JAN DENEF

Poles of *p*-adic complex powers and Newton polyhedra

Groupe de travail d'analyse ultramétrique, tome 12, nº 1 (1984-1985), exp. nº 17, p. 1-3
<http://www.numdam.org/item?id=GAU_1984-1985__12_1_A10_0>

Groupe d'étude d'Analyse ultramétrique
(Y. AMICE, G. CHRISTOL, P. ROBBA)
12e année, 1984/85, n° 17, 3 p.

25 mars 1985

POLES OF p-ADIC COMPLEX POWERS AND NEWTON POLYHEDRA

by Jan DENEF (*)

Abstract.

Let p denote a fixed prime number, $\mathbb{Z}_p$ the ring of p-adic integers, and $\mathbb{Q}_p$ the field of p-adic numbers. Let $x = (x_1, \ldots, x_n)$ and $f(x) \in \mathbb{Q}_p[x]$. Let Φ be a Schwartz-Bruhat function on $\mathbb{Q}_p^n$ (i. e. a locally constant function with compact support. The *p-adic complex power of* f is defined by

$$Z_f(s, \Phi) = \int_{\mathbb{Q}_p^n} |f(x)|^s \, \Phi(x) \, |dx| ,$$

for $s \in \mathbb{C}$, $\mathrm{Re}(s) > 0$, where $|f(x)|$ denotes the p-adic absolute value of $f(x)$, and $|dx|$ the Haar measure on $\mathbb{Q}_p^n$ so normalized that $\mathbb{Z}_p^n$ has measure 1.

THEOREM (IGUSA [1]).

(i) $Z_f(s, \Phi)$ *is a rational function of* p^{-s}.

(ii) *The poles of* $Z_f(s, \Phi)$ *are of the form* $s = -1$ or $s = -\frac{\nu}{N} + \frac{2\pi i}{\log p} \frac{k}{N}$, *where* N, $\nu \in \mathbb{N}_{>0}$, $k \in \mathbb{Z}$ *and the* (N, ν) *are among the numerical data of an embedded resolution of singularities of the hypersurface* $f = 0$ (see [1], p. 86).

(iii) *The multiplicity of the poles is* $\leq n$.

In the known examples, only very few of the numerical data (N, ν) of the resolution give rise to actual poles. In the Archimedean case (i. e. $\mathbb{Q}_p$ replaced by $\mathbb{R}$ in the definition of $Z_f(s, \Phi)$) this is explained by topological considerations involving monodromy (see MALGRANGE [3], p. 428). It is an open problem whether an analogue of this holds for $\mathbb{Q}_p$.

In this summary, we will state some results about the poles of $Z_f(s, \Phi)$ in the case that f is non-degenerated with respect to its Newton polyhedron. First, we need some notation and definitions.

For all what follows suppose that $f(0) = 0$ and $\frac{\partial f(0)}{\partial x_i} = 0$ for $i = 1, \ldots, n$, i. e. the origin is a singular point of the hypersurface $f = 0$. Write

$$f(x) = \sum_{n \in \mathbb{N}^n} a_n x^n ,$$

and let $S = \{n \in \mathbb{N}^n ; \ a_n \neq 0\}$. *Newton's polyhedron* $\Gamma(f)$ *of* f is the convex

(*) Jan DENEF, Departement Wiskunde, Faculteit Wetenschappen, Katholicke Universiteit Leuven, 200-B Celestijnenlaan, B-3030 LEUVEN (Belgique).

hull of

$$\bigcup_{n \in S} (n + \mathbb{R}_+^n) ,$$

where $\mathbb{R}_+ = \{x \in \mathbb{R} \; ; \; x \geq 0\}$. If σ is a face of $\Gamma(f)$, we put $f_\sigma = \sum_{n \in \sigma} a_n x^n$. One says that f is <u>non-degenerated with respect to its Newton polyhedron</u> if, for every compact face σ of $\Gamma(f)$, we have for all $x \in \mathbb{Q}_p^n$ that

$$\frac{\partial f_\sigma(x)}{\partial x_1} = \dots = \frac{\partial f_\sigma(x)}{\partial x_n} = 0 \Longrightarrow (x_i = 0 \text{ for some } i).$$

This notion is interesting, because in the set of all polynomials with a given Newton polyhedron Γ , the degenerate ones form a proper algebraic subset.

