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ON THE TATE CONSTANT

by Bernard DWORK (*)

Chapter I

1. Introduction. - Our work on the relation between the congruence zeta function and p-adic analysis began in February 1958 with the suggestion of J. TATE that his constant C (described below) may be constructed by p-adic analytic methods. (For an alternate description of C , see [Dw 5], Introduction (0.1)).

Let k be a field of characteristic zero complete with respect to a discrete valuation, with valuation ring $\mathcal{O}$ and residue class field $\bar{k} = \mathcal{O}/\mathfrak{p}$. Let A be an elliptic curve defined over k by an equation

(1) $$y^2 + (a_1 x + a_2) y = x^3 + a_3 x^2 + a_4 x + a_5$$

where the $a_i \in \mathcal{O}$.

Letting $x = ty$, we find

$$y^3 t^3 + y^2 (- 1 - a_1 t + a_3 t^2) + y (- a_2 + a_4 t) + a_5 = 0$$

and hence there exists a unique solution in $k((t))$ for y with a pole of order 3 at $t = 0$. This solution is of the form

(2) $$y = t^{-3} + B_{-2} t^{-2} + \dots$$

and the coefficients B_i lie in $\mathcal{O}$. Clearly

(3) $$x = t^{-2} + B_{-2} t^{-1} + \dots .$$

Let

(4) $$du = - dx/(2y + a_1 x + a_2)$$

a differential of the first kind on A . In terms of the uniformizing parameter $t = x/y$ at infinity, we have after integration

(5) $$u = t + \frac{1}{2} D_1 t^2 + \frac{1}{3} D_2 t^3 + \dots$$

where the D_i lie in $\mathcal{O}$.

2. THEOREM.

Part 1. - If the reduced curve $\bar{A}$ defined over $\bar{k}$ is non-singular and has Hasse invariant not zero, then there exists a unit C in the maximal unramified extension K of k such that $\exp Cu$ ($\in K[[t]]$) has, in fact, integral coefficients

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in K .

Part 2. - If $\bar{k}$ is finite, i. e. k is a p-adic field, then the unit root of the zeta function of the reduced curve $\bar{A}$ is $C^{\sigma-1}$, where σ is the Frobenius automorphism of K over k .

3. Explanation of the Tate's theorem. - Let

$$S = \{(x , y) \in A ; \ |x| > 1\} .$$

In terms of the uniformizing parameter, $t = x/y$,

$$x = t^{-2} + A_{-1} t^{-1} + A_0 + \dots$$

and so S is parametrized by $t \in D(0 , 1^-)$. The map of $t \mapsto u(t)$ gives a homomorphism of S into k_+ .

Since k is of characteristic zero,

$$u(t) = 0$$

for each $P \in S$ which is division point. Since u is a one to one map of $D(0 , |\pi|^-)$ onto itself, $u(t) = 0$ can only be valid for $t_0 = 0$ if $t_0 \in D(0 , |\pi|^-)$. On the other hand, it is shown by LUTZ [L] that, for $P \in S$,

$$\operatorname{ord} t(pP) \geqslant \operatorname{Min}(1 + \operatorname{ord} t(P) , \ 4 \operatorname{ord} t(P))$$

and hence if $u(t(P_0)) = 0$, then $t(p^v P_0) \in D(0 , |\pi|^-)$ for suitable v and so P_0 is a p^{th} power division point.

Since $1 + \mathfrak{p}$ ($\mathfrak{p} = D(0 , 1^-)$) does have points of finite order, TATE sought an isomorphism of S into $1 + \mathfrak{p}$ such as $t \mapsto \exp \theta.u(t)$. If one exists with integral coefficients then it is invertible and gives an isomorphism of S_K with $1 + \mathfrak{p}_K$ where K is a complete field containing $k(\theta)$. The exact sequence

$$0 \to S \xrightarrow{inj} A \xrightarrow{red} \bar{A} \to 0$$

together with the fact that for $(\ell , p) = 1$ both A and $\bar{A}$ have ℓ^2 points of order ℓ shows again that the only division points in S are of p power order. If there are p points of order p in $\bar{A}$ then there are only p in S (as there are in $1 + \mathfrak{p}$) and so the isomorphism of Tate could (and infact does) exist. If $\bar{A}$ has no points of order p then there are p^2 such points in S and then the suggested isomorphism is impossible. This explains the role of the Hasse invariant.

