Groupe de travail d'analyse ultramétrique

Siegfried Bosch

Stable reduction and rigid analytic uniformization of abelian varieties

Groupe de travail d'analyse ultramétrique, tome 11 (1983-1984), exp. nº 20, p. 1-4
<http://www.numdam.org/item?id=GAU_1983-1984__11__A12_0>

Groupe d'étude d'Analyse ultramétrique
(Y. AMICE, G. CHRISTOL, P. ROBBA)
11e année, 1983/84, n° 20, 4 p.
16 avril 1984

STABLE REDUCTION AND RIGID ANALYTIC UNIFORMIZATION OF ABELIAN VARIETIES

by Siegfried BOSCH (*)
[Univ. Münster]

This is a report on joint work with W. Lütkebohmert [BL], where we have given a purely analytic approach to the uniformization of curves and abelian varieties.

In classical uniformization theory over the field of complex numbers, one shows that each non-singular projective curve can be analytically represented as a quotient Ω/Γ with an open subset $\Omega \subset \underset{\sim}{P}^1$ and a subgroup $\Gamma \subset PGL(2, \underset{\sim}{C})$ acting discontinuously on Ω. Similarly, each abelian variety can be represented as an analytic torus which, from the multiplicative point of view, is defined as a quotient $(\underset{\sim}{C}^*)^g/\Gamma$ by a discrete subgroup $\Gamma \subset (\underset{\sim}{C}^*)^g$, free of rank g. Over a non-archimedean ground field k, the situation is quite different. The curves and abelian varieties which admit a good uniformization as above constitute only a small open part in the corresponding moduli spaces (see [M1] and [M2]). On the other hand, there are curves and abelian varieties admitting no non-trivial uniformization at all. This behavior is related to the phenomenon of good reduction, which has no classical counterpart. In general, good uniformization and good reduction occur in a mixed way. It is for this reason that uniformization theory is substantially more complicated in the non-archimedean case.

1. Uniformization via algebraic geometry.

The basic tools in uniformization theory over discretely valued fields are the stable reduction theorem of Deligne-Mumford [DM] and the semi-abelian reduction theorem of Grothendieck [SGA7]. In order to explain these results, consider a discrete valuation ring R, and let k be its field of fractions.

Let A be an abelian variety over k, and denote by $\mathfrak{A}$ its Néron model ([N], [R1]). Then the theorem of Grothendieck says that (modulo finite separable extension of the ground field) the identity component of the special fibre $\mathfrak{A}_s$ of $\mathfrak{A}$ is semi-abelian ; i. e. $\mathfrak{A}_s^0$ is an extension of an abelian variety by a multiplicative group. Likewise, if C is a smooth geometrically connected projective curve of genus $\geqslant 2$ over k, one can consider its minimal model $\mathcal{C}$ over R. (See [Ab] or [Li] for the existence of a regular model and [Sh] for the minimality.) The result of Deligne-Mumford asserts (again, modulo finite extension of the ground field) that the special fibre $\mathcal{C}_s$ of $\mathcal{C}$ has only ordinary double points as

(*) Siegfried BOSCH, Mathematisches Institut der Universität, 62 Einsteinstrasse, D-4400 MÜNSTER (Allemagne fédérale).

singularities. Both results are more or less equivalent. In fact, the proof of the stable reduction theorem in [DM] uses the semi-abelian reduction of the Jacobian J of C and the fact that the "identity component" of the Néron model of J represents the functor $\mathrm{Pic}^0(\mathcal{C}/\mathrm{Spec}\ R)$ (see [R3]). Apart from uniformization theory, there are far-reaching applications of both theorems in the theory of moduli and also in number theory.

