathfrak{X}$ <u>l. f. t. and flat over</u> K^o <u>such that</u> $\mathfrak{X}_\eta = X_\eta$, <u>such that the special fibre</u> $\mathfrak{X}_s$ of $\mathfrak{X}$ <u>is reduced, and such that</u> $\bar{\mathfrak{X}} = \bar{X}$. <u>Furthermore,</u> $\mathfrak{X}$ <u>is smooth over</u> K^o <u>if and only if</u> X_η <u>is smooth over</u> K <u>and</u> $\bar{X}$ <u>is smooth over</u> k^o .

2. Uniformization of abelian varieties.

The construction of Néron models will be based on the results of [BL 2]. So let us recall the necessary facts about the uniformization of abelian varieties, and let us adapt them to the case of a not necessarily algebraically closed ground field k. The field $K \subset k$ is as in sect. 1. Let A be an abelian variety over K, and, for any extension L of K, write A_L for $A \otimes_K L$. Let A^{an} be the analytification of A. Then A^{an} is a rigid analytic group over k.

PROPOSITION 2.1. - <u>There is a unique open analytic subgroup</u> $\bar{A}$ <u>of</u> A^{an} <u>which is a (geometrically) connected (quasi-compact) formal analytic group satisfying the following property.</u>

<u>There exists a finite separable extension</u> ℓ <u>of</u> k <u>such that</u> $\bar{A}_\ell := \bar{A} \otimes_k \ell$ <u>is smooth over</u> ℓ^o <u>and has semi-abelian reduction.</u>

The group $\bar{A}$ has the following universal mapping property.

PROPOSITION 2.2. - <u>Let</u> X <u>be a connected formal analytic variety, smooth over</u> k^o. <u>If</u> $\varphi : X \longrightarrow A^{an}$ <u>is an analytic morphism such that</u> $\operatorname{im}\varphi \cap \bar{A} \neq \emptyset$, <u>then</u> $\operatorname{im}\varphi \subset \bar{A}$ <u>and</u> $\varphi : X \longrightarrow \bar{A}$ <u>is a formal morphism.</u>

For the discussion of Néron models the notion of étale points is basic. Let X be an analytic variety over k. A closed point $x \in X$ is called <u>étale</u> if the field $k(x)$ is an unramified extension of k. Thereby we mean that $k(x)$ is separable over k, that the extension of the valuation from k to $k(x)$ has ramification index 1, and that the residue extension of $k(x)/k$ is separable. These conditions are equivalent to the fact that the valuation ring of $k(x)$ is étale over k^o. The set of étale points in X will be denoted by X^{et}. It is clear that morphisms send étale points to étale points. Furthermore, for any unramified extension ℓ of k, one obtains a surjective map $X_\ell^{et} \longrightarrow X^{et}$.

LEMMA 2.3. - <u>Let</u> X <u>be a formal analytic variety over</u> k <u>which is smooth over</u> k^o. <u>Then</u> X^{et} <u>is formally dense in</u> X.

<u>Proof</u>. - Due to 1.1, we can view X as a formal scheme smooth over k^o. Since the special fibre $X_s = \tilde{X}$ is smooth over $\tilde{k}$, the set of closed points, whose residue field is separable over $\tilde{k}$, is Zariski-dense in $\tilde{X}$. By the lifting property of smoothness, such points lift to étale points over the complete field k,

q. e. d.

If $\bar{A}$ is smooth over k^o, we see by 2.3 that $A^{et} \cap \bar{A}$ is formally dense in $\bar{A}$. We want to use the uniformozation of A^{an} in order to obtain a precise knowledge about the position of all points in A^{et}. We need the following facts from [BL 2].

2.4. - <u>The group</u> $\bar{A}$ <u>contains a unique closed subgroup</u> $\bar{T}$ <u>such that</u> $\bar{T}$ <u>is a maximal affinoid torus in</u> $\bar{A}$ (i. e., <u>after extending the ground field</u> k <u>suitably,</u>

$\bar{T}$ becomes isomorphic to a product of multiplicative groups $\mathbb{G}_m$). If $\bar{A}$ is smooth over k^o , the torus $\bar{T}$ is smooth over k^o , and the closed immersion $\bar{T} \hookrightarrow \bar{A}$ reduces to the closed immersion $\tilde{T} \hookrightarrow \tilde{A}$, where $\tilde{T}$ is the maximal affine torus in $\tilde{A}$.