4. Proof of part 2 of Tate's theorem.

In 1958 (unpublished), we obtained a proof of part 2 of Tate's theorem for the Legendre model

$$y^2 = x(1 - x)(t - x) .$$

Using $t = 1/\sqrt{x}$ as parameter at ∞ , we may write

$$du = \sum_{n=0}^{\infty} t^{2n} D_{2n}(\lambda) \qquad (4.1)$$

where

(4.2) $$D_{2n} = (-1)^n \sum_{i=0}^{n} \binom{-1/2}{n-i} \binom{-1/2}{i} \lambda^i .$$

If λ lies in an unramified extension of $\mathbb{Q}_p$ then the existance of C (again in an unramified extension) is equivalent (by the Dieudonné criterion [Dw 1]) to congruences

(4.3) $$C^{\sigma-1} D_{mp^s-1}(\sigma\lambda) \equiv D_{mp^{s+1}-1}(\lambda) \mod p^{s+1}$$

for all $s \in \mathbb{N}$, $(m, p) = 1$, where σ is the absolute Frobenius. The consistency of these conditions is demonstrated by means of the formal congruences

(4.4) $$D_{mp^{s+1}-1}(\lambda) \equiv (-1)^{(p-1)/2} D_{mp^s-1}(\lambda)\, F(\lambda)/F(\lambda^p) \mod p^{s+1} \mathbb{Z}_p[[\lambda]]$$

where

$$F(\lambda) = F(\tfrac{1}{2}, \tfrac{1}{2} ; 1, \lambda) = \sum ((\tfrac{1}{2})_j/j\,!)^2 \lambda^j .$$

By means of these congruences, we showed that $F(\lambda)/F(\lambda^p)$ extends to an analytic element f on the Hasse domain

(4.5) $$H = \{\lambda ; |D_{p-1}(\lambda)| \geqslant 1\} .$$

Congruences similar to (4.4) are treated else where [Dw 2], [Dw 4].

Thus if $\sigma \lambda_0 = \lambda_0^p$, i. e. λ is a Teichmüller representative of its residue class then $C(\lambda_0) \in K$ is to be chosen so that

(4.6) $$C(\lambda_0)^{\sigma-1} = f(\lambda_0) .$$

More generally if $\lambda = \lambda_0 + \lambda_1$, $|\lambda_1| < 1$. Then we must put

(4.7) $$C(\lambda) = C(\lambda_0)/[1 + \sum_{s=1}^{\infty} \eta_s(\lambda_0) \lambda_1^s]$$

where

$$\eta_s = s\,!^{-1} F^{(s)}/F .$$

The point being that η_s is an analytic element on H whose restriction to $D(0, 1^-)$ is as indicated. This may also be expressed by the condition

(4.8) $$C(\lambda) = C(\lambda_0)/v(\lambda)$$

where v is the unique branch of $F(1/2, 1/2, 1, \lambda)$ at λ_0 i. e. the unique solution of the corresponding second order differential equation which is bounded on $D(\lambda_0, 1^-)$ and such that

$$v(\lambda_0) = 1 .$$

In appendix B, we indicate how these results should be generalised to curves with ordinary reduction.

5. Heuristics. - We assume the reduced curve is ordinary.

If ω_1 , ω_2 are "eigenvectors" of Forbenius, i. e.

$$\omega^{\sigma\Phi}_{1,\lambda^p} = p\,\omega_{1,\lambda} + d\,\xi_1$$

$$\omega^{\sigma\Phi}_{2,\lambda^p} = \omega_{2,\lambda} + d\,\xi_2$$

where ξ_1 , ξ_2 are daggerized algebraic functions on A , and we think of σ as operating on coefficients of the differential forms while Φ represents $x \to x^p$, then upon integration, setting $I_{2,\lambda} = \int \omega_{2,\lambda}$, a local abelian integral, we obtain

(5.1) $$I^{\sigma\varphi}_{1,\lambda^p} = p\,I_{1,\lambda} + \xi_1$$

(5.2) $$I^{\sigma\varphi}_{2,\lambda^p} = I_{2,\lambda} + \xi_2 \,.$$

We are tempted to deduce Tate's theorem by applying Dieudonne's criterion to (5.1). There are two questions :

(5.3) Is ξ_1 bounded by p on a generic disk ?

(5.4) $I_{1,\lambda}$ need not be an integral of the first kind ?

Our purpose is to show how these objections may be met by means of the theory of normalized solution matrices of the hypergeometric differential equation as explained in Chapter 9 of [Dw 5].

Chapter II.

We shall consider the hypergeometric differential equation

(1.0) $$\frac{d}{d\lambda}(u_1 , u_2) = (u_1 , u_2)\begin{pmatrix} -\frac{c}{\lambda} & \frac{c-a}{1-\lambda} \\ \frac{c-b}{\lambda} & \frac{a+b-c}{1-\lambda} \end{pmatrix}$$

in a split case of period one. By this, we mean that a , b , c are elements of $\mathbb{Q}$ whose denominators divide $p - 1$, and such that after replacement by minimum representative mod 1 , c does not lie on the real interval connecting a and b . To fix ideas, we assume that we have the Type 1 situation, i. e.

