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ON THE STRATIFICATION OF THE MODULI SPACE OF MUMFORD CURVES

by Frank HERRLICH (*)

The space $\mathfrak{M}_g$ of Mumford curves of genus g is an open analytic subspace of the algebraic quasiprojective variety $\mathfrak{M}_g$ of smooth curves of genus g. POPP [5] introduced a stratification of $\mathfrak{M}_g$ which was generalized to arbitrary characteristic by LØNSTED [3]. In this note, we develop a generalized p-adic Teichmüller theory to study the intersection of strata of $\mathfrak{M}$ with $\mathfrak{M}_g$. The proofs are rather sketchy, more details will be given in a paper which is in preparation.

1. - We begin with a brief review of the moduli theory of Mumford curves : Let k be an algebraically closed complete nonarchimedean field. For $i = 1 , 2$, let $\Gamma_i \subset PGL_2(k)$ be a Schottky group, and Ω_i the set of ordinary points of Γ_i.

Then $C_i := \Omega_i/\Gamma_i$ is analytically isomorphic with a non-singular projective curve of genus $g_i = \text{rank } \Gamma_i$, and it is well known that the Mumford curves C_1 and C_2 are isomorphic if, and only if, the Schottky groups Γ_1 and Γ_2 are conjugate subgroups of $PGL_2(k)$.

Therefore the main goal of the moduli theory, the classification of isomorphy classes of Mumford curves, reduces to the classification of conjugacy classes of Schottky groups. This is performed by p-adic Teichmüller theory (cf. [1]).

Let $g \geq 1$ be an integer, F_g an (abstract) free (nonabelian) group of rank g. Let

$$S_g := \text{Hom}\,(F_g\ , PGL_2(k))$$

and

$$T_g := \{\tau \in S_g\ ;\ \tau \text{ injective and } \tau(F_g) \text{ discontinuous}\}\ .$$

S_g is an affine algebraic k-variety of dimension $3g$ (in fact, isomorphic with $(PGL_2(k))^n$), and T_g is an analytic subset of S_g, given by the infinitely many inequalities

$$\tau \in T_g \iff |\text{tr } \tau(\omega)| > 1 \text{ for every } \omega \in F_g$$

where $\text{tr } \gamma := (\text{trace } \tilde{\gamma})^2/\det \tilde{\gamma}$ for any representative $\tilde{\gamma} \in GL_2(k)$ for the fractional linear transformation $\gamma \in PGL_2(k)$.

On S_g we have a natural action of $\text{Aut } F_g$ on the right, and of

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$$\mathrm{Aut}\,(\mathrm{PGL}_2(k)) = \mathrm{PGL}_2(k)$$

on the left. T_g is invariant for both groups. $PGL_2(k)$ is a reductive (even semi-simple) algebraic group (of dimension 3), so the quotient $PGL_2(k)\backslash S_g$ is again algebraic and of dimension $3g-3$. Aut F_g is a discrete group. Furthermore the two group actions commute, so Aut F_g acts also on $PGL_2(k)\backslash S_g$, and this action is discontinuous on

$$\bar{T}_g := PGL_2(k)\backslash T_g$$

($\bar{T}_g$ is called the p-adic Teichmüller space). The inner automorphisms of F_g act trivially on $\bar{T}_g$, so we have an action of Aut' F_g := Aut F_g/(inner automorphisms) on $\bar{T}_g$.

In this way, we impose an analytic structure on

$$\mathfrak{M}_g := \bar{T}_g/\mathrm{Aut}'\,F_g = PGL_2(k)\backslash T_g/\mathrm{Aut}\,F_g .$$

By what was said above the points of $\mathfrak{M}_g$ correspond to the isomorphy classes of Mumford curves of genus g.

$\mathfrak{M}_g$ is a subset of $\mathfrak{M}_g$, the moduli space of all nonsingular curves of genus g over k (as constructed by MUMFORD), so $\mathfrak{M}_g$ inherits an analytic structure as subspace of $\mathfrak{M}_g$. LÜTKEBOHMERT [4] has shown that this analytic structure is the same as the one defined above.