2.5. – Assume that $\bar{A}$ is smooth over k^o and that $\bar{T}$ splits over k . Then the closed immersion $\bar{T} \hookrightarrow \bar{A}$ extends to an analytic homomorphism $T \to A^{an}$, where T is the affine torus over k containing $\bar{T}$ as subgroup of units, and the open immersion $\bar{A} \hookrightarrow A^{an}$ extends to a surjective covering map $p : \hat{A} \to A^{an}$, where $\hat{A} := \bar{A} \times T/(\text{diagonal})$ is the universal covering of A

$$\begin{tikzcd}
 & & & \Gamma \arrow[d] \\
 & T \arrow[rr, hook] & & \hat{A} \arrow[dd, "p"] \\
\bar{T} \arrow[ur, hook] \arrow[rr, hook] & & \bar{A} \arrow[ur, hook] \arrow[dr, hook] & \\
 & & & A^{an}
\end{tikzcd}$$

The kernel $\Gamma := \ker p$ is a discrete subgroup of $\hat{A}$.

We will characterize the situation in 2.5 by saying that the uniformization of A is defined over k . In this case we know (up to unramified extension of k) that $\bar{A}$ is locally a product of $\bar{T}$ and a locally closed subvariety of $\bar{A}$; hence $\hat{A}$ is locally a product of T and a locally closed subvariety of $\bar{A}$. Since the étale points of tori are easy to determine, we obtain the following information on the étale points of $\hat{A}$ and A .

PROPOSITION 2.6. – Assume that the uniformization of A is defined over k . Denote by $T(k)$ the group of k-rational points in T . Then the lattice Γ consists of k-rational points in $\hat{A}$. The set of étale points in $\hat{A}$ satisfies

$$\hat{A}^{et} = T(k) . (\bar{A} \cap A^{et}) \quad \text{and} \quad p(\hat{A}^{et}) = A^{et} .$$

In particular $\Gamma \subset T(k).\bar{A}$, and A^{et} is contained in a finite union of translates of $\bar{A}$ by points in $p(T(k))$.

THEOREM 2.7. – Let $\bar{A}$ be smooth over k^o . Then there exists a unique open analytic subgroup $\bar{A}^{et}$ of A^{an} , which is a quasi-compact formal analytic group smooth over k^o , and which satisfies the following universal mapping property.

For each formal analytic variety X smooth over k^o , and each analytic map $\varphi : X \to A^{an}$, the image im φ is contained in $\bar{A}^{et}$, and $\varphi : X \to \bar{A}^{et}$ is a formal morphism.

Furthermore, $\bar{A}$ is the identity component of $\bar{A}^{et}$, and A^{et} is formally dense in $\bar{A}^{et}$. In particular, if the uniformization of A is defined over k, each component of $\bar{A}^{et}$ contains a k-rational point, and the surjection $\hat{A} \to (|k_a^*|)^t$, where k_a is the algebraic closure of k, gives rise to a bijection

$$\bar{A}^{et}/\bar{A} \xrightarrow{\sim} (|k^*|)^t/\Lambda ,$$

where Λ is the image of the lattice $\Gamma \subset \hat{A}$.

Proof. - By means of Galois theory, the general case is reduced to the case where the uniformization of A is defined over k. So let us assume that we are in the situation of 2.5. Then, due to 2.6, the union of all translates of $\bar{A}$ by the k-rational points of $p(T(k))$ yields a formal group $\bar{A}^{et}$ as required ; namely, the mapping property follows easily from 2.2, since each component of a given X, which is smooth over k^o, contains an étale point by 2.3,

q. e. d.

3. Néron models and Grothendieck's semi-abelian reduction.

In this section we will use the results of sect. 1, in particular Propositions 1.3 and 1.4 in order to discuss Néron models of abelian varieties in terms of rigid analysis. The necessary facts about the analytic structure of abelian varieties have been gathered in sect. 2. The fields $K \subset k$ are as in sect. 1.

Definition 3.1. - Let X be a scheme smooth over K. A scheme $\mathfrak{X}$ over K^o is called a Néron model of X if the following conditions are satisfied

(i) $\mathfrak{X}_\eta = X$

(ii) $\mathfrak{X}$ is smooth over K^o.

(iii) If $\mathfrak{Y}$ is a scheme, smooth over K^o and $\varphi_\eta : \mathfrak{Y}_\eta \to \mathfrak{X}_\eta$ is a K-morphism, then φ_η extends uniquely to a K^o-morphism $\varphi : \mathfrak{Y} \to \mathfrak{X}$.