(1.1) $$1 > \mathrm{Max}(a , b) \geqslant \mathrm{Min}(a , b) > c > 0 \,.$$

The Type II case can be treated similarly. We shall restrict λ to the region

(1.2) $$|\lambda| = |\lambda - 1| = 1 \,.$$

We recall L_λ = analytic functions on the complement of sets of the type

$$D(0 , \epsilon_0) \cup D(1 , \epsilon_1) \cup D(\lambda^{-1} , \epsilon_{\lambda^{-1}}) \cup D(\infty , \epsilon_\infty)$$

(were ϵ_i is less than the distance from i to the remaining elements of $\{0, 1, \lambda^{-1}, \infty\}$, distance from ∞ to be computed in terms of $1/x$).

$f = x^{b-1}(1-x)^{c-b}(1-\lambda x)^{-a}$,

$G = xf(\lambda, x)/x^p f(\lambda^p, x^p)$,

$E = x(\partial/\partial x)$,

$D_{j,\lambda} = (xf)^{-1} \circ E \circ xf$,

$\alpha = \psi \circ G$,

$\beta = G^{-1} \circ \Phi$,

$\Phi x = x^p$,

$(\psi \xi)(x) = \sum p^{-1} \xi(y)$, the sum being over $\{y | y^p = x\}$,

$W_\lambda = L_\lambda / D_{f,\lambda} L_\lambda$,

Hasse domain = set of all residue classes where (1.0) has a bounded solution,

Gauss norm is relative to the x variable.

We start with [Dw 2] (6.3.2), which we write in the form

$$\alpha\begin{pmatrix} 1 \\ \dfrac{1}{1-x} \end{pmatrix} = A^t \begin{pmatrix} 1 \\ \dfrac{1}{1-x} \end{pmatrix} + D_{t,\lambda^p}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \tag{1.3}$$

where $y_1, y_2 \in L_{\lambda^p}$.

We show (chap. III) below that

$$\mathrm{Max}(|y_1|_{Gauss}, |y_2|_{Gauss}) \leqslant |p| . \tag{1.4}$$

We apply β to (1.3) and deduce

$$\begin{pmatrix} 1 \\ \dfrac{1}{1-x} \end{pmatrix} + (\beta\alpha - 1)\begin{pmatrix} 1 \\ \dfrac{1}{1-x} \end{pmatrix} = A^t \beta \begin{pmatrix} 1 \\ \dfrac{1}{1-x} \end{pmatrix} + D_{j,\lambda}\begin{pmatrix} \beta y_1 \\ \beta y_2 \end{pmatrix} \cdot \frac{1}{p} . \tag{1.5}$$

(1.6). PROPOSITION. – *If* $z \in L_\lambda$, $|z|_{Gauss} \leqslant 1$ *then*

$$(\beta\alpha - 1) z = D_{j,\lambda} w \tag{1.6.1}$$

where $w \in I_\lambda$, $|w|_{Gauss} \leqslant 1$.

Proof. – The mapping α induces a map of W_λ into W_{λ^p} with β as inverse. Hence equation (1.6) with $w \in L_\lambda$ is trivial. We need only check the Gauss norm. For this, we need only find a formula for w valid on an annulus

$$1 - \epsilon < |x| < 1 . \tag{1.6.2}$$

The formula for f shows that we may write

$$x f_\lambda = x^b g_\lambda(x) , \quad g \in \mathbb{Z}_p[\Lambda][[x]] .$$

Then

$$G = x^{-\mu_p} g_\lambda(x)/g_{\lambda^p}(x^p)$$

and so

$$\begin{aligned} \beta \circ \alpha &= x^{\mu_b} \frac{g^\Phi_{\lambda^p}}{g_\lambda} \circ \Phi \circ \psi \circ \frac{g_\lambda}{g^\Phi_{\lambda^p}} x^{-\mu_b} \\ &= x^{\mu_b} \frac{1}{g_\lambda} \circ \psi \circ g_\lambda x^{-\mu_b} \end{aligned}$$

i. e.

$$D_f w = x^{\mu_b} \frac{1}{g_\lambda} (\Phi \psi - 1) g_\lambda x^{-\mu_b} z$$

and so

$$E(x^b g_\lambda w) = x^{p^b} (\Phi \psi - 1) g_\lambda x^{-\mu_b} z . \tag{1.6.3}$$

Now

$$g_\lambda x^{-\mu_b} z = \sum_{n=-\infty}^{\infty} \gamma_m x^n$$

a representation valid on (1.6.2), and the boundary norm (as $\epsilon \to 0$) being bounded by unity, we have

$$|\gamma_m| \leqslant 1 \quad \forall m .$$

Since $(\Phi \psi - 1)$ annihilates x^m if $p|m$, we find

$$z = - x^{-b} \frac{1}{g_\lambda} \sum_{\substack{m=-\infty \\ p\nmid m}}^{+\infty} \gamma_m x^{m+pb}/(m + pb) . \tag{1.6.4}$$

(There is no constant of integration as z is single valued in the annulus (1.6. 2).) The assertion now follows from the boundary norm of g_λ . This completes the proof.

