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SINGULAR RESIDUE CLASSES WHICH ARE ORDINARY FOR $F(a,b,c,\lambda)$

by Bernard DWORK (*)

In early work [2], on the p-adic theory $(p \neq 2)$ of the differential equation

$$\lambda(1-\lambda)\frac{du^2}{d\lambda^2} + (1-2\lambda)\frac{du}{d\lambda} - \frac{1}{4}u = 0 ,$$

a critical role was played by the existance of ratio τ of formal solutions such that

(1) $$\exp \tau(\lambda) \in \underset{\sim}{Z}_p[[\lambda]] .$$

Indeed if $F(\lambda)$ denotes the unique solution holomorphic and taking the value 1 at $\lambda = 0$, (hence $F(\lambda) = F(\frac{1}{2}, \frac{1}{2}, 1, \lambda)$) then there exists a second solution

$$u = F(\lambda)(\log \lambda + \lambda\, \vartheta(\lambda))$$

defined uniquely by the condition that $\vartheta \in \underset{\sim}{Q}[[\lambda]]$. It is known that if we set

$$16\, q = \lambda\, e^{\vartheta(\lambda)} ,$$

then

$$\lambda = 16\, q \prod_{\nu=1}^{\infty} \left(\frac{1+q^{2\nu}}{1+q^{2\nu-1}}\right)^8 .$$

In a subsequent article [3], an elementary proof of (1), independent of the theory of elliptic modular functions wa presented. However insofar as the hypergeometric function ${}_2F_1(a,b,c,\lambda)$ is concerned, this second treatment was restricted to the case of logarithmic singularity, i. e. $c \in \underset{\sim}{Z}$.

In more recent work [4], the behavior of the Frobenius matrix was carefully computed on the singular disk $D(0, 1^-)$, but the question of normalized solution matrix on the singular disk and the question of whether singular disks are ordinary was not treated. The object of the present note is to respond to these questions for the hypergeometric function.

The usual condition for being ordinary, the non-vanishing of the Hasse invariant is not quite appropriate in the present situation. A better definition involves not having a too high order of zero at $\lambda = 0$. A precise definition of ordinary singular disk is given below (1.11). We show that our definition is consistent with the

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usual one in terms of the existence of unique bounded solutions of the differential equation and in terms of special solutions of the Riccati equation. We do not give a detailed examination of supersingular, singular disks comparable of that of [4], chapter 16, for supersingular, nonsingular disks. In § 5, we pose the question of whether the canonical lifting extends to all of the ordinary, singular disk. The terminology used here is that of references [4], [5]. We take this opportunity to observe that in [4], Theorem 25, in case 4 the values given for θ_1, θ_4, $B_4(0)$, B_1 should all be divided by p. This error does not appear in [5], Theorem 4.

1. Review of previous work and definition of ordinary singular disk.

We study the differential equation

$$(1.1)_{(a,b,c)} \qquad \frac{d}{d\lambda}(u_1, u_2) = (u_1, u_2)\begin{pmatrix} -c, & c-a \\ c-b, & a+b-c \end{pmatrix}\begin{pmatrix} 1/\lambda & 0 \\ 0 & 1/(1-\lambda) \end{pmatrix}.$$

with (a,b,c) subject to the conditions

$$(1.2) \qquad (a, b, c) \in Q \cap \underset{\sim}{Z}_p$$

$(a, b) \in (0, 1)$, $c \in (0, 1]$, $a \neq c \neq b$.

We choose $\ell \in N$ such that

$$(p^{\ell} - 1)(a, b, c) \in \underset{\sim}{Z}^3 .$$

We use B in the sense of [4], equation (9.1.1.2), as matrix of the mapping $\frac{1}{p^m}\alpha_0^* \circ \dots \circ \alpha_{\ell-1}^*$ of $K_{f,\lambda^{p^{\ell}}}$ into $K_{f,\lambda}$ relative to the basis $\{1^*, (\frac{1}{1-x})^*\}$. We know that

$$(1.3) \qquad (u_1, u_2) \longrightarrow (u_1, u_2)^{\sigma^{\ell}\varphi^{\ell}} B$$

is an endomorphism of the solution space of $(1.1)_{(a,b,c)}$. Here m is the number of steps of type 4 in the sequence $\alpha_0^*, \dots, \alpha_{\ell-1}^*$.