2. -- The stratification of the algebraic moduli variety $\mathfrak{M}_g$ has been carried out by POPP [5] (in characteristic 0) and LØNSTED [3]. The results are as follows :

By means of the Torelli map one identifies $\mathfrak{M}_g$ with the subspace $J_g \subset \mathfrak{U}_g$, i. e. the subspace of the moduli variety of polarized abelian varieties of dimension g which corresponds to Jacobians of nonsingular curves of genus g. Then one considers the space $J_{g,n}$ of Jacobians with level-n-structure, $n \geqslant 3$. It is known that $J_{g,n}$ is nonsingular, and that J_g is the quotient of $J_{g,n}$ by a finite group which is isomorphic to $GL_{2g}(\mathbb{Z}/n\mathbb{Z})$. The stabilizers of this action are the automorphism groups of the corresponding curves ($g \geqslant 4$; exception for $g = 3$ and hyperelliptic curves, and for $g = 2$), and the strata of $J_g \cong \mathfrak{M}_g$ are the (images of the) fixed point sets of the different subgroups of $GL_{2g}(\mathbb{Z}/n\mathbb{Z})$. Now, in the p-adic case we have the period map $q : \bar{T}_g \to H_g$ from the Teichmüller space into the Siegel upper half plane, and the quotient $\mathfrak{A}_g$ of H_g by the group $GL_g(\mathbb{Z})$ is the analytic space of polarized abelian varieties which are k-analytic tori. Furthermore the congruence subgroup $\Gamma_{g,n} = \{\gamma \in GL_g(\mathbb{Z}) \; ; \; \gamma \equiv \begin{pmatrix}1&0\\0&1\end{pmatrix} \bmod n\}$ acts without fixed points on H_g, and $H_g/\Gamma_{g,n}$ can be embedded into $J_{g,n}$ in such a way that the two group actions coincide. This shows that the (images of the) fixed point sets in $\bar{T}_g$ of subgroups of the Teichmüller modular group Aut' F_g are the intersections of $\mathfrak{M}_g$ with the strata of $\mathfrak{M}_g$. Instead of subgroups of Aut' F_g we can of course consider subgroups of $\mathrm{Aut}\ F_g$ which contain the group of inner automorphisms. The fixed point sets of conjugated subgroups of $\mathrm{Aut}\ F_g$ have the same image in the quotient space $\mathfrak{M}_g$, and subgroups of $\mathrm{Aut}\ F_g$ containing F_g are isomorphic if, and only if, they are conjugate. Furthermore the stabilizer subgroup of $\mathrm{Aut}\ F_g$ of a point in $\bar{T}_g$ is isomorphic with the normalizer of the corresponding Schottky group in $PGL_2(k)$. This shows :

PROPOSITION 1. - Let $C = \Omega/\Gamma$, $C' = \Omega'/\Gamma'$ be Mumford curves of genus $g \geq 4$ with Schottky groups Γ, Γ'. Let N, N' be the normalizers of Γ and Γ' in $PGL_2(k)$. Then C and C' are in the same stratum of $\mathfrak{M}_g$ if, and only if, there is an isomorphism $\alpha : N \to N'$ such that $\alpha(\Gamma) = \Gamma'$ (we call N, N' (Γ,Γ')-isomorphic in that case).

This proposition makes it possible to study the intersections $S \cap \mathfrak{M}_g$, S stratum of $\mathfrak{M}_g$, by a Teichmüller theory which is very similar to the one explained in section 1.

3. - For any finitely generated group N, let

$$S(N) := \mathrm{Hom}(N, PGL_2(k))$$

and

$$T(N) := \{\tau \in S(N) \ ; \ \tau \text{ injective and } \tau(N) \text{ discontinuous}\} .$$

If N is generated by n elements, $S(N)$ is an algebraic subspace of the affine variety $(PGL_2(k))^n$ whose defining equations come from the relations among the n generators.

$T(N)$ is given by the inequalities

$$|\mathrm{tr}\ \tau(x)| > 1 \quad \text{for every } x \in N \text{ of infinite order.}$$

So $T(N)$ is an analytic subspace of $S(N)$ (which may, of course, be empty).

