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DIFFERENTIALS OF THE SECOND KIND FOR FAMILIES OF MUMFORD CURVES

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The space of everywhere meromorphic differentials on a Mumford curve M of genus g which can be integrated on the universal covering of M is a space of codimension g in the full space of meromorphic differentials on M. This fact allows to conclude that the Gauss-Manin connection associated to an analytic family of Schottky groups has g linearly independent horizontal elements which are defined everywhere on the parameter space of the family. I will give a sketch of the proof for this result.

1. ζ-functions and differentials of the second kind.

Let K be an algebraically closed field together with a complete non-archimedean valuation. Let Γ be a Schottky subgroup of the group $PGL_2(K)$ of fractional linear transformations of the Riemann surface $\underset{\sim}{P} = K \cup \{\infty\}$ over K. Let Z be the domain of ordinary points of Γ, see [GP], Chap. I, § 4.

THEOREM 1. - <u>Let</u> $h(z)$ <u>be a rational function on</u> $\underset{\sim}{P}$, <u>whose poles all lie in</u> Z <u>and let</u> $z_0 \in Z$ <u>be an ordinary point for</u> Γ. <u>Then the series</u>

$$\zeta(h\ ;\ z_0\ ;\ z) := \sum_{\substack{\gamma\in\Gamma \\ h(\gamma(z_0))\neq\infty}} (h(\gamma(z)) - h(\gamma(z_0))) + \sum_{\substack{\gamma\in\Gamma \\ h(\gamma(z_0))=\infty}} h(\gamma(z))$$

<u>is as a function of</u> z <u>uniformly convergent on any affinoid subdomain of</u> Z. <u>Its limit is a meromorphic function on</u> Z.

A proof of this result appears in [G], (1).

Let now I be the K-vectorspace of those meromorphic functions $f(z)$ on Z for which

$$f(\gamma z) - f(z) \in K$$

for all $\gamma \in \Gamma$.

The differential df of a function from I is Γ-invariant and is thus a differential of the Mumford curve $M = Z/\Gamma$.

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Denote by H the K-vectorspace of rational functions on $\underset{\sim}{P}$ whose poles all lie in Z . One can show that any $f \in I$ is obtained as $\xi(h , z_0 ; z)$ with $h \in H$, see [G], (2).

Let Hom(Γ , K) be the K-vectorspace of group homomorphisms $c : \Gamma \to K$. If we fix a basis $\alpha_1 , \dots , \alpha_g$ of the free group Γ, we obtain a canonical isomorphism $\mathrm{Hom}(\Gamma , K) \tilde{\to} K^g$ when we map c onto the g-tuple $(c(\alpha_1) , \dots , c(\alpha_g))$.

For any $f \in I$ we denote by P(f) the group homomorphism $\Gamma \to K$ given by

$$P(f)(\gamma) = f(\gamma z) - f(z) .$$

Then $P(f)(\gamma)$ is the period of the differential df with respect to the "cycle" γ .

The mapping

$$P : I \to \mathrm{Hom}(\Gamma , K)$$

is K-linear whose kernel consists of the field of Γ-invariant meromorphic functions on Z which is the field of rational functions on the curve M . One can prove that the mapping $P : I \to \mathrm{Hom}(\Gamma , K)$ is surjective, see [G], (3).

THEOREM 2. - <u>Let</u> $\alpha_1 , \dots , \alpha_g$ <u>be a basis of</u> Γ . <u>Then there exist functions</u> $f_1 , \dots , f_g \in I$ <u>such that</u> $P(f_i)(\alpha_j) = \delta_{ij} = \begin{cases} 1 : i = j \\ 0 : i \neq j \end{cases}$.

A meromorphic differential $\omega = f dz$ on Z is called to be of the second kind if for any point $a \in Z$ there is a meromorphic function $h_a(z)$ on Z such that $\omega - dh_a$ is analytic in a .

Denote by Ω_2 the K-vectorspace of Γ-invariant differentials on Z of the second kind. The proof of the following theorem is given in [G], (4).