It is clear that the Néron model is unique (up to canonical isomorphism) if it exists. Furthermore, $\mathfrak{X}$ is a group scheme if X is a group scheme. The main result we want to prove combines the existence of Néron models for abelian varieties [N], [R] with Grothendieck's semi-abelian reduction [SGA 7].

Semi-Abelian Reduction Theorem 3.2. - Let A be an abelian variety over K. Then there exists a finite separable extension L of K such that the valuation extends uniquely from L to K and such that

(a) $A_L := A \otimes L$ admits a Néron model $\mathfrak{A}$ f. t. over L^o.

(b) $\mathfrak{A}$ has semi-abelian reduction, i. e., the identity component $\mathfrak{A}_s^o$ of the special fibre $\mathfrak{A}_s$ of $\mathfrak{A}$ is an extension of an abelian variety by a torus.

It is shown in [N], [R] that A has always a Néron model over K^o ; however, this model does not necessarily have semi-abelian reduction. Considering the analytification A^{an} of A we have seen in 2.1 that the open analytic subgroup $\bar{A} \subset A^{an}$, after replacing k by a finite separable extension ℓ, becomes smooth over ℓ^o. By Krasner's Lemma (in the version of [BGR], 3.4.2/3), ℓ can be interpreted as the completion of a finite separable extension L of K with a unique valuation extending the valuation on K. Thus the assertion of 3.2 will be clear if we prove:

THEOREM 3.3. - <u>Let</u> A <u>be an abelian variety over</u> K, <u>and consider the open analytic subgroup</u> $\bar{A}$ <u>of</u> A^{an} <u>which has been defined in 2.1. Then the following are equivalent</u>

(a) $\bar{A}$ <u>is smooth over</u> k^o.

(b) <u>The Néron model</u> $\mathcal{A}$ <u>of</u> A <u>exists, is f. t. over</u> K^o, <u>and has semi-abelian reduction.</u>

<u>Furthermore, if (a) and (b) are satisfied, the analytic subgroup</u> $\bar{\mathcal{A}} \subset A^{an}$ <u>associated to the formal completion</u> $\hat{\mathcal{A}}$ <u>of</u> A <u>coincides canonically with the group</u> $\bar{A}^{et}$ <u>of 2.7, whose identity component is</u> $\bar{A}$.

We want to deduce the assertions of 3.3 from the results of sects. 1 and 2 be means of the following lemma which involves the group $\bar{A}^{et}$.

LEMMA 3.4. - <u>Let</u> A <u>be as in 3.3, and assume that</u> $\bar{A}$ <u>is smooth over</u> k^o. <u>Then there exists an affine open covering</u> $\{U_i\}_{i=1,\dots,n}$ <u>of</u> A <u>as well as an affinoid formal open covering</u> $\{\bar{U}_i\}_{i=1,\dots,n}$ <u>of</u> $\bar{A}^{et}$ <u>such that</u> $\bar{U}_i \subset U_i^{an}$ <u>and</u> $\mathcal{O}(U_i)$ <u>is dense in</u> $\mathcal{O}(\bar{U}_i)$ <u>for all</u> i.

First, let us indicate how 3.3 is derived from 3.4. Assume that $\bar{A}$ is smooth over k^o. Then we use the coverings of 3.4 and obtain by 1.4 a scheme $\mathcal{A}$ f. t. over K^o which extends A. The formal analytic variety $\bar{\mathcal{A}}$ associated to $\mathcal{A}$ coincides with $\bar{A}^{et}$; hence $\mathcal{A}$ is smooth over K^o. Furthermore, the identity component $\mathcal{A}_s^o$ of the special fibre $\mathcal{A}_s$ of $\mathcal{A}$ coincides with the reduction of $\bar{A}$ (cf. 2.1), and we see that $\mathcal{A}_s^o$ is semi-abelian. Thus, in order to show that $\mathcal{A}$ is the Néron model of A, it remains to verify the universal mapping property (iii) of 3.1. However, the latter is a consequence of 1.3 and the universal mapping property of $\bar{A}^{et}$ mentioned in 2.7.

Conversely, assume that condition (b) of 3.4 is satisfied. Then $\bar{\mathcal{A}}$ is an open analytic subgroup of A^{an}, which is smooth over k^o as a formal analytic group. Since the reduction of its identity component $\bar{\mathcal{A}}^o$ coincides with $\mathcal{A}_s^o$ and hence is semi-abelian, we see by 2.1 that $\bar{A} = \bar{\mathcal{A}}^o$. In particular, $\bar{A}$ is smooth over k^o. Thereby we have reduced the proof of 3.3 to the proof of 3.4.