Assuming that k is complete, the universal covering $\hat{C}$ of C can be easily constructed as follows. Consider $\bar{\mathcal{C}}$, the formal completion of $\mathcal{C}$. Then $\bar{\mathcal{C}}$ is a formal scheme over R or, from the rigid analytic viewpoint, a formal analytic variety over k. In fact, $\bar{\mathcal{C}}$ may be interpreted as the analytification of C (again denoted by C), and the formal structure of $\bar{\mathcal{C}}$ gives rise to a reduction map $\pi : C \to \tilde{C} = \mathcal{C}_s$. The formal fibre $\pi^{-1}(\tilde{x})$ over a closed (rational) point $\tilde{x} \in \tilde{C}$ is an open disc if $\tilde{x}$ is non-singular, and an open annulus if $\tilde{x}$ is an ordinary double point. This leads to a geometric description of C. Namely, the π-inverse of the non-singular locus $\tilde{C} - \mathrm{Sing}\ \tilde{C}$ is a disjoint union of components which are smooth over R and thus simply connected (in the sense of rigid analysis; see [BL], II, 8.12). The curve C is obtained by connecting these components by means of the annuli $\pi^{-1}(\tilde{x})$, $\tilde{x} \in \mathrm{Sing}\ \tilde{C}$. Likewise, the universal covering $\hat{C}$ of C is constructed by resolving all loops which are generated by this process. If $\tilde{C}$ has only rational components (this is the case of Mumford's split degenerate reduction [M1], C has a good uniformization as discussed above. Namely, $\hat{C}$ can be viewed as an open analytic subvariety of $\underset{\sim}{P}^1$, the automorphisms of $\hat{C}$ over C being fractional linear transformations. The uniformization of abelian varieties A is more complicated (see [R2]). Here, analogous to the case of curves, a fundamental role is played by the formal completion $\bar{\mathfrak{a}}$ of the Néron model of A, or to be more precise, by the identity component $\bar{\mathfrak{a}}^0$ of $\bar{\mathfrak{a}}$.

2. Uniformization of curves via rigid analysis.

It is a surprising fact, first realized by VAN DER PUT [P], that the formal completion $\bar{\mathcal{C}}$ of the minimal model $\mathcal{C}$ of C can be constructed by a direct analytic method (without knowing $\mathcal{C}$). Thus proceeding as in section 1, a purely analytic approach to the uniformization of curves is obtained by proving the following analytic version of the stable reduction theorem (for arbitrary non-archimedean ground fields, modulo finite separable field extension) :

<u>The curve</u> C <u>can be viewed as a formal analytic variety with associated reduction</u> $\pi : C \to \tilde{C}$ <u>such that</u> $\tilde{C}$ <u>has at most ordinary double points as singularities.</u>

The proof of this result in [BL] uses the key fact that for arbitrary reductions $\pi : C \to \tilde{C}$, the periphery of a fibre $\pi^{-1}(\tilde{x})$ is a disjoint union of annuli. Namely, one starts with an arbitrary reduction $\pi : C \to \tilde{C}$ and refines π inductively by using the technique of blowing up, in such a way that all bad singularities of

$\tilde{C}$ disappear.

3. Uniformization of abelian varieties via rigid analysis.

As an application of the analytic stable reduction theorem, it is shown, in [BL], how to obtain the uniformization of abelian varieties (over arbitrary non-archimedean ground fields k, modulo finite separable field extension). Namely, consider the Jacobian J of a smooth geometrically connected projective curve C. The stable reduction theorem provides deep insight into the analytic structure of C. Thereby it is possible to construct line bundles with prescribed properties and to carry out explicit computations. One constructs an open analytic subgroup $\bar{J}$ of J, the group of normalized line bundles on C; it has a canonical reduction $\tilde{J}$ which is isomorphic to the Jacobian of $\tilde{C}$. Hence $\tilde{J}$ is an extension of an abelian variety by a multiplicative group. If the valuation of the ground field k is discrete, $\bar{J}$ may be interpreted as the identity component of the formal completion of the Néron model of J.

In order to construct the universal covering of J, one looks at the analytic cohomology group $H^1(C, \mathbb{Z})$. It is free of rank $r \leq g$ (= genus of C); in fact, the rank r reflects the number of loops in C as discussed in section 1. Using Picard functors, one interprets $H^1(C, \mathbb{Z})$ as the $\mathbb{Z}$-module of analytic group homomorphisms $\mathbb{G}_m \dashrightarrow J$ or $\bar{\mathbb{G}}_m \dashrightarrow \bar{J}$, where $\mathbb{G}_m$ denotes the multiplicative group over k and where $\bar{\mathbb{G}}_m$ is its subgroup of "units". Thus there is a closed subgroup $\bar{\mathbb{G}}_m^r \hookrightarrow \bar{J}$ which reduces to the multiplicative part of $\tilde{J}$ and which may be extended to an analytic homomorphism $\mathbb{G}_m^r \dashrightarrow J$. Then $\hat{J} := \mathbb{G}_m^r \times \bar{J}/(\text{diagonal})$ is the universal covering of J. The projection map $\hat{J} \dashrightarrow J$ has a discrete kernel Γ which is free of rank r, so that $J = \hat{J}/\Gamma$. In particular, the following assertions are equivalent :

(i) C is a Mumford curve,

(ii) rank $H^1(C, \mathbb{Z}) = g$,

(iii) J is an analytic torus.