THEOREM 3. - $\Omega_2 = \Omega_1 \oplus dI$ <u>where</u> Ω_1 <u>is the</u> g-<u>dimensional</u> K-<u>vectorspace of analytic differentials on</u> M .

2. Families of Schottky groups.

Let S be a rigid analytic space over K , see [BGR], Chap. 9. We consider the projective line over S , namely the product space $\underset{\sim}{P} \times S$ together with the projection π onto the second factor.

Denote by $\mathrm{Aut}_S(P \times S)$ the group of those bianalytic mapping $\gamma : \underset{\sim}{P} \times S \to \underset{\sim}{P} \times S$ which are compatible with π (i. e. $\gamma \circ \pi = \pi$).

One can prove that there is an admissible covering $\mathfrak{S} = (S_i)_{i \in I}$ of S such $\gamma | S_i$ is a fractional-linear transformation over S_i which means that there is a matrix

$$\begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} \in GL_2(\mathcal{O}(S_i)) ,$$

where $\mathcal{O}(S_i)$ is the K-algebra of analytic functions on S_i such that

$$(\gamma|S_i)(s\ ,\ z) = \frac{a_i(s) \times z + b_i(s)}{c_i(s) \times z + d_i(s)}\ .$$

For any point $s \in S$ we obtain a canonical homomorphism $\mathrm{Aut}_S(\underset{\sim}{P} \times S) \to PGL_2(K)$ by restricting $\gamma \in \mathrm{Aut}_S(\underset{\sim}{P} \times S)$ to the subspace $\underset{\sim}{P} \times \{s\}$ of $\underset{\sim}{P} \times S$. We denote the restriction of γ to $\underset{\sim}{P}\{s\}$ by σ_s.

<u>Definition</u>. – A subgroup $\Gamma \subset \mathrm{Aut}_S(\underset{\sim}{P} \times S)$ is called a Schottky group over S (or a family of Schottky groups parametrized by S) if for any point $s \in S$ the restriction of the canonical homomorphism $\mathrm{Aut}_S(\underset{\sim}{P} \times S) \to PGL_2(K)$ to Γ gives an isomorphism from Γ to a Schottky group Γ_s of $PGL_2(K)$.

Let now Γ be a Schottky group over S. The proof of the following result will be given elsewhere.

THEOREM 4. – <u>There exists an admissible subdomain</u> Z <u>of</u> $\underset{\sim}{P} \times S$ <u>such that for any</u> $s \in S$ <u>the intersection</u> $Z \cap (\underset{\sim}{P} \times \{s\})$ <u>is the domain of ordinary points for the Schottky groups</u> Γ_s. <u>If</u> S <u>is an affinoid space there is an affinoid subdomain</u> $F \subset Z$ <u>such that</u>

$$\cup_{\gamma\in\Gamma}\ \gamma(F) = Z$$

$$\gamma(F)\ \cap\ F = \text{empty for almost all}\ \ \gamma \in \Gamma\ .$$

<u>If</u> S <u>is irreducible, then so is the domain</u> Z.

COROLLARY. – $Z/\Gamma \to S$ <u>is an analytic family of Mumford curves.</u>

From now on let S be irreducible and H be the $\mathcal{O}(S)$-algebra of meromorphic functions on $\underset{\sim}{P} \times S$ whose poles and points of indeterminancy all lie in Z.

Let $z_0 : S \to Z$ be an analytic mapping such that $\pi \circ z_0 = id_S$ and $h \in H$. Let h_s be the restriction of h onto $\underset{\sim}{P} \times \{s\}$. Then there is a meromorphic function $\xi(h\ ;\ z_0\ ;\ s\ ,\ z)$ on Z such that the restriction of $\xi(h\ ;\ z_0\ ;\ s\ ,\ z)$ onto $\underset{\sim}{P} \times \{s\}$ equals $\xi(h_s\ ;\ z_0(s)\ ;\ z)$. Let I_S be $\mathcal{O}(S)$-module of meromorphic functions $f(s\ ,\ z)$ on Z for which $f \circ \gamma - f \in \mathcal{O}(S)$ for all $\gamma \in \Gamma$. Let $\mathrm{Hom}(\Gamma,\ (S))$ be the free $\mathcal{O}(S)$-module of rank g of all group homomorphisms $c : \Gamma \to \mathcal{O}(S)$.