We shall assume

$$(1.4) \qquad \alpha_{\ell-1}^* \text{ is split step of type } 1 \text{ (resp. type } 2) .$$

As in [4], Chapter 24, we set if $c \neq 1$

$$U(\lambda) = \begin{pmatrix} (c-b)\, F(a,b,1+c,\lambda) & c\, F(a,b,c,\lambda) \\ (1-c)\, F(a-c,b-c,1-c,\lambda) & (c-a)\,\lambda\, F(a+1-c,b+1-c,2-c,\lambda) \end{pmatrix}$$

(resp.

$$U(\lambda) = \begin{pmatrix} (1-b)\,F(a\,,\,b\,,\,2\,,\,\lambda) & F(a\,,\,b\,,\,1\,,\,\lambda) \\ (1-b)\,H(a\,,\,b\,,\,2\,,\,\lambda) & H(a\,,\,b\,,\,1\,,\,\lambda) \end{pmatrix}$$

if $c = 1$ where $H \in \lambda\ \mathbb{Q}[[\lambda]]$ is defined by the following condition). The choice of U is such that

$$\begin{pmatrix} 1 & 0 \\ 0 & \lambda^{-c} \end{pmatrix} U(\lambda)$$

(resp.

$$\begin{pmatrix} 1 & 0 \\ \log\lambda & 1 \end{pmatrix} U(\lambda)\ )$$

is a solution matrix of $(1.1)_{(a,b,c)}$.

It is known [4], Chapter 4, that aside from possible poles at 0 , ∞ , the matrix B is analytic for

$$(1.5) \qquad |\lambda - 1| > |\pi| = |p|^{1/(p-1)}\ .$$

In fact, there is no pole at $\lambda = 0$. This can be deduced from the explicit calculations of [5] (refining [5], (3.15.1), by replacing the factor $\lambda^{1+\mu_2-p}$ by $\lambda^{1+\mu_2}$).

The situation at $\lambda = 0$ may also be explained by the method of [4], Chapter 24, using the calculation of constants in Chapters 25, 26 of that work. In this connection, it is useful ot recall that the steps $\alpha_0^*\ , \ldots , \alpha_{s-1}^*$ are either all logarithmic or all non-logarithmic.

1.6 LEMMA. - <u>If</u> $c \neq 1$ (<u>resp.</u> $c = 1$) ,

$$U(\lambda^{p^s})\ B = \begin{pmatrix} e_1 & 0 \\ 0 & e_4\ \lambda^{c(p^s-1)} \end{pmatrix} U(\lambda) \quad (\underline{\text{resp.}}\quad e\begin{pmatrix} 1 & 0 \\ \gamma & p^s \end{pmatrix}\ U(\lambda))$$

where $e_1\ ,\ e_4 \in \mathbb{Q}_p$

$$\mathrm{ord}_p\ e_1 = s_1\ (=\ \underline{\text{the number of steps of type}}\ \ 1\ )$$

$$\mathrm{ord}_p\ e_4 = s_2\ (=\ \underline{\text{the number of steps of type}}\ \ 2\ )$$

(<u>resp.</u> $e = (-1)^{b(p^s-1)}$, $\gamma \in \mathbb{Q}_p$).. <u>Note that</u> $s_0 = s_1 + s_2$, <u>the number of split steps</u>.

There is no need to give the completely elementary proof, except to remark that the calculation of $\operatorname{ord}_p e_1$ results from the explicit formula for θ_1 ([4], Theorem 25) which must be repeated for each step $\alpha_0^*, \ldots, \alpha_{\ell-1}^*$ together with the formula ([4] p. 246)

$$(1.7) \qquad (p-1) \operatorname{ord} \gamma_p(-t+py, y) = t$$

provided $t \in \{0, 1, \ldots, p-1\}$. Thus in computing $\operatorname{ord} \gamma_p(x, y)$ with $x, y \in (\underset{\sim}{Z}_p \cap \underset{\sim}{Q}) - \underset{\sim}{Z}$, $py - x \in \underset{\sim}{Z}$, we must use the translation formula [4], (21.4.3), to reduce to the situation in which $py - x$ is a positive integer bounded by $p - 1$. Also we must remember to remove a factor, p, for each step of type 4.