If $T(N) \neq \emptyset$, we know by the structure theory of finitely generated p-adic discontinuous groups [2] hat N contains a free normal subgroup F of finite index and finite rank g. For any such pair (N, F), where N is finitely generated and F is a normal free subgroup of finite index, we define

$$\mathrm{Aut}_F(N) := \{\alpha \in \mathrm{Aut}\ N \ ; \ \alpha(F) = F\} .$$

Now we may form the quotient

$$\mathfrak{M}(N, F) := PGL_2(k)\backslash T(N)/\mathrm{Aut}_F(n) .$$

This is the quotient of an analytic space by a discrete group and can therefore be given an analytic structure (we shall exhibit a fundamental domain in section 5).

PROPOSITION 2. - Let $F < N$ finitely generated groups, F free of rank $g \geq 0$, such that $(N : F) < \infty$.

(i) $\mathfrak{M}(N, F)$ <u>is the space of Mumford curves</u> $C = \Omega/\Gamma$ <u>of genus</u> g <u>such that the normalizer</u> $N(\Gamma)$ <u>contains a subgroup which is</u> (Γ, F)-<u>isomorphic with</u> N .

(ii) <u>If</u> $g \geqslant 4$, <u>there is a stratum</u> S <u>of</u> $\mathcal{M}_g$ <u>such that</u>

$$S \cap \mathfrak{M}_g = \mathfrak{M}(N , F)\backslash\{C \in \mathfrak{M}_g \ ; \ \mathrm{Aut}\, C \supsetneqq N/F\} .$$

(iii) <u>If</u> $g \geqslant 4$, <u>for every stratum</u> S <u>of</u> $\mathcal{M}_g$ <u>there exists a unique pair</u> (N,F) <u>as above such that</u> $S \cap \mathfrak{M}_g$ <u>is as in</u> (ii).

4. - In order to calculate the dimension of $\mathfrak{M}(N , F)$ (and therefore of $S \cap \mathfrak{M}_g$ which by Proposition 2 (ii) is an open subset of $\mathfrak{M}(N , F)$), we need some information on the structure of the possible groups N . If $\mathfrak{M}(N , F) \neq \emptyset$, then there is a discontinuous subgroup $\tilde{N} \subset PGL_2(k)$ which is isomorphic with N . $\tilde{N}$ acts on a locally finite tree $B(\tilde{N})$ such that the quotient graph $Q(\tilde{N}) = B(\tilde{N})/\tilde{N}$ is a finite graph (see [2] e. g.) which becomes a graph of groups by assigning the stabilizer group of a preimage to every vertex and edge of $Q(\tilde{N})$ (with the obvious inclusion maps). Then from the Bass-Serre structure theorem for groups acting on trees, we find the following presentation for $\tilde{N}$ (and therefore for N) :
N is generated by

(i) the vertex groups of $Q(\tilde{N})$,

(ii) one free generator γ_{ℓ} for every (unoriented edge ℓ of $Q(\tilde{N})$ not contained in a fixed maximal subtree.

The defining relations are amalgamation of the edge groups in the corresponding vertex groups (resp. identification after conjugation by γ_1 , if the edge 1 is not in the maximal subtree). Of course, the presentation of N as the fundamental group of a graph of groups is not unique ; but the following lemma, from combinatorial group theory (whose proof will be given elsewhere), shows that the relevant vertex and edge groups are unique.

LEMMA 1. - <u>Let</u> Q , Q' <u>be reduced graphs of groups with isomorphic fundamental group</u>. <u>Then there is a finite sequence</u> $Q = Q_0 , \dots , Q_n = Q'$ <u>of reduced graphs of groups such that for</u> $i = 1 , \dots , n$ <u>either</u> Q_i <u>is derived from</u> Q_{i-1} <u>by an admissible contraction of an edge</u>, <u>or conversely</u> Q_{i-1} <u>from</u> Q_i .

(Here we call a graph of groups reduced if for every vertex v either there are at least 3 edges starting at v or there is an edge starting at v with edge group strictly contained in the vertex group of v ; a contraction of an edge ℓ is called admissible if ℓ is not a loop and if the edge group of ℓ is isomorph with the vertex group of one of the end points of ℓ .)

In characteristic 0 , the possible vertex groups are the cyclic and dihedral groups and the exceptional groups A_4 , S_4 and A_5 (in characteristic p there are also subgroups of $PGL_2(\mathbb{F}_{p^n})$ possible). A nontrivial cyclic subgroup of

$PGL_2(k)$ is determined by its two fixed points, all other finite subgroups by three parameters (they are generated by two elements of finite order with one relation between them). Finally, a free generator of N is a hyperbolic element of $PGL_2(k)$ and therefore determined by 3 parameters (fixed points and multiplier).