It follows from (1.4), (1.6) that there exist $z_1 , z_2 \in L_\lambda$

$$\operatorname{Max} (|z_1|_{Gauss} , |z_2|_{Gauss}) \leqslant 1 \tag{1.7.1}$$

such that

$$\begin{pmatrix} 1 \\ \frac{1}{1 - \gamma} \end{pmatrix} = A^t \beta \begin{pmatrix} 1 \\ \frac{1}{1 - x} \end{pmatrix} + D_{f,\lambda} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} . \tag{1.7.2}$$

Our object is to find "eigen vectors" of β . For this purpose, we ask for an invertible matrix $Y(\lambda)$, defined on a disk $D(\lambda_0 , 1^-)$ for which the differential equation is not super singular, such that

$$Y(\lambda) A^t = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} Y^\sigma(\lambda^p) \tag{1.8}$$

The σ (= absolute Frobenius) referring to the fact that Y may be defined over a maximal unramified extension of $\mathbb{Q}_p$. This condition is equivalent of

(1.9) $$(Y^{\sigma}(\lambda^p))^{*} A = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} Y^{*}$$

$(Y^{*} = (Y^{\text{transpose}})^{-1})$ and so we may take Y^{*} to be the normalized solution matrix of (1.0) on $D(\lambda_0 , 1^-)$.

Thus we may satisfy (1.8) by setting

(1.10) $$Y^{*} = \begin{pmatrix} 1 & 0 \\ \tau & 1 \end{pmatrix} \begin{pmatrix} u & 0 \\ 0 & \hat{u} \end{pmatrix} \begin{pmatrix} 1 & \eta \\ 0 & 1 \end{pmatrix}$$

where u , $\hat{u}$, η , τ satisfy the following conditions [Dw 1] (9.6)

(1.11.1) η is analytic element bounded by 1 on the Hasse domain of (1.0).

(1.11.2) $u(1 , \eta)$ is the unique bounded solution of (1.0) on $D(\lambda_0 , 1^-)$.

(1.11.3) $u(\lambda)/u^{\sigma}(\lambda^p)$ extends to an analytic element on the Hasse domain.

(1.11.4) $\tau^{\sigma}(\lambda^p) \equiv p\ \tau(\lambda) \mod p \quad (\forall\ \lambda \in D(\lambda_0 , 1^-))$.

(1.11.5) $\hat{u}$ = wronskian/u .

(1.11.6) u , $\hat{u}$ take on unit values throughout $D(\lambda_0 , 1^-)$.

Putting

(1.12) $$Y(\lambda) \begin{pmatrix} 1 \\ \frac{1}{1-x} \end{pmatrix} \overset{\text{def}}{=} \begin{pmatrix} \xi_{1,\lambda} \\ \xi_{2,\lambda} \end{pmatrix} = \vec{\xi}_{\lambda}(x)$$

and multiplying (1.7) by Y and using (1.8),

(1.13) $$\vec{\xi}_{\lambda} = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \beta\ \vec{\xi}^{\sigma}_{\lambda^p} (x^p) + D_{f,\lambda}\ Y(\lambda) \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} .$$

Putting

(1.13.1) $$\begin{pmatrix} \hat{z}_1 \\ z_2 \end{pmatrix} = \hat{z} = Y(\lambda) \begin{pmatrix} \hat{z}_1 \\ z_2 \end{pmatrix}$$

we may rewrite (1.13) in the form

(1.14) $$x\ f_{\lambda}\ \vec{\xi}_{\lambda} = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} (x\ f_{\lambda^p}\ \xi^{\sigma}_{\lambda^p})^{\phi} + E\ x\ f_{\lambda}\ \hat{z} .$$

Applying E^{-1} to both sides and writting as function on a general residue class $D(x_0 , 1^-)$ in x time (with $x_0^{\sigma} = x_0^p$)

$$I_{j,\lambda}(x) = \int_{x_0}^{x} f\ \xi_{j,\lambda}\ dx ,$$

we obtain

(1.15) $$\begin{aligned} I_{1,\lambda}(x) &= \frac{1}{p}\, I^{\sigma}_{1,\lambda^p} (x^p) + x\ f_{\lambda}\ \hat{z}_1 \\ I_{2,\lambda}(x) &= I^{\sigma}_{2,\lambda^p} (x^p) + x\ f_{\lambda}\ \hat{z}_2 \end{aligned}$$

(where I^σ is function on $D(x_0^\sigma , 1^-)$).