Let $P(f)(\gamma) := f \circ \gamma = f$. Then $P(f) \in \mathrm{Hom}(\Gamma\ ,\ \mathcal{O}(S))$.

THEOREM 5. – <u>Let</u> $\alpha_1\ ,\ \dots\ ,\ \alpha_g$ <u>be a basis of</u> Γ. <u>There is an admissible covering</u> $(S_i)_{i\in I}$ <u>of</u> S <u>and for any</u> i <u>there are functions</u> $f_1\ ,\ \dots\ ,\ f_g \in I_{S_i}$ <u>such that</u>

$$P(f_j)(\alpha_l) = \delta_{jl}\ .$$

Let $\Omega_2 = \Omega_{2M/S}$ denote the sheaf on S whose set of sections on an admissible open domain $U \subseteq S$ are the K-vectorspace of Γ-invariant differentials relative to $Z \to S$, of the second kind on $Z_U = Z \cap (\underset{\sim}{P} \times U)$.

Let Ω_{ex} be the subsheaf of Ω_2 of exact differentials and H^1_{DR} be the quotient sheaf Ω_2/Ω_{ex} .

THEOREM 6. - H^1_{DR} <u>is a free coherent module over the structure sheaf</u> $\mathcal{O}_S$ <u>on</u> S <u>of rank</u> $2g$. <u>There is a canonical decomposition</u>

$$H^1_{DR} = \overline{dI} \oplus \Omega_1$$

<u>where</u> Ω_1 <u>is the subsheaf of</u> Ω_2 <u>of analytic differentials and</u> $\overline{dI}$ <u>is the sheaf of cohomology classes of differentials of the form</u> df <u>with</u> $f \in I$. $\overline{dI}$ <u>and</u> Ω_1 <u>are free modules of rank</u> g <u>over</u> $\mathcal{O}_S$.

Sketch of proof : In order ot prove that Ω_1 is free of rank g , we have to observe that for any $\gamma \in \Gamma$ there is a canonical differential $\omega_\alpha = (du_\alpha/u_\alpha) \in \Omega$, where u_α is defined on Z as in [GP], Chap. 2. While the u_α are unique up to a unit from $\mathcal{O}_S$, the differential ω_α is unique. If $\alpha_1 , \dots , \alpha_g$ is a basis of Γ , then $\omega_{\alpha_1} , \dots , \omega_{\alpha_g}$ is a basis for Ω_1 .

The result concerning $\overline{dI}$ follows from Theorem 4. While the function $f_j^{(i)}$ depends on the index i , we find that $df_j^{(i)} - df_j^{(1)}$ are in the intersection $S_i \cap S_1$ the differential of a Γ-invariant function and thus the cohomology class of $df_j^{(i)}$ equals the cohomology class of $df_j^{(i)}$. Thus they constitute a basis element of $\overline{dI}$ [G], (Satz 6), we conclude the proof.

3. Gauss-Manin-Connection.

Let ∇ be the Gauss-Manin connection for the analytic family $M = Z/\Gamma \to S$ of Mumford curves, see [KO], [K], [D]. Thus for any vector field D on S there is an extension ∇_D on the module sheaf $H^1_{DR}(M/S)$.

THEOREM 7. - <u>The restriction</u> $\nabla|\overline{dI}$ <u>of</u> ∇ <u>onto</u> $\overline{dI}$ <u>is trivial, i. e. there is a basis of horizontal elements in</u> $\overline{dI}$.

Sketch of proof : The result is local in nature. If $\mathfrak{S} = (S_i)$ is an admissible covering of S and if we have proved the result for the family over S_i for all i , the proof is complete.

