GROUPE DE TRAVAIL D'ANALYSE ULTRAMÉTRIQUE

STEVEN SPERBER
YASUTAKA SIBUYA

On the p-adic continuation of the logarithmic derivative of certain hypergeometric functions

Groupe de travail d'analyse ultramétrique, tome 12, nº 1 (1984-1985), exp. nº 4, p. 1-5
<http://www.numdam.org/item?id=GAU_1984-1985__12_1_A3_0>

Groupe d'étude d'Analyse ultramétrique
(Y. AMICE, G. CHRISTOL, P. ROBBA)
12e année, 1984/85, nº 4, 5 p.
5 novembre 1984

ON THE p-ADIC CONTINUATION OF THE LOGARITHMIC DERIVATIVE OF CERTAIN HYPERGEOMETRIC FUNCTIONS

by Steven SPERBER and Yasutaka SIBUYA (*)

1. Introduction.

In the p-adic study of Kloosterman and multiple Kloosterman sums [4], [6], a p-adic cohomology space is constructed which has the structure of an F-crystal. The connection has, as its associated scalar differential equation, the equation for the hypergeometric function,

$$ {}_0F_N(1\ ,\ \ldots\ ,\ 1\ ;\ \pi^{N+1}\ x) = \sum_{j=0}^{\infty} (\pi^{N+1}\ x)^j/(j!)^{N+1} $$

where $\pi^{p-1} = -p$. One of the fundamental results of the theory is that the logarithmic derivative of this function has a p-adic analytic continuation to the closed disk $D(0\ ,\ 1^+)$. This continuation enables one to prove that the function

$$ E(x) = {}_0F_N(1\ ,\ \ldots\ ,\ 1\ ;\ \pi^{N+1}\ x)/{}_0F_N(1\ ,\ \ldots\ ,\ 1\ ;\ \pi^{N+1}\ x^p) $$

also continues to the closed disk $D(0\ ,\ 1^+)$. Furthermore the unit root of the L-function assciated with the (multiple) Kloosterman sum

$$ K_{N,m}(\bar{x}\ ,\ \psi) = \sum_{t_i \in F_{q^m}^x} \psi . \mathrm{Tr}_{F_{q^m}/F_p}(\bar{t}_1 + \ldots + \bar{t}_N + \bar{x}(\bar{t}_1\ \ldots\ \bar{t}_N)^{-1}) $$

(where $\bar{x} \in F_q$, ψ is a non-trivial additive character of F_p determined by the unique primitive p^{th} root of 1 which is congruent to $1 + \pi \bmod \pi^2$) is **then** given by

$$ E(x)\ E(x^p)\ \ldots\ E(x^{q/p}) $$

where $x = x^q$ is the Teichmuller lifting of $\bar{x}$.

Our purpose in the present paper is to give a non-cohomological elementary proof of this continuation. We use the following notation. $\underset{\sim}{N}$ denotes the set of natural numbers. If $n = \sum_{j=0}^{r} \alpha_j\ p^j \in \underset{\sim}{N}$, then $S(n) = \sum_{j=0}^{r} \alpha_j$, and $T(n) = [\log n/\log p]$, so that $T(n) = r$ if $\alpha_r \neq 0$.

2. Fundamental lemma

The following lemma provides us with the basic estimate which will be utilized in the proof of our main result in the next section.

(*) Steven SPERBER and Yasutaka SIBUYA, School of Mathematics, University of Minnesota, MINNEAPOLIS, MN 55455. (Etats-Unis).

2.1. LEMMA. - *Let*

$$f_N(k) = \frac{N+1}{N}\frac{S(k)}{p-1} + \frac{1}{N}T(k) .$$

Let $n_h \in \underset{\sim}{N}$ *for* $1 \leqslant h \leqslant M$; *let* $n = \sum_{h=1}^{M} n_h$. *Let* $\beta_h \in Z_{\geqslant 0}$, *and* $\sum_{h=1}^{M} \beta_h = N+1-M$.