Let σ be a facet (i. e. an $n - 1$ dimensional face) of $\Gamma(f)$. Let the supporting hyperplane of σ have equation

$$a_1 x_1 + a_2 x_2 + \dots + a_n x_n = N ,$$

with a_i , $N \in \mathbb{N}$ and $\gcd(a_1 , \dots , a_n , N) = 1$. Then, we will use the following notation

$$N(\sigma) = N , \quad \nu(\sigma) = a_1 + a_2 + \dots + a_n .$$

If f is non-degenerated with respect to its Newton polyhedron, VARČENKO [4] (in the Archimedean case, but the p-adic case is entirely similar) has given a procedure to compute a set of candidates for the poles of $Z_f(s , \Phi)$ by using toroidal resolution of singularities. The candidates come from the numerical data of the resolution. It turns out that most of these candidates actually are not poles. This was proved by LICHTIN and MEUSER [2], in the case that $n = 2$, by showing that there is cancellation between the contributions of the exceptional divisors. It seems difficult to generalize their method of proof to the case $n > 2$. However, we could treat the case that n is arbitrary, by not constructing a full resolution of singularities, but using a weaker construction which introduces only a small number of numerical data. More precisely, we proved Theorem 1 and 2 below :

THEOREM 1. - <u>Suppose that</u> f <u>is non-degenerated with respect to its Newton polyhedron, and that the support of</u> Φ <u>is contained in a sufficiently small neighbourhood of</u> 0 . <u>If</u> s <u>is a pole of</u> $Z_f(s , \Phi)$, <u>then</u> $s = -1$ <u>or</u>

$$s = - \frac{\nu(\sigma)}{N(\sigma)} + \frac{2\pi i}{\log p} \frac{k}{N(\sigma)}$$

<u>for some facet</u> σ <u>of</u> $\Gamma(f)$ <u>and</u> $k \in \mathbb{Z}$.

The case $n = 2$ of this theorem is due to LICHTIN and MEUSER [2]. The pairs $(N(\sigma) , \nu(\sigma))$ are among the numerical data of a toroidal resolution of singularities of $f = 0$, but are only a <u>few</u> of them.

Analogy with the Archimedean case [5] (where the poles are related to the eigenvalues of monodromy) let us to suspect :

THEOREM 2. - Suppose that f is non-degenerated with respect to its Newton polyhedron, and that the support of Φ is contained in a sufficiently small neighbourhood of 0.

Let σ be a facet of $\Gamma(f)$ which is a pyramid with base contained in a coordinate hyperplane and top at distance 1 from that hyperplane. Suppose that there is no other facet with the same value for $\frac{\nu(\sigma)}{N(\sigma)}$, and that $\frac{\nu(\sigma)}{N(\sigma)} \neq 1$.

Then there is no pole s of $Z_f(s, \Phi)$ with $\mathrm{Re}(s) = -\frac{\nu(\sigma)}{N(\sigma)}$.

We also abtained explicit formulas for $Z_f(s, \Phi)$ in the case that the a_i are p-adic units, and that the reduction of f mod p is non-degenerated w. r. t. $\Gamma(f)$. These formulas, and the proofs of Theorems 1 and 2 will be contained in a paper which is currently in preparation.

REFERENCES

[1] IGUSA (J.-I.). - Lectures on forms of higher degree. - Berlin, Heidelberg, New York, Springer-Verlag, 1978 (Tata Institut of fundamental Research. Lectures on Mathematics and Physics, 59).

[2] LICHTIN (B.) and MEUSER (D.). - Poles of a local zeta function and Newton polygons, Comp. Math., Groningen, t. 55, 1985, p. 313-332.

[3] MALGRANGE (B.). - Intégrales asymptotiques et monodromie, Ann. scient. Ec. Norm. Sup., 4e série, t. 7, 1974, p. 405-430.

[4] VARČENKO (A. N.). - Newton polyhedra and estimates of oscillatory integrals [in Russian], Funkcional. Anal. i Priložen., t. 10, 1976, fasc. 3, p. 13-38 ; [in English], Functional Anal. and its Appl., t. 10, 1977, p. 175-196.

[5] VARČENKO (A. N.). - Zeta-function of monodromy and Newton's diagram, Invent. Math., Berlin, t. 37, 1976, p. 253-262.