Using Theorem 5 we may therefore assume that there are function $f_1 , \dots , f_g \in I$ such that $P(f_i)(\alpha_j) = \delta_{ij}$, where $\alpha_1 , \dots , \alpha_g$ is a basis of Γ . We have to show that $\nabla_D(\overline{df}_i) = 0$ where $\overline{df}_i$ is the cohomology class of df_i in H^1_{DR} . Now by the very definition of ∇_D we know that $\nabla_D(\overline{df}_i) = d(\hat{D}f_i)$ where $\hat{D}$ is an extension of the derivation D to the field of meromorphic function on M with $\hat{D}(x) = 0$ for a meromorphic function x on M which is not a meromorphic function on S . (= is not constant on all the curves of the family $M \to S$).

We are done if we can show that $\hat{D}f_i$ is Γ-invariant. This seems obvious as $(\hat{D}f_i) \circ \alpha_j = \hat{D}(f_i \circ \alpha_j) = \hat{D}(f_i + \delta_{ij}) = \hat{D}(f_i)$.

The problem with this argument is that $\hat{D}$ is defined only on the field of meromorphic functions of M and f_i is not in it. But one can define a unique extension of $\hat{D}$ to a vector field on Z which does justify the above line of argument as soon as we have shown

$$(\hat{D}f_i) \circ \alpha = \hat{D}(f_i \circ \alpha) .$$

But $D'(f) := (\hat{D}(f \circ \alpha)) \circ \alpha^{-1} - \hat{D}(f)$ is an analytic vector field on Z with $D'(f) \equiv 0$ for all meromorphic functions on M . Thus $D' \equiv 0$ and

$$(\hat{D}f_i) \circ \alpha = \hat{D}(f_i \circ \alpha) .$$

4. Elliptic case.

The first nontrivial example is the family of Tate curves which has been studied by a number of authors, see for example [R], [Rb], [K], [DR].

Assume that char $K \neq 2$.

$$S = \{q \in K : \ 0 < |q| < 1\}$$

$$Z = \{(q , z) \in K^2 : \ q \in S , \ z \in K - \{0\}\}$$

$\alpha(q , z) := (q , qz)$ is a bianalytic map $Z \to Z$. Let Γ be the transformation group generated by α . Then $M = Z/\Gamma \to S$ is the universal family of Tate curves.

The de Rham cohomology space H^1_{DR} for the family $M \to S$ is freely generated over the structure sheaf on S by the class τ_1 of the analytic differential (dz/z) and by the class τ_2 of the meromorphic differential $d\xi$ where

$$\xi(q , z) = \frac{1}{1 - z} + \sum_{n=1}^{\infty} \left(\frac{1}{1 - q^n z} - \frac{1}{1 - q^n z^{-1}}\right)$$

$$= \frac{1}{1 - z} + \sum_{n=1}^{\infty} \left(\frac{q^n z}{1 - q^n z} - \frac{q^n z^{-1}}{1 - q^n z^{-1}}\right)$$

for which holds

$$\xi(q , qz) - \xi(q , z) = 1$$

$$\xi(q , z^{-1}) = 1 - \xi(q , z)$$

$$\xi(q , - 1) = \frac{1}{2} \text{ if char } K \neq 2$$

$$\xi(q , \pi) = 1 \text{ if } \pi^2 = q .$$

Denote by $(\partial/\partial q)$ (resp. $(\partial/\partial z)$ the canonical partial derivatives with respect to the first (resp. second) variable of $Z = S \times K^*$.

$$\Phi = z \frac{\partial \xi}{\partial z}$$

$$\Phi' = z \frac{\partial \Phi}{\partial z} .$$

Then Φ , Φ' are Γ-invariant meromorphic functions on Z and the following equation holds

$$\Phi'^2 = 4(\Phi - e_1)(\Phi - e_2)(\Phi - e_3)$$

where $e_1 = \Phi(q , -1)$, $e_2 = \Phi(q , \pi)$, $e_3 = \Phi(q , -\pi)$ with π a fixed square root of q .