Then

$$\sum_{h=1}^{M} f_N(n_h) + \frac{1}{N}\sum_{h=1}^{M} \beta_h \text{ ord } n_h \geqslant f_N(n) + \text{ord } n .$$

Proof. - Write $n_h = \sum_{k=s}^{t} \alpha_k^{(h)} p^k$ where $\alpha_s^{(h)} \neq 0$, for some $h \in \{1 , \dots , M\}$, and where $\alpha_t^{(h)} \neq 0$, for some $h \in \{1 , \dots , M\}$. Thus $\min_{1\leqslant h\leqslant M}\{\text{ord } n_h\} = s$, and $\max_{1\leqslant h\leqslant M} T(n_h) = t$. We define inductively integers $\{r_{j+1} , \gamma_j\}_{j=s}^{\mu}$ where $\mu \geqslant t$, $r_j \geqslant 0$, $p > \gamma_j \geqslant 0$, as follows

$$(2.2) \qquad \begin{cases} \sum_{h=1}^{M} \alpha_s^{(h)} = \gamma_s + r_{s+1}\, p & \text{if } j = s \\ r_j + \sum_{h=1}^{M} \alpha_j^{(h)} = \gamma_j + r_{j+1}\, p & \text{if } j > s . \end{cases}$$

Note that for $j > t$, we set $\alpha_j^{(h)} = 0$ for all h . We define $\mu = t$ if $r_{t+1} = 0$; otherwise we define μ to be the smallest integer k , $k > t$, such that $r_{k+1} = 0$. Thus $r_h \geqslant 1$, for $t < h \leqslant \mu$, and $r_{\mu+1} = 0$. Note that $\sum_{h=s}^{t} \sum_{j=1}^{M} \alpha_h^{(j)} = \sum_{i=1}^{M} S(n_i)$. Also, multiplying (2.2) by p^j and adding these equations from $j = s$ to $j = t$ yields $n = \sum_{h=s}^{\mu} \gamma_h\, p^h$ so that

$$(2.3) \qquad S(n) = \sum_{h=s}^{\mu} \gamma_h , \quad T(n) = \mu .$$

Similarly, simply adding equations in (2.2) from $j = s$ to $j = t$, yields

$$(2.4) \qquad \frac{1}{p-1}\left(\sum_{h=1}^{M} S(n_h) - S(n)\right) = \sum_{h=s+1}^{\mu} r_h .$$

We claim

$$(2.5) \qquad \text{ord } n \leqslant s + \lambda$$

where $\lambda \geqslant 0$ satisfies $s + \lambda + 1 = \inf\{j ; j > s , r_j = 0\}$. We have assumed $\alpha^{(j)} \neq 0$, for some j ; thus $\gamma_s \neq 0$ or $r_{s+1} \neq 0$. If $r_{s+1} = 0$, then $\gamma_s \neq 0$ so that ord $n = s$ and (2.5) holds. Assume $r_{s+1} \neq 0$, so $\lambda \geqslant 1$. Let $r_{s+\lambda+1} = 0$ and $r_{s+k} \neq 0$ for $1 \leqslant k \leqslant \lambda$. Then (2.2) implies $\gamma_{s+\lambda} \neq 0$ (since $r_{s+\lambda} + \sum_{j=1}^{M} \alpha_{s+\lambda}^{(j)} > 0$), so that ord $n \leqslant s + \lambda$, as claimed in (2.5).

It remains to show

$$(2.6) \qquad \frac{N+1}{N} \sum_{h=s+1}^{\mu} r_h + - \left(\sum_{h=1}^{M} T(n_h) - \mu + \sum_{h=1}^{M} \beta_h \text{ ord } n_h\right) \geqslant s + \lambda$$

Since $r_h \geq 1$ for $s+1 \leq h \leq s+\lambda$, therefore $\sum_{h=s+1}^{\mu} r_h \geq \lambda$. Also ord $n_h \geq s$ for all h, and $T(n_h) = t$ for some h, and $T(n_h) \geq s$ for all h. Thus

$$\frac{N+1}{N}\sum_{h=s+1}^{\mu} r_h + \frac{1}{N}\left(\sum_{h=1}^{M} T(n_h) - \mu + \sum_{h=1}^{M} \beta_h \operatorname{ord} n_h\right)$$

$$\geq \lambda + s + \frac{1}{N}\left(\sum_{h=s+1}^{\mu} r_h + t - \mu\right).$$

But $r_h \geq 1$, for $t+1 \leq h \leq \mu$, so that $\sum_{h=s+1}^{\mu} r_h \geq \mu - t$ which establishes (2.6) and completes the proof of the lemma.

q. e. d.