With the notations

$c_v = c_v(N) =$ number of nontrivial cyclic vertex groups of N

$c_e = c_e(N) =$ " edge "

$d_v(d_e) =$ " noncyclic vertex (edge) groups of N

$f = f(N) =$ " free generators of N

(= cyclomatic number of $Q(N)$)

we have the following :

THEOREM 1. - Let $F < N$ as in Proposition 2. If $T(N) \neq \emptyset$, then

$$\dim \mathfrak{M}(N, F) = 3f + 3d_v + 2c_v - 3d_e - 2c_e - 3 .$$

(- 3 is the dimension of $PGL_2(k)$).

5. - In this section, we shall sketch the proof of :

THEOREM 2. - Let $F < N$ as in Proposition 2, $T(N) \neq \emptyset$. Then (N, F) is connected.

From this theorem one easily deduces :

COROLLARY. - For any stratum S of $\mathfrak{M}_g$, the intersection $S \cap \mathfrak{M}_g$ is either empty or connected.

To prove theorem 2, we have to embed $T(N)$ into some analytic space k^n, n sufficiently large ; i. e. we need coordinates on $T(N)$.

For this purpose, we fix a set A of generators of N which comes from the presentation of N as the fundamental group of a graph of groups as explained in the previous section. Now every $\tau \in T(N)$ is of course determined by the set $\tau(A)$, and every $\tau(a)$, $a \in A$, is given by its two fixed points in $\mathbb{P}^1(k)$ and its multiplier. If $\tau(a)$ is hyperbolic, one of the fixed points is attracting and the other repelling ; but if a is of finite order d the fixed points cannot be distinguished, and the multiplier takes on only a finite number of values (primitive d-th roots of unity). It is convenient to begin with a space $T_o(N)$ where the coordinates are the fixed points and multipliers. $T(N)$ will then be the quotient of $T_o(N)$ by the action of the group $(\mathbb{Z}/2\mathbb{Z})^{r+2s}$, where $r + 2s$ is the number of elements of finite order in A.

To be more precise, let

$$A = \{a_1 , \dots , a_r ; b_1 , c_1 , \dots , b_s , c_s ; d_1 , \dots , d_f\}$$

where $r = c_v$, $s = d_v$ in the notation of Theorem 1 ; i. e. the a_i generate cyclic vertex groups, b_i and c_i generate a noncyclic vertex group and the d_i are of infinite order. As coordinates, we have the fixed points x_i , y_i of a_i , x_{i+r} , y_{i+r} of b_i , z_i of c_i (the other fixed point of c_i can be calculated from the relation between c_i and b_i) and x_{i+r+s} , y_{i+r+s} of d_i as well as the multipliers t_i of d_i . We omit the multipliers of the elements of finite order : They give rise to a finite number of isomorphic connected components of our space $T_\vartheta(N)$, which are all identified by the action of Aut N .

Now we introduce a fundamental domain in $T_\vartheta(N)$ for the action of Aut N (as $\mathrm{Aut}_F(N)$ has finite index in Aut(N) if F has finite index in N , a finite number of translates of this domain will be a fundamental domain for $\mathrm{Aut}_F(N)$.)

We call $\tau \in T(N)$ geometric (or $\tau(A)$ a geometric set of generators) if $\{\tau(a) ; a \in A\}$ comes from the Bass-Serre presentation of $\tau(N)$ as the fundamental group of $B(\tau(N))/\tau(N)$. Obviously for every discontinuous subgroup of $PGL_2(k)$, isomorphic with N , there is a geometric set of generators, so that

$$B(N) := \{\tau \in T(N) ; \tau \text{ geometric}\}$$

is a fundamental domain in $T_o(N)$ for the action of Aut N .