As in the treatment of [4], Chapter 24, we may use this last lemma to determine the value of B at $\lambda = 0$.

1.8. LEMMA.

$$B = \begin{pmatrix} \lambda^{c(p^{\ell}-1)} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} K_1 & \lambda K_2 \\ B_3 & B_4 \end{pmatrix}$$

$$\begin{pmatrix} K_1 & \lambda K_2 \\ B_3 & B_4 \end{pmatrix}(0) = \begin{pmatrix} e_4 & 0 \\ e_1 \dfrac{c-b}{c} & e_1 \end{pmatrix}$$

(<u>if</u> $c = 1$ <u>then</u> $e_1 = e$, $e_4 = ep^{\ell}$).

<u>Aside for possible poles at infinity</u> K_1, K_2, B_3, B_4 <u>are analytic on the set</u> (1.5).

<u>Proof</u>. - For $\ell = 1$ this is shown in [4], Chapter 24. The generalization to $\ell \neq 1$ is trivial.

We now recall the mod p type calculations of B.

1.9. LEMMA. - <u>There exists a</u> 2×2 <u>matrix</u> $\bar{B}$ <u>with coefficients in</u> $\underset{\sim}{F}_p[\lambda, \frac{1}{1-\lambda}]$ <u>such that for</u> $|\lambda| = 1 = |\lambda - 1|$

$$B(\lambda) \bmod p = \bar{B} \; (= \begin{pmatrix} \bar{B}_1 & \bar{B}_2 \\ \bar{B}_3 & \bar{B}_4 \end{pmatrix}) .$$

<u>Furthermore if</u> $\alpha_{\ell-1}^*$ <u>is of type</u> 1 (<u>resp. type</u> 2) <u>then the 2nd row</u> (<u>resp. the first row</u>) <u>of</u> $\bar{B}$ <u>is trivial and neither</u> B_1 <u>nor</u> B_2 (<u>resp. neither</u> $\bar{B}_3$ <u>nor</u> $\bar{B}_4$) <u>is trivial</u>.

<u>Proof</u>. - This is shown [4], (9.1.4), subject to the further conditions ([4] (6.6.4)). These last conditions were used in verifying [4], Theorem 6.6. These hypotheses are

eliminated in the calculation of [5], § 4.

1.10. COROLLARY. - $\lambda^{c(p^{\ell}-1)}$ <u>divides</u> $\bar{B}_1$ in $\underset{\sim}{F}_p[\lambda , \frac{1}{\lambda - 1}]$.

(It follows from the lemma that B is bounded by unity on the generic disk and hence the Taylor series expansion of $\bar{B}$ may be deduced from that of B by reduction mod p .)

We are now prepared to define supersingularity for the singular disk $D(0 , 1^-)$.

1.11. <u>Definition</u>. - We say that $D(0 , 1^-)$ is ordinary for $(1.1)_{(a,b,c)}$ if $\alpha^*_{\ell-1}$ is split of type 1 (resp. type 2) and $\bar{B}_1/\lambda^{c(p^{\ell}-1)}$ (resp. $\bar{B}_4$) not zero at $\lambda = 0$.

We recall ([4], Theorem 9.6), the Hasse domain, S_H , is the union of all residue classes ($\neq 0$, $\bar{1}$, ∞) such that $B_1(\lambda)$ (resp. $B_4(\lambda)$) is a unit. Aside from the trivial factor, our condition for $D(0 , 1^-)$ is formally of the same type. However we note that under our definition, by Lemma 1.6, $D(0 , 1^-)$ is ordinary if, and only if, the sequence $\alpha^*_0 , \dots , \alpha^*_{\ell-1}$ has at least one split step and all other split steps are of the same type.

We extend the symbol $H_0(S_H)$ and let $S'_H = S_H \cup D(0 , 1^-)$ if $D(0 , 1^-)$ is ordinary, and let $H_0(S'_H)$ denote the ring of analytic elements on S'_H which are bounded by unity.