In order to prepare the proof of 3.4, we need an auxiliary result.

LEMMA 3.5. - <u>In the situation of 3.4, consider a non-empty affine open subvariety</u> $U \subset A$ <u>and a connected component</u> E <u>of the group</u> $\bar{A}^{et}$. <u>Then there exist a non-</u>

empty affine open subvariety $U' \subset U$ and a non-empty affinoid formal open subvariety $\bar{U} \subset E$ such that $\bar{U} \subset U'^{an}$ and such that $\mathcal{O}(U')$ is dense in $\mathcal{O}(\bar{U})$.

Proof of 3.5. - The variety U^{an} contains a non-empty affinoid formal open subvariety F of E. Fixing a closed immersion $U \hookrightarrow \mathbb{A}^n_K$ and looking at its analytification $U^{an} \hookrightarrow \mathbb{A}^n_K$, we write $U^o := U^{an} \cap \mathbb{B}^n_K$. Then U^o is affinoid, and we may assume $F \subset U^o$. By a result of Gerritzen and Grauert ([BGR], 7.3.5/3), F is a finite union of rational subdomains in U^o, and we can use the following elementary fact

Let $\{V_i\}_{i\in I}$ be an admissible open covering of an affinoid variety V. Then there exists an index $i \in I$ such that V_i contains a formal subdomain of V.

Thereby we may assume that F itself is a rational subdomain in U^o or, equivalently, a Weierstrass domain in a Laurent domain in U^o. Since analytic functions on U^o can be approximated by algebraic functions on U, we may replace U by a suitable affine open subvariety and thereby assume that $\bar{U} := F$ is a Weierstrass domain in U^o. Then $\mathcal{O}(U)$ is dense in $\mathcal{O}(\bar{U})$ as required,

q. e. d.

If there are enough K-rational points in A, the coverings of 3.4 can be constructed by translating the sets U' and $\bar{U}$ obtained in 3.5. However, in general this procedure will not work unless the field K is extended. If $K \neq k$, the fact that the valuation does not necessarily extend uniquely from K to an algebraic extension L makes it impossible to descend from L to K by means of Galois theory. To fix these troubles, one uses ample divisors on A. We will establish 3.4 by generalizing the assertion of 3.5.

LEMMA 3.6. - In the situation of 3.4, consider a connected component E of the group $\bar{A}^{et}$ and a point $x \in E$. Then there exists an affine open subvariety $U \subset A$ and an affinoid formal open subvariety $\bar{U} \subset E$ such that $x \in \bar{U} \subset U^{an}$ and such that $\mathcal{O}(U)$ is dense in $\mathcal{O}(\bar{U})$.

Proof of 3.6. - Choose an effective divisor D on A, which is ample. Then $V := A - \operatorname{supp} D$ is affine open in A. Applying 3.5, we may assume there is an affinoid formal open subvariety $\bar{V} \subset E$ such that $\bar{V} \subset V^{an}$ and $\mathcal{O}(V)$ is dense in $\mathcal{O}(\bar{V})$. Next, let ℓ be a finite Galois extension of k such that $\bar{x}$ decomposes into ℓ-radical points $x_1, \dots, x_r$ over ℓ. Then $x_1, \dots, x_r \in E_\ell := E \otimes \ell$. Furthermore,

$$\bar{A}_\ell \cap \bigcap_{i=1}^r x_i^{-1} \bar{V}_\ell \cap \bigcap_{i=1}^r x_i (\bar{V}_\ell)^{-1}$$

is a formal open subvariety in $\bar{A}_\ell$ and contains a point z; extending ℓ if necessary, we may assume that z is ℓ-radical. Then

$$x_1, \dots, x_r \in z\bar{V}_\ell \cap z^{-1} \bar{V}_\ell ,$$

and if $G = G(\ell/k)$ denotes the Galois group of ℓ over k, we have

$$x_1 , \ldots , x_r \in \bigcap_{\sigma\in G}(z^\sigma \bar{V}_\ell \cap (z^\sigma)^{-1} \bar{V}_\ell) .$$

By Galois descend, the variety on the right hand side is extension of an affinoid formal open subvariety $\bar{U} \subset E$ containing x.