Since

(1.16) $$Y = \begin{pmatrix} 1 & -\tau \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1/u & 0 \\ 0 & 1/\hat{u} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\eta & 1 \end{pmatrix} .$$

It follows from (1.13.1), that

(1.17) $$\hat{z}_2 = - \frac{1}{\hat{u}(\lambda)} (z_1 \eta(\lambda) - z_2)$$

and hence, by (1.7.1), (1.11.1), (1.11.6),

(1.17.1) $$|\hat{z}_2|_{Gauss} \leq 1 .$$

Thus, by (1.15),

(1.18) $$|I_{2,\lambda}(x)| \leq 1 \quad \text{for all} \quad x \in D(x_0 , 1^-) .$$

We now put

(1.19) $$J_\lambda(x) = \frac{1}{u(\lambda)} \int f_\lambda(x)\, dx .$$

<u>For</u> Type I, <u>this is the unique</u> (up to factor independent of x) <u>integral of first kind</u> associated with the differentials in the integral representation of $F(a , b , c , \lambda)$ and its derivatives. [Dw 1] (chapter 14). By (1.12) ,

(1.20) $$\xi_{1,\lambda} = \frac{1}{u(\lambda)} \cdot 1 - \tau(\lambda)\, \xi_{2,\lambda}$$ $$\xi_{2,\lambda} = \frac{1}{\hat{u}(\lambda)} \left(- \eta(\lambda)\, 1 + \frac{1}{1 - x}\right)$$

and so

(1.21) $$I_{1,\lambda}(x) = J_\lambda(x) - \tau(\lambda)\, I_{2,\lambda}(x) .$$

Multiplying the second equation of (1.15) by $\tau(\lambda)$, and adding to the first

(1.22) $$J_\lambda(x) = \frac{1}{p} J_\lambda^\sigma(x^p) + H(\lambda , x)$$

where

(1.22.1) $$H(\lambda , x) = \left(\tau(\lambda) - \tau^\sigma(\lambda^p) \frac{1}{p}\right) I^\sigma_{2,\lambda^p} (x^p) + x\, f_\lambda \frac{z_1}{u(\lambda)} .$$

We observe that, by (1.18), (1.11.4), (1.7.1),

(1.23) $$|H(\lambda , x)| \leq 1$$

on $D(\lambda_0 , 1^-) \times D(x_0 , 1^-)$. By the Dieudonné's conditions we now have the first part of Tate's theorem

(1.24) $$\exp J_\lambda(x) \in \mathcal{O}_K[[x - x_0]]$$

where K is a sufficiently large field containing λ . The second part of Tate's theorem is also demonstrated since $(u , u\eta)$ is an "eigenvector" of a semilinear transformation with matrix A corresponding to eigenvalue 1 . Using Adolphson's explanation in the appendix of [Dw 1], we may deduce the connection between $(u(\lambda_0))^{1-\sigma}$ and the unit reciprocal root of the corresponding L-function. In the next section, we complete the treatment by verifying (1.4).

Chapter III

1. Our object is to verify the estimates of II (1.4).

LEMME. - *Let* $L = \mathbb{Q}(\lambda)[x, x^{-1}, (1-x)^{-1}, (1-\lambda x)^{-1}]$. *For* $s \geq 2$ *there exist* $\alpha_s, \beta_s, \gamma_s, \delta_s \in \mathbb{Q}(\lambda)$ *and* $\xi_s, \eta_s \in L$ *such that*

(1.1) $$(1-x)^{-s} = \alpha_s 1 + \beta_s(1-x)^{-1} + D_{f,\lambda}\, \xi_s$$

(1.2) $$(1-\lambda x)^{-s} = \gamma_s 1 + \delta_s(1-x)^{-1} + D_{f,\lambda}\, \eta_s$$

and subject to II (1.2),

(1.3) $$\mathrm{Max}(|\alpha_s|, |\beta_s|, |\xi_s|_{Gauss}) \leq \sup_{0\leq m\leq s-1} \frac{1}{|b-c+m|}$$

(1.4) $$\mathrm{Max}(|\gamma_s|, |\delta_s|, |\eta_s|_{Gauss}) \leq \sup_{0\leq m\leq s-1} \frac{1}{|a+m|} .$$

Proof. - Equation (1.1) follows from [Dw 1] (1.2). In terms of [Dw 1] (2.3.5.10)

(1.5) $$\alpha_s = \langle T_1^{-s}, 1^* \rangle$$

(1.6) $$\beta_s = \langle T_1^{-s}, (T_1^{-1})^* \rangle$$

and so α_s (resp. β_s) is given by the coefficient of T_1^{s-1} in the formula for ξ_1 in [Dw 1] (p. 25) with $A = 0$, $B = \lambda(a-c)$ (resp. $A = b-c$, $B = A(c-b)$) .

The estimates for $|\alpha_s|$, $|\beta_s|$ follow from this formula. A second proof of these estimates will appear below.