If $N = F_g$, B(N) is the set of Schottky bases. We shall show that B(N) is the finite (disjoint) union of the spaces B(Q) where Q runs through the set of labelled graphs of groups with fundamental group N . A labelling of such a graph of groups consists of a maximal subtree P of Q and a map ν from the set B of fixed points of the elements of A and all elements in the vertex groups to the set of vertices Q such that

(i) for every edge not in P the end points are labelled by the fixed points of a generator of infinite order,

(ii) if $b_1 , b_2 \in B$ belong to $\alpha \in N$ then the unique simple path in P from $\nu(b_1)$ to $\nu(b_2)$ is the fixed point set of α in P .

(Here we assume for simplicity that N contains no elements of order p = residue characteristic of k .)

B(Q) is now given by inequalities of three different types, all involving the cross ratio

$$CR(x_1 , x_2 , x_3 , x_4) := \frac{x_1 - x_3}{x_1 - x_4} \frac{x_2 - x_4}{x_2 - x_3}$$

of four distinct points $x_1 , \dots , x_4$ in $\mathbb{P}^1(k)$:

(a) Let $B = \{b_1 , \dots , b_\rho\}$ for some ρ . For $b_i , b_j \in B$, let π_{ij} denote the projection in P onto the oriented path in P from $\nu(b_i)$ to $\nu(b_j)$. We say that $b_k <_{ij} b_\ell$ (or $=_{ij}$ or $>_{ij}$) if $\pi_{ij}(\nu(b_k))$ is closer to $\nu(b_i)$ than $\pi_{ij}(\nu(b_\ell))$ (resp. is equal to or is closer to $\nu(b_j)$). Then every point in $B(Q)$ satisfies the (in-)equalities

$$|CR(b_i , b_j , b_k , b_\ell)| \begin{cases} < 1 ; & b_k <_{ij} b_\ell , \\ = 1 ; & b_k =_{ij} b_\ell , \\ > 1 ; & b_k >_{ij} b_\ell . \end{cases}$$

(b) For every i, $r + s < i \leq r + s + f$ (i. e. every free generator of N) let e_i denote the corresponding edge in $Q \backslash P$ (the fixed points and the multipliers of the corresponding transformation were already denoted x_i, y_i and t_i).

Then for every pair b_j, b_k of elements of B which do not belong to an element in the stabilizer of e_i we have the following inequalities

$$|t_i| < |CR(x_i , y_i , b_j , b_k)| < |t_i|^{-1} .$$

(c) Let v be a vertex of Q, $\alpha \in N$ in the vertex group of v, b_i, $b_j \in B$ the fixed points of α. For every b_k, $b_\ell \in B$, such that

$$\pi_{ij}(\nu(b_k)) = \pi_{ij}(\nu(b_\ell)) = v ,$$

we must have

$$|CR(b_i , b_j , \alpha(b_k) , b_\ell) - 1| = 1$$

(this condition assures that α does not map an edge of Q starting at v onto another edge of Q).

LEMMA 2. - $B(Q)$ <u>is precisely the set of points</u>

$$(x_1 , y_1 , \ldots , x_{r+s+f} , y_{r+s+f} , z_1 , \ldots , z_s , t_1 , \ldots , t_f)$$

<u>satisfying all the inequalities in</u> (a), (b) <u>and</u> (c) <u>and the condition that all</u> x_i, y_i, z_i <u>be different.</u>

This lemma shows that $B(Q)$ is connected (even a Stein domain). Lemma 1 showed that one can come from one Q to another by a sequence of admissible contractions. After a contraction some projections coincide so that some inequalities in (a) become equalities. This shows that the union of the two domain is connected and so $B(N)$ is connected and also the quotient $\mathfrak{M}(N , F)$ is connected.

6. - To illustrate the concepts of the previous sections by an example we shall study two strata of $\mathfrak{M}_3$ which will turn out to be especially nice ($g = 3$ was not allowed in Proposition 2 because the hyperelliptic curves without further automorphism are not singular points of $\mathfrak{M}_3$; but the situation is very similar to the general case : only the uniqueness of the group N fails to be true, but there is

still a unique maximal group). Consider the fundamental group N of the following graph of groups :

$$S_3 \quad o\overset{1}{\text{———}}o \quad C_2$$

N is isomorphic to the free product $S_3 * C_2$. Let S_3 be generated by elements τ and σ such that $\tau^3 = \sigma^2 = (\tau\sigma)^2 = 1$, and let $C_2 (= \underset{\sim}{Z}/2\underset{\sim}{Z})$ be generated by an involution α.