Furthermore, J has good reduction, if C has good reduction.

Since, up to isogeny, any abelian variety A can be embedded into a product of Jacobian varieties, the above uniformization of Jacobians implies the uniformization of A. Namely, one constructs the analogues $\bar{A}$ and $\hat{A}$ of the groups $\bar{J}$ and $\hat{J}$, and shows $A = \hat{A}/\Gamma$. In this case, the rank of Γ has to be interpreted as the rank of the cohomology group $H^1(A', \mathbb{Z})$, where A' is the dual abelian variety of A.

4. Further applications of the analytic method.

Using simple algebraization techniques, the methods of [BL] seem to yield new proofs for the facts from algebraic geometry mentioned in section 1, namely for the following results :

- Existence of minimal models for curves,
- Existence of Néron models for abelian varieties,
- Stable reduction theorem of Deligne-Mumford,
- Semi-abelian reduction theorem of Grothendieck.

BIBLIOGRAPHY

[Ab] ABHYANKAR (Shreeram S.). - Resolution of singularities of arithmetical surfaces, "Arithmetical algebraic geometry", Proceedings of a Conference held at Purdue University 1963, p. 111-152. - New York, Harper and Row Publishers, 1965 (Harper's Series in Modern Mathematics).

[BL] BOSCH (S.) and LÜTKEBOHMERT (W.). - Stable reduction and uniformization of abelian varieties, I, II (to appear).

[DM] DELIGNE (P.) and MUMFORD (D.). - The irreducibility of the space of curves of given genus. - Paris, Presses universitaires de France, 1969 (Institut des hautes Etudes scientifiques, Publications mathématiques, 36, p. 75-110).

[Li] LIPMAN (Joseph). - Desingularization of two-dimensional schemes, Annals of Math., Series 2, t. 107, 1978, p. 151-207.

[M1] MUMFORD (David). - An analytic construction of degenerating curves over complete local rings, Compos. Math., Groningen, t. 24, 1972, p. 129-174.

[M2] MUMFORD (David). - An analytic construction of degenerating abelian varieties over complete rings, Compos. Math., Groningen, t. 24, 1972, p. 239-272.

[N] NÉRON (André). - Modèles minimaux des variétés abéliennes sur les corps locaux et globaux. - Paris, Presses universitaires de France, 1964 (Institut des hautes Etudes scientifiques, Publications mathématiques, 21).

[P] VAN DER PUT (M.). - Stable reductions of algebraic curves, Report Rijksuniversiteit Groningen, 1981.

[R1] RAYNAUD (Michel). - Modèles de Néron, C. R. Acad. Sc. Paris, t. 262, 1966, série A, p. 345-347.

[R2] RAYNAUD (Michel). - Variétés abéliennes et géométrie rigide, "Actes du Congrès international des Mathématiciens", 1970 [Nice], vol. 1, p. 473-477. - Paris, Gauthier-Villars, 1971.

[R3] RAYNAUD (Michel). - Spécialisation du foncteur de Picard, C. R. Acad. Sc. Paris, t. 264, 1967, série A, p. 941-943 ; p. 1001-1004.

[SGA7] GROTHENDIECK (A.). - Groupes de monodromie en géométrie algébrique (SGA 7, I). - Berlin, Heidelberg, New York, Springer-Verlag, 1972 (Lecture Notes in Mathematics, 288).

[Sh] SHAFAREVITCH [ŠAFAREVIČ] (J. R.). - Lectures on minimal models and birational transformations of two dimensional schemes. - Bombay, Tata Institute of fundamental Research, 1966 (Tata Institute of fundamental Research ; Lectures on Mathematics, 37).