The proof of (1.1) [Dw 1] (p. 10) shows that $\xi_s \in T_1^{-1}\, \mathbb{Q}(\lambda)[T_1^{-1}]$ and is of degree bounded by $s-1$ as polynomial in T_1^{-1} .

On the other hand, we may solve (1.1) for ξ_s by writting the solution in the form

(1.7) $$-T_1 \frac{d}{dT_1}(x f_\lambda \xi_s) = T_1 f[T_1^{-s} - \alpha_s - \beta_s T_1^{-1}]$$

and so

(1.8) $$-\xi_s = \theta[T_1^{-s} - \alpha_s - \beta_s T_1^{-1}]$$

where θ is defined by

(1.9) $$\theta = (1-T_1)^{-b}(1-t_1 T_1)^{a} T_1^{b-c} \left(T_1 \frac{d}{dT_1}\right)^{-1} T_1^{1+c-b} (1-T_1)^{b-1}(1-t_1 T_1)^{-a}$$

as endomorphism of $\mathbb{Q}(\lambda)((T_1))$. (Since $b-c \notin \mathbb{Z}$, there is no ambiguity due to possible constant of integration. Since both $\theta 1$ dans $\theta T_1^{\pm 1}$ lie in $\mathbb{Q}(\lambda)[[T_1]]$, we conclude that

(1.10) $$- \xi_s = P_1 \theta T_1^{-s}$$

where P_1 denotes the principal part at $x = 1$. Writting

(1.11.1) $$(1 - T_1)^{b-1} (1 - t_1, T_1)^{-a} = \sum_{m=0}^{\infty} g_m T_1^m \in \mathbb{Z}_p[[T_1]]$$

(1.11.2) $$(1 - T_1)^{-b} (1 - t_1, T_1)^{a} = \sum_{m=0}^{\infty} h_m T_1^m \in \mathbb{Z}_p[[T_1]]$$

we compute

(1.12) $$- \xi_s = P_1 \sum_{m,m'=0}^{\infty} h_{m'} g_m T_1^{m'+b-c} (T_1 \frac{d}{dT_1})^{-1} T_1^{c-b+m+1-s}$$

$$= \sum_{j=1}^{s-1} T_1^{-j} \sum h_{m'} g_m \frac{1}{c - b + m + 1 - s},$$

the inner sum being over all pairs m, $m' \geqslant 0$ such that

(1.12.1) $$m + m' = s - 1 - j .$$

Estimate (1.3) for ξ_s follows from (1.12) and the fact that g_m, $h_{m'}$ lie in $\mathbb{Z}_p$. A second proof of the estimates (1.3) for α_s, β_s now follows from (1.1). The proof of (1.4) follows by the same methods.

2. The proof of II (1.4) follows the proceedure of [Dw 1] (chapter 6). We write

(2.1) $$\psi T^{-\mu_b} T_1^{\mu_b - \mu_c} T_\lambda^{\mu_a} (\frac{1}{T_1^{-1}} - 1) = \begin{pmatrix} \chi_1 & \chi_3 \\ \chi_2 & \chi_4 \end{pmatrix} \begin{pmatrix} 1 \\ T_1^{-1} \end{pmatrix} + D_{f,\lambda^p} \begin{pmatrix} Y_{1,1} \\ Y_{2,1} \end{pmatrix}$$

(2.2) $$\psi M \begin{pmatrix} 1 \\ T_1^{-1} \end{pmatrix} = \begin{pmatrix} \rho_1 & \rho_3 \\ \rho_2 & \rho_4 \end{pmatrix} \begin{pmatrix} 1 \\ T^{-1} \end{pmatrix} + D_{f,\lambda^p} \begin{pmatrix} Y_{1,2} \\ Y_{2,2} \end{pmatrix}$$

where

$$M = G^{(1)} G^{(2)} - 1 .$$

The matrix χ was computed [Dw 1] (6.4) and subject to certain conditions it is shown [Dw 1] (6.1) that the matrix ρ is bounded by $|p|$. For our present purpose, we note that

(2.3) $$Y_i = Y_{i1} + Y_{i2} \quad (i = 1 , 2) .$$

For case 1, it is shown [Dw 1] (p. 100) that

(2.4) $$Y_{i,1} = 0 .$$

We apply [Dw 1] (6.1) to equation (2.2). We may here take

(2.5.1) $$\ell_0 = (p - 1) b , \quad \ell_1 = 0 = \ell_\infty$$

and

(2.5.2) $$j_0 = j_\infty = 1 ; \; j_1 = 4 .$$

By [Dw 1] (6.1.15), we have

(2.6) $$\mu_{0,s} = 0 \quad \text{for} \quad s \geqslant 1$$

(2.7) $$\mu_{\infty,s} = 0 \quad \text{for} \quad s \geqslant 1 .$$

Thus in [Dw 1] (6.1.12), we have in the representation of (say) ψM ,

(2.8.1) $$K_0 = 0 ,$$

(2.8.2) $$K_\infty = \mu_{\infty,0} \quad \text{(a constant)} .$$

On the other hand, by definition

(2.8.3) $$K_1 = \sum_{s=1}^{\infty} \mu_{1,s} T_1^{-s}$$

(2.8.4) $$K_{1/\lambda^p} = \sum_{s=1}^{\infty} \mu_{1/\lambda^p,s} T_{1/\lambda^p}^{s}$$