N contains the free normal subgroup F generated by

$$\gamma_1 = \alpha\sigma \ , \quad \gamma_2 = \tau\alpha\sigma\tau^{-1} \ , \quad \gamma_3 = \tau^{-1}\alpha\sigma\tau \ .$$

Aside from inner automorphisms $\mathrm{Aut}\, N$ contains only the automorphisms of S_3. Only the automorphism which fixes σ and sends τ to τ^{-1} leaves F unchanged, so

$$\mathrm{Aut}_F(N)/(\text{inner automorphism}) = \{\mathrm{id} \ , \ \tau \mapsto \tau^{-1}\} \cong C_2 \ .$$

By Theorem 1, we have

$$\dim \mathfrak{M}(N \ , F) = 3 + 2 - 3 = 2 \ .$$

The other possible reduced graphs of groups with fundamental group N are

$$S_3 \quad o\overset{C_3}{\text{—}}\overset{C_3}{o}\overset{1}{\text{———}}o \quad C_2 \quad \text{and} \quad S_3 \quad o\overset{C_2}{\text{—}}\overset{C_2}{o}\overset{1}{\text{———}}o \quad C_2$$

To explicitly calculate $\mathfrak{M}(N \ , F)$, we begin with $T_0(N)$ and divide out the action of $PGL_2(k)$ (which we may do by the commutativity of the group actions). This means that we assume τ to have fixed points $0 \ , \infty$ and σ the fixed points $1 \ , -1$; if ρ denotes a primitive third root of unity, the fixed points of $\sigma\tau$ and $\sigma\tau^{-1}$ are $\rho \ , -\rho$ and $\rho^{-1} \ , -\rho^{-1}$. We denote fhe fixed points of α by x and y.

The possible labellings of the graphs of groups are

$$\underset{0,\infty,1,-1,\rho,-\rho,\rho^{-1},-\rho^{-1}}{S_3} \quad o\overset{1}{\text{———}}o \quad \underset{x,y}{C_2}$$

$$\underset{0,1,-1,\rho,-\rho,\rho^{-1},-\rho^{-1}}{S_3} \quad o\overset{C_3}{\text{—}}\underset{\infty}{\overset{C_3}{o}}\overset{1}{\text{———}}o \quad \underset{x,y}{C_2}$$

$$\underset{\infty,1,-1,\rho,-\rho,\rho^{-1},-\rho^{-1}}{o}\overset{C_3}{\text{—}}\underset{0}{\overset{C_3}{o}}\overset{1}{\text{———}}o \quad \underset{x,y}{C_2}$$

and so on (6 possibilities for the third graph of groups).

In all cases, the inequalities describing $B(Q)$ can be determined easily by condition (a) of Lemma 2. A straightforward calculation shows that the union of all the $B(Q)$, i. e.

$$PGL_2(k)\backslash T_o(N) =: \bar{T}_o(N)$$

is given by

$\bar{T}_o(N) = \{(x , y) \in k^2 ; \quad x \neq y$

and $|x - y| < \min (|x| , |x+1| , |x-1| , |x-\rho| , |x+\rho| , |x-\rho^{-1}| , |x+\rho^{-1}|)\}$

To come from $\bar{T}_o(N)$ to $\mathfrak{M}(N , F)$, we have to divide by the action of C_2^3 (identification of the fixed points of a transformation) and by the action of $\tau \longmapsto \tau^{-1}$. But the latter automorphism acts in the same way as interchanging 0 and ∞ . As we are already in $PGL_2(k)\backslash T_o(N)$ we have to normalize the actions of the automorphisms by a suitable linear fractional transformation.

Then $\Phi_1 : 0 \longmapsto \infty$, $\Phi_2 : 1 \longmapsto -1$, $\Phi_3 : x \longmapsto y$ act in the following way on x and y :

	Φ_1	Φ_2	Φ_3
x	x^{-1}	$-x$	y
y	y^{-1}	$-y$	x

The invariants are easily seen to be generated by $v' = \frac{x}{y} + \frac{y}{x}$ and $w' = xy + \frac{1}{xy}$. To have nicer inequalities, we prefer as coordinates on $\mathfrak{M}(N , F)$

$$v := v' - 2 = \frac{(x - y)^2}{xy} \quad \text{and} \quad w := w' - 4 = \frac{(xy - 1)^2}{xy} .$$

Then from $|x - y| < |x| = |y|$, we have $0 < |v| < 1$.