If we put

$$x := \frac{\Phi - e_1}{e_2 - e_1}$$

$$y := \frac{\Phi'}{2(e_2 - e_1)^{3/2}}$$

then

$$y^2 = x(x - 1)(x - \lambda)$$

with

$$\lambda = \frac{e_3 - e_1}{e_2 - e_1} = x(q , -\pi)$$

which is the Legendre normal form for the family of Tate curves.

Let

$$D_q := \frac{\partial}{\partial q} - \frac{\dot{x}}{x'} \frac{\partial}{\partial z}$$

where

$$\dot{x} = \frac{\partial x}{\partial q} , \quad x' = \frac{\partial x}{\partial z} .$$

We claim that the vector field D_q coincides with the vector field $\hat{D}$ for $D = (\partial/\partial q)$ in the proof of Theorem 7.

$$D_q(x) = 0 = \hat{D}(x)$$

$$D_q(f) = \frac{\partial f}{\partial q} = \hat{D}(F) \quad \text{if } f \text{ is analytic on } S .$$

Thus $\hat{D} = D_q$.

Let $\nabla = (\nabla_{\partial/\partial q})$. Then $\nabla(f dx) = D_q(f)\, dx$ by definition of $(\nabla_{\partial/\partial q})$.

One can by direct computation show that

$$\nabla(df) = d(D_q(f))$$

and that $D_q(\xi) = (\partial\xi/\partial q) - (\dot{x}/x')(\partial\xi/\partial z)$ is Γ-invariant.

This proves that $\nabla(\tau_2) = 0$ which gives a more direct proof of Theorem 7 for the family of Tate curves.

Let σ_1 (resp. σ_2) be the cohomology class of $(dx/2y)$ (resp. $x(dx/2y)$) .

Then σ_1 , σ_2 is a basis of H^1_{DR} . Let

$$\overline{d\xi} = \tau_2 = A.\sigma_1 + B\sigma_2 .$$

THEOREM 8.

$$A = \frac{\Phi(q , - 1)}{\sqrt{\Phi(q , \pi) - \Phi(q , - 1)}}$$

$$B = \sqrt{\Phi(q , \pi) - \Phi(q , - 1)}$$

<u>and</u> $\frac{A}{B}$ <u>as a function of</u> λ <u>can be given by</u>

$$\frac{A}{B} = 2\lambda[(1 - \lambda) \frac{F'}{F} + \frac{1}{2}]$$

<u>where</u>

$$F(\lambda) = {}_2F_1(\tfrac{1}{2} , \tfrac{1}{2} ; \ 1 ; \ 1 - \lambda)$$

$$= \sum_{n=0}^{\infty} (-\frac{(1/2)_n}{n !})^2 (1 - \lambda)^n .$$

Sketch of proof : The proof of the first part is given by a small computation. One can use the characterization of elements τ in H^1_{DR} with $\nabla(\tau) = 0$ given in [P], (7.11), (ii), to prove the second part.

We find that $\tau_2 = \lambda(1 - \lambda) \frac{\partial f}{\partial \lambda} .\sigma_1 - \lambda(1 - \lambda) f\nabla(\sigma_1)$ where f satisfies the hypergeometric equation

$$\lambda(1 - \lambda) \frac{\partial^2 f}{\partial f^2} + (1 - 2\lambda) \frac{\partial f}{\partial \lambda} - \frac{1}{4} f = 0 .$$

Here one has to use the fact that the map $\pi \to \lambda(\pi) = x(q , - \pi)$ gives a bianalytic map from S onto $\{\lambda : |1 - \lambda| < |2|\}$.

Thus the inverse mapping $\pi(\lambda)$ is an analytic function of λ .

Now we conclude that $f = c.F(\lambda)$ as f is analytic on $\{\lambda : |1 - \lambda| < |2|\}$ with a constant $c \in K^*$ which can be determined by letting $\lambda \to 1$ (i. e. $\pi \to 0$).

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