and by [Dw 1] (lemma 5.2, and 6.1.16),

(2.9) $$\sup_{s\geqslant 1} \{|\mu_{1,s}| , |\mu_{1/\lambda^p,s}| , |\mu_{\infty,0}|\} \leqslant 1/p$$

(2.10) $$\sup_{s\geqslant 1} \{|\mu_{1,s}| , |\mu_{1/\lambda^p,s}|\} \leqslant p\, |\pi|^{ps} .$$

Thus, by lemme 1, we have

(2.11) $$Y_{1,2} = \sum_{s=1}^{\infty} \mu_{1,s}\, \xi_s + \sum_{s=1}^{\infty} \mu_{1/\lambda^p,s}\, \eta_s .$$

Precisely as in [Dw 1] (6.1), we may deduce $|Y_{1,2}| \leqslant |p|$ by means of equations (2.9) - (2.11) subject to conditions [Dw 1] (6.1.8) which reduce here to the conditions

(2.12) $$|a| = |b - c| = |a - c| = 1 .$$

These conditions are a consequence of II (1.1) and the condition that (a , b , c) be of period one. This completes the proof.

Appendix A.

The theorem of Tate for elliptic curves in Legendre normal form requires (a , b , c) = (1/2 , 1/2 , 1) . This is of type II. In the notation of [Dw 1] (9.5.1), the normalized bounded solution of II (1.0) is $(\bar{u}\,\bar{\eta} , \bar{u})$.

Here, equation II (1.12) must be written

$$\begin{pmatrix} \bar{u}\,\bar{\eta} , & \bar{u} \\ \hat{\bar{u}} + \bar{\tau}\,\bar{u}\,\bar{\eta} , & \bar{\tau}\,\bar{u} \end{pmatrix} \begin{pmatrix} 1 \\ \frac{1}{1-x} \end{pmatrix} = \begin{pmatrix} \xi_{1,\lambda} \\ \xi_{2,\lambda} \end{pmatrix}$$

and so

$$I_{1,\lambda} = \int \omega_{1,\lambda}\, dx$$

$$I_{2,\lambda} = \int \omega_{2,\lambda}\, dx$$

$$\omega_{1,\lambda} = \frac{1}{\bar{u}(\lambda)} \frac{1}{1-x} f\, dx - \bar{\tau}(\lambda)\, \omega_{2,\lambda}$$

$$\omega_{2,\lambda} = \frac{1}{\bar{u}(\lambda)} \left(fdx - \bar{\eta}(\lambda) \frac{1}{1-x} fdx\right) .$$

Here $\frac{1}{1-x}$ fdx is the unique differential of the first kind and putting

$$J_\lambda(x) = \frac{1}{\bar{\mu}(\lambda)} \int \frac{1}{1-x} fdx$$

we obtain the analogue of II (1.22), (1.23). Here $\bar{u}$ is a branch of $F(1/2, 1/2, 1, \lambda)$ and $1/\bar{u}(\lambda)$ is the constant of Tate.

Appendix B.

Families of curves with ordinary reduction.

For the split case of period greater than one, we must leave the situation involving a two dimensional piece of cohomology. For this reason, we briefly sketch how the theory extends to curves. We consider a family $f(\lambda, x, Y) = 0$ of (possibly singular) plane curves with generic ordinary reduction.

There exists a basis $\{\omega_{1,\lambda}, \dots, \omega_{g,\lambda}\}$ for the differentials of the first kind together with set of representatives $\omega_{g+1,\lambda}, \dots, \omega_{2g,\lambda}$ of a basis of differentials of the second kind modulo exact + d.f.k. such that

(1)
$$\begin{pmatrix} \omega_{1,\lambda} \\ \vdots \\ \omega_{2g,\lambda} \end{pmatrix} = \frac{1}{p} A^t \begin{pmatrix} \omega_{1,\lambda^p} \\ \vdots \\ \omega_{2g,\lambda^p} \end{pmatrix}^{\Phi} + d \begin{pmatrix} z_1 \\ \vdots \\ z_{2g} \end{pmatrix}$$

where the z_i are daggerized algebraic functions with Gauss norm bounded by unity, Furthermore

$$\omega_1 \wedge \omega_2 = 0_1 \pm 1$$

so that the pairing matrix is

$$\begin{pmatrix} 0 & I_g \\ I_g & 0 \end{pmatrix}.$$

The matrix A is an over convergent $2g \times 2g$ matrix function of λ and

(2)
$$A \equiv \begin{pmatrix} \bar{A}_1 , & \bar{A}_2 \\ 0 , & 0 \end{pmatrix} \bmod p$$

where $\bar{A}_1$ is the Hasse-Witt matrix. (Furthermore

$$\begin{pmatrix} I_g & \\ & p^{-1} I_g \end{pmatrix} A$$

is invertible mod p).