If $|x| \neq 1$, we have $|w'| = |w' - 4| > 1$. If $|x| = 1$, let

$$y = x + \varepsilon , \quad |\varepsilon| < \min_{i=o}^{2} |x \pm \rho^i| ;$$

then

$$|w| = |w' - 4| = |xy - 1|^2 = |x^2 - 1 + \varepsilon x|^2$$

$$= |(x - 1)(x + 1) + \varepsilon x|^2 = |x - 1|^2 |x + 1|^2 > |\varepsilon|^2 = |v| .$$

In the same way we see that if $|x| = 1$

$$|w + 3|^2 = |(xy)^2 + xy + 1|^2 = |x^4 + x^2 + 1 + \varepsilon(x + 2x^3 + \varepsilon x^2)|^2$$

$$= |(x - \rho)(x + \rho)(x - \rho^{-1})(x + \rho^{-1})|^2 > |\varepsilon|^2 = |v| .$$

Therefore we have

$$\mathfrak{M}(N , F) = \{(v , w) \in k^2 ; \quad 0 < |v| < \min(1 , |w| , |w + 3|^2)\} .$$

$\mathfrak{M}(N , F)$ is not a single stratum of $\mathfrak{M}_3$, but the union of two strata :

It contains a one-dimensional subspace of curves with bigger automorphism group than $N/F \cong S_3$.

Namely, if the fixed points x and y of α satisfy $xy = -1$, i. e. $y = \frac{1}{x}$, the transformation $\Phi(z) := -\frac{1}{z}$ generates with τ and σ a finite group D_6 , and with α a Klein four group D_2 . One checks immediately that

$$\Phi\gamma_1 \Phi^{-1} = \gamma_1 , \Phi\gamma_2 \Phi^{-1} = \gamma_3 , \Phi\gamma_3 \Phi^{-1} = \gamma_2 ,$$

so F is normal in the group N' generated by N and Φ , and $N/F \cong D_6$. x and $y = -x^{-1}$ define a point in $T_o(N)$ if, and only if,

$$|x^2 + 1| < 1$$

(i. e. $|x + i| < 1$ or $|x - i| < 1$ for $i = \sqrt{-1}$) .

The variables v and w of $\mathfrak{M}(N , F)$ become

$$v = -(\frac{x^2 + 1}{x})^2 , \quad w = -4 ,$$

and the inequality to be satisfied is

$$0 < |v| < 1 .$$

The corresponding (unique) graph of groups for N' is

$$D_6 \circ\!\!\overset{C_2}{\text{-----}}\!\!\circ D_2 .$$

We have shown that

$$\mathfrak{M}(N' , F) = \{v \in k ; 0 < |v| < 1\} = \{(v , w) \in \mathfrak{M}(N , F) : w = -4\} .$$

There is no group N'' containing N' such that F is normal in N'' and $T(N'') \neq \emptyset$. This means that $\mathfrak{M}(N' , F)$ and $\mathfrak{M}(N , F) \setminus \mathfrak{M}(N' , F)$ are indeed the intersections of two strata of Ω_3 with $\mathfrak{M}_3$.

REFERENCES

[1] GERRITZEN (Lothar). - p-Adic Teichmüller space and Siegel halfspace, Groupe d'étude d'Analyse ultramétrique, 9e année, 1981/82, n° 26, 15 p.

[2] HERRLICH (Frank). - Endlich erzeugbare p-adische diskontinuierliche Gruppen, Arch. der Math., t. 35, 1980, p. 505-515.

[3] LØNSTED (Knud). - The structure of some finite quotients and moduli for curves, Commun. in Algebra, t. 8, 1980, p. 1335-1370.

[4] LUTKEBOHMERT (Werner). - Ein globaler Starrheitssatz für Mumfordkurven, J. für reine und angew. Math., t. 340, 1983, p. 118-139.

[5] POPP (Herbert). - Stratifikation von Quotientenmannigfaltigkeiten (in Charakteristik 0) und insbesondere der Modulmannigfaltigkeiten für Kurven, J. für reine und angew. Math., t. 250, 1971, p. 12-41.