3. Main result.

Consider the linear differential equation

(3.1) $$\delta^{N+1} y = xy \quad (\delta = xd/dx).$$

We are interested in the non-linear differential equation satisfied by $\eta = \delta y/y$, where y is any solution of (3.1). If we denote $\eta_i = \delta^i y/y$ for $i \geq 1$, with $\eta_1 = \eta$, then $\eta_{i+1} = \delta\eta_i + \eta\eta_i$ so that if y satisfies (3.1), namely $\eta_{N+1} = x$, then η satisfies $(\delta + \eta)^N(\eta) = x$, or more explicitly

(3.2) $$\delta^N \eta + \sum C(i)\, \eta^{i_1} (\delta\eta)^{i_2} \cdots (\delta^{N-1}\eta)^{i_N} = x$$

where $C(i) \in Z_{\geq 0}$, and the sum is taken over N-tuples $i = (i_1, \ldots, i_N)$ of non-negative integers satisfying

(3.3) $$\sum_{k=1}^{N} k i_k = N+1.$$

If $N \geq 0$, then (3.2) has a unique formal power-series solution $\eta(x) = \sum_{i=1}^{\infty} a_i x^i$ with zero constant term. In fact, the coefficients $\{a_i\}_{i\geq 1}$ of this unique formal power-series solution satisfy the recursion

(3.4) $$\begin{cases} a_1 = 1 \\ n^N a_n = \sum C(i) \sum \prod_{h=1}^{M} a_{n_h} n_h^{k_h} \end{cases}$$

where $I_0 = 0$, $I_k = \sum_{j=1}^{k} i_j$ for $1 \leq k \leq N$; $M(= I_N) = \sum_{j=1}^{N} i_j$; $k_h = k$, for $I_k + 1 \leq h \leq I_{k+1}$; the outer sum is as before, (3.3); the inner sum runs over M-tuples of positive integers $(n_1, \ldots, n_M)$ satisfying $\sum_{h=1}^{M} n_h = n$. Note that $\sum_{h=1}^{M} k_h = \sum_{k=1}^{N-1} k i_{k+1} = N+1-M$. We scale our coefficients via $b_n = (n!)^{N+1} a_n$ and effect this by multiplying (3.4) by $(n!)^{N+1}$, yielding

(3.5) $$\begin{cases} b_1 = 1 \\ n^N b_n = \sum C(i) \sum (n!/\prod_{h=1}^{M} n_h!)^{N+1} \prod_{h=1}^{M} b_{n_h} n_h^{k_h}, \quad n > 1. \end{cases}$$

We now show by induction that

(3.6) $$\operatorname{ord} b_k \geq T(k), \text{ for all } k.$$

Assume (3.6) for $k < n$. Then by (3.5), (3.6) is established for $k = n$ provided

$$\sum_{h=1}^{M} T(n_h) + (N+1)(\sum_{h=1}^{M} S(n_h) - S(n)) + \sum_{h=1}^{M} k_h \text{ ord } n_h \geq T(n) + N \text{ ord } n .$$

But this is precisely the assertion of lemma 2.1. Summarizing, we have proved the following result.

3.7. THEOREM. - Let $\eta(x) = \sum_{i=1}^{\infty} a_i x^i$ be the unique power-series solution with zero constant term satisfying (3.2). Then $\text{ord}(n!\, a_n) \geq T(n)$.