Similarly

$$\bigcap_{\sigma\in G}(z^\sigma V_\ell \cap (z^\sigma)^{-1} V_\ell)$$

is extension of an affine open subvariety $U' \subset A_{k'}$, where $\bar{U} \subset U'$. Using some standard arguments on complete tensor products and the fact that $\mathcal{O}(V)$ is dense in $\mathcal{O}(\bar{V})$, one easily shows $\mathcal{O}(U')$ is dense in $\mathcal{O}(\bar{U})$. This verifies the assertion of 3.6 already in the case where $K = k$ is complete.

So far we have not really needed the ample divisor D, and we will use it now in order to approximate U' by a variety defined over K. Translating the divisor $D_\ell = D \otimes \ell$ by z and z^{-1}, where z is as above, one obtains effective divisors $D_\ell(z)$ and $D_\ell(z^{-1})$ such that, by the theorem of the square, their sum is linearly equivalent to $2D_\ell$. Then

$$\sum_{\sigma\in G} D_\ell(z^\sigma) + D_\ell((z^\sigma)^{-1})$$

is extension of a divisor D_k' on A_k such that

$$\operatorname{supp} D_k' = A_k - U_k'$$

and such that D_k' is linearly equivalent to a multiple of $D_k = D \otimes k$. Replacing D by a multiple of itself, we may assume D_k' linearly equivalent to D_k and D very ample. Let $\mathcal{L} \simeq \mathcal{O}(D)$ be the bundle associated to D. Then

$$\mathcal{L} \otimes k \simeq (D_k) \simeq (D_k') ,$$

and we have

$$\Gamma(\mathcal{L} \otimes k , A_k) \simeq \Gamma(\mathcal{L} , A) \otimes k$$

for the vector spaces of global sections. Interpreting $\mathcal{L} \otimes k$ as the sheaf $\mathcal{O}(D_k')$, the constant function $s_0' \equiv 1$ gives rise to a global section of $\mathcal{L} \otimes k$ which generates $\mathcal{L} \otimes k$ over U_k' and, in particular, over $\bar{U}$. Approximating s_0' by a section $s_0 \in \mathcal{L}(A)$, we consider the affine open subvariety $U \subset A$, where s_0 generates $\mathcal{L}$. Then $\bar{U} \subset U^{an}$ if the approximation is good enough. Furthermore, $\mathcal{O}(U)$ is dense in $\mathcal{O}(\bar{U})$, since there is an affinoid generating system of $\mathcal{O}(\bar{U})$ in $\mathcal{O}(U_k')$ and hence, by approximation in $\mathcal{O}(U)$. This concludes the proof of 3.6.

It is now easy to construct the coverings needed in 3.4. Namely, for each $x \in \bar{A}^{et}$ one constructs varieties $\bar{U}$ and U as in 3.6. Then finitely many of the varieties $\bar{U}$ must cover $\bar{A}^{et}$ since this group is quasi-compact. Adding affine open subvarieties $U \subset A$ and empty varieties $\bar{U}$, one can guarantee that the varieties U cover A. Thereby the assertions of 3.4 and thus also of 3.3 and 3.2 are clear,

q. e. d.

Remark 3.7. - We have seen that the condtruction of the Néron model $\mathfrak{A}$ of A works well when the group $\bar{A} \subset A^{an}$ of 2.1 is smooth over k^o. If this is not the case, there is a maximal connected open subgroup $A' \subset \bar{A}$ which is smooth over k^o. Then one can interpret the group $\bar{A}^{et}$ of 2.7 (which has been constructed only in the case where $\bar{A}$ is smooth over k^o) as the group generated by A' and all étale points of A^{an}. Algebraizing $\bar{A}^{et}$ in literally the same way as exercised in this section, one obtains the Néron model $\mathfrak{A}$ of A over K^o. Of course, $\mathfrak{A}$ will not have semi-abelian reduction since A', the identity component of its formal completion, does not have semi-abelian reduction.

Remark 3.8. - The construction of Néron models with semi-abelian reduction requires in most cases an extension of the ground field K as stated in 3.2. However, there is a valuable criterion, due to Raynaud, saying that such an extension is unnecessary if, for some $n \geqslant 3$ prime to the residue characteristic of K, the n-torsion points of the abelian variety A are rational ([SGA 7], exposé IX, 4.7). One can prove a similar criterion for the group $\bar{A}^{et}$ and thereby deduce Raynaud's criterion from the results of this section.

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