In terms of the theory of normalised period matrices [Dw 3], we have

(3) $$Y(\lambda) = \begin{pmatrix} I_g & 0 \\ T & I_g \end{pmatrix} \begin{pmatrix} U & 0 \\ 0 & U^* \end{pmatrix} \begin{pmatrix} I_g & \eta \\ 0 & I_g \end{pmatrix}$$

and

(4) $$Y^\sigma(\lambda^p)\ A = \begin{pmatrix} I_g & \\ & p\ I_g \end{pmatrix} Y(\lambda)\ ,$$

multiplying (1) on the left by $Y(\lambda)^*$, we obtain

(5) $$Y(\lambda)^* \begin{pmatrix} \omega_{1,\lambda} \\ \vdots \\ \omega_{2g,\lambda} \end{pmatrix} = \frac{1}{p} \begin{pmatrix} I_g & \\ & p\ I_g \end{pmatrix} Y^\sigma(\lambda^p)^* \begin{pmatrix} \omega^\sigma_{1,\lambda^p} \\ \vdots \\ \omega^\sigma_{2g,\lambda^p} \end{pmatrix}^\Phi + d\ Y(\lambda)^* \begin{pmatrix} z_1 \\ \vdots \\ z_{2g} \end{pmatrix}$$

Putting

$$\bar\omega_{1,\lambda} = \begin{pmatrix} \omega_{1,\lambda} \\ \omega_{g,\lambda} \end{pmatrix}, \quad \bar\omega_{2,\lambda} = \begin{pmatrix} \omega_{g+1,\lambda} \\ \omega_{2g,\lambda} \end{pmatrix},$$

we set

(5.1) $$\hat\omega_{2,\lambda} = -U\eta\ \bar\omega_{1,\lambda} + U\ \bar\omega_{2,\lambda}$$

(5.2) $$\hat\omega_{1,\lambda} = U^*\ \omega_{1,\lambda} - T(\lambda)\ \bar\omega_{2,\lambda}$$

(5.3) $$\tilde z_1 = U^*\ \bar z_1 - T(\lambda)\ \tilde z_2$$

(5.4) $$\tilde z_2 = -U\eta\ \bar z_1 + U\ \bar z_2$$

it being understood that

$$\bar z_1 = \begin{pmatrix} z_1 \\ \vdots \\ z_g \end{pmatrix}, \quad \bar z_2 = \begin{pmatrix} z_{g+1} \\ \vdots \\ z_{2g} \end{pmatrix}.$$

We conclude that

(6.1) $$\hat\omega_{1,\lambda} = p^{-1}(\hat\omega_{1,\lambda^p})^\Phi + d\ \tilde z_1$$

(6.2) $$\hat\omega_{2,\lambda} = (\hat\omega^\sigma_{2,\lambda^p})^\Phi + d\ \tilde z_2\ .$$

It follows from (5.4) that $\tilde z_2$ is bounded by 1 .

Setting

$$I_{i,\lambda} = \int_{x_0}^{x} \hat\omega_{i,\lambda} \qquad i = 1\ ,\ 2$$

we obtain

(7.1) $$I_{1,\lambda} = p^{-1}\ I^{\sigma\Phi}_{1,\lambda^p} + \tilde z_1$$

(7.2) $$I_{2,\lambda} = I^{\sigma\Phi}_{2,\lambda^p} + \tilde{z}_2 .$$

We deduce that $I_{2,\lambda}(x)$ is bounded by 1 on $D(x_0, 1^-)$.

We put

$$J_\lambda(x) = U(\lambda)^* \int_{x_0}^{x} \bar{\omega}_{1,\lambda} ,$$

a g-tuple of abelian integrals of the first kind we deduce from (6) that

(8) $$J_\lambda(x) = p^{-1} J^{\sigma}_{\lambda^p}(x^p) + H(\lambda, x)$$

where

$$H(\lambda, x) = I^{\sigma\Phi}_{2,\lambda^p}(T(\lambda) - p^{-1} T^{\sigma}(\lambda^p)) + U(\lambda)^* \bar{z}_1 .$$

By the theory of normalized period matrices we deduce that $H(\lambda, x)$ is bounded by unity.

The completes our sketch of the generalization of Tate's theorem.

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