In [4] and [6], an F-crystal is constructed to study the p-adic properties of the Kloosterman and multiple Kloosterman sums. We consider the case of the multiple Kloosterman sums

(3.8) $$K_{N,m}(\bar{x}, \psi) = \sum \psi.\text{Tr}_{F_{q^m}/F_p}(\bar{t}_1 + \ldots + \bar{t}_N + \bar{x}(\bar{t}_1 \ldots \bar{t}_N)^{-1})$$

where the sum runs over $(\bar{t}_1, \ldots, \bar{t}_N) \in (F^x_{q^m})^N$, ψ is the non-trivial additive character of F_p (where $\psi(1)$ is the unique p^{th} root of 1 congruent to $1 + \pi$ mod π^2), and $\text{Tr}_{F_{q^m}/F_p}$ denotes the trace. In this case the scalar differential equation associated with the connection is

(3.9) $$\delta^{N+1} z = \pi^{N+1} xz ,$$

the differential equation of the hypergeometric function

$$F(x) = {}_0F_N(1, \ldots, 1 ; \pi^{N+1} x) = \sum_{j=0}^{\infty} (\pi^{N+1} x)^j/(j!)^{N+1}$$

where π is a uniformizer of the p-adic field $Q_p(\zeta_p)$ of p^{th} roots of 1 which satisfies $\pi^{p-1} = -p$.

3.10. COROLLARY. - If $\tilde{\eta}(x) = \sum_{n=1}^{\infty} c_n x^n (= \delta F(x)/F(x))$ is the unique power-series solution of the Riccati equation associated with (3.9), namely $(\delta + \tilde{\eta})^N \tilde{\eta} = \pi^{N+1} x$, with zero constant term, then $\tilde{\eta}(x)$ continues to $D(0, 1^+)$.

Proof. - In fact, $\eta(x) = \tilde{\eta}(x/\pi^{N+1})$ is the unique power-series solution of (3.2), so that

(3.11) $$\begin{cases} c_n = \pi^{n(N+1)} a_n = \pi^{n(N+1)} b_n/(n!)^{N+1} , \quad (b_n \in Q) , \\ \\ \text{ord } c_n \geq [\log n/\log p] + (N+1)\, S(n)/(p-1) \end{cases}$$

which establishes the corollary, describes the rate of convergence with p and gives a logarithmic lower bound for the rate of convergence on the closed disk.

Remark (i). - The method is somewhat more general than the particular example given here. In (3.1) and (3.2), the function x on the right side of the equation can be replaced by any power-series $\sum_{i=1}^{\infty} d_i x^i$ with $\text{ord } d_n \geq T(n)$, or more generally

$$\text{ord } d_n \geq T(n) + \text{ord}(n^N/(n!)^{N+1}) .$$

Remark (ii). - In [1] and [2] somewhat more complicated examples are studied and continuation is proved cohomologically. We do not know how to modify the argument to prove continuation in these examples using an elementary approach. Similarly, we do not know if such an approach can prove continuation of the logarithmic derivative of ${}_2F_1(1/2 , 1/2 , 1 ; x)$ as in [3] and [5].

Remark (iii). - The estimates we derive in the present article are related to the radius of meromorphy of the unit-root zeta function for multiple Kloosterman sums. The existence of an excellent lifting of Frobenius for Kloosterman sums [4] implies that in this case $(N = 1)$ the unit-root zeta function has infinite radius of meromorphy. The estimates (3.11) may be useful in getting a good estimate of the radius of meromorphy for the unit-root zeta function when $N > 1$.

REFERENCES

[1] ADOLPHSON (A.) and SPERBER (S.). - Twisted Kloosterman sums and p-adic Bessel functions, Amer. J. of Math., t. 106, 1984, p. 549-591.

[2] CARPENTIER (M. J.). - p-adic cohomology for generalized hyperkloosterman sums, Thesis, University of Minnesota, 1985.

[3] DWORK (B.). - p-adic cycles. - Paris, Presses universitaires de France, 1969 (Institut des hautes Études scientifiques, Publications mathématiques, 37, p. 27-116.

[4] DWORK (B.). - Bessel functions as p-adic functions of the argument, Duke math. J., t. 41, 1974, p. 711-738.

[5] DWORK (B.). - Lectures on p-adic differential equations. - Berlin, Heidelberg, New York, Springer-Verlag, 1982 (Grundlehren der mathematischen Wissenschaften 253).

[6] SPERBER (S.). - p-adic hypergeometric functions and their cohomology, Duke math. J., t. 44, 1977, p. 535-589.