1.12. THEOREM. - <u>If</u> $\alpha^*_{\ell-1}$ <u>is of type 1</u> (<u>resp.</u> <u>type 2</u>) <u>and if</u> $D(0 , 1^-)$ <u>is ordinary then the fixed point</u> η , (<u>resp.</u> $\bar{\eta}$) <u>in</u> $H_0(S_H)$ <u>of</u>

1.12.1 $$w \dashrightarrow \frac{B_2 + B_4 w^{\varphi^{\ell}}}{B_1 + B_3 w^{\varphi^{\ell}}} \quad \underline{\text{if}} \quad \alpha^*_{\ell-1} \quad \underline{\text{is of type 1}}$$

(<u>resp.</u> $$w \dashrightarrow \frac{B_3 + B_1 w^{\varphi^{\ell}}}{B_4 + B_2 w^{\varphi^{\ell}}} \quad \underline{\text{if}} \quad \alpha^*_{\ell-1} \quad \underline{\text{is of type 2}})$$

<u>extends to an element of</u> $H_0(S'_H)$ <u>and</u>

$$\bar{\eta}(0) = 0 \quad (\underline{\text{resp.}} \quad \eta(0) = (c - b)/c) .$$

<u>Proof</u>. - To fix ideas we restrict our attention to the type 1 case. By lemmas 1.8, 1.9,

(1.12.2) $$B = \begin{pmatrix} \lambda^{c(p^{\ell}-1)} & 0 \\ 0 & p \end{pmatrix} \begin{pmatrix} K_1 & \lambda K_2 \\ K_3 & K_4 \end{pmatrix}$$

where each K_2 and $1/K_1$ lie in $H_0(S_H^!)$.

Putting $w = \lambda\,\hat{w}$, we reduce the problem to that of finding the fixed point of

$$\hat{w} \dashrightarrow \frac{K_2 + pK_4\ \lambda^{(1-c)(p^{\ell}-1)}\ \hat{w}^{\varphi^{\ell}}}{K_1 + pK_3\ \lambda^{(1-c)(p^{\ell}-1)}\ \lambda\hat{w}^{\varphi^{\ell}}} \ .$$

This mapping is clearly contractive on $H_0(S_H^!)$ and so the asserted fixed point on $H_0(S_H^!)$ exists, clearly coinciding on S_H with the fixed point demonstrated in [4], Theorem 9.6. This completes the proof.

1.13. COROLLARY.

$$\eta = \frac{c-a}{1-c}\ \frac{\lambda\, F(a+1-c\ ,\ b+1-c\ ,\ 2-c\ ,\ \lambda)}{F(a-c\ ,\ b-c\ ,\ 1-c\ ,\ \lambda)} \ .$$

$$\left(\text{resp.}\quad \bar{\eta} = \frac{c-b}{c}\ \frac{F(a\ ,\ b\ ,\ 1+c\ ,\ \lambda)}{F(a\ ,\ b\ ,\ c\ ,\ \lambda)}\right) .$$

<u>Proof</u>. - We know that η (resp. $\bar{\eta}$) satisfies the condition that $(u_1\ ,\ \eta\, u_1)$ (resp. $(\bar{\eta}\, u_2\ ,\ u_2)$) is a solution of (1.1) for suitable u_1 (resp. u_2) (cf. [4], Theorem 9.6). Hence η (resp. $\bar{\eta}$) are solutions of a Riccati equation. Our formula for U gives us two solutions for this Riccati equation. To make sure that we have the correct solution it is enough to check the initial value. This completes the proof.

2. Normalized solution matrix.

Let $D(0\ ,\ 1^-)$ be ordinary for (1.1). To fix ideas, let $\alpha^*_{\ell-1}$ be of type 1. We know that $(u_1\ ,\ \eta u_1)$ is a solution of (1.1) with $\lambda^c\, u \in \mathcal{Q}[[\lambda]]$. In fact,

$$(2.1)\qquad (u_1\ ,\ \eta u_1)^{\varphi^{\ell}}\ B = e_4(u_1\ ,\ \eta u_1)\ ,$$

and by hypothesis e_4 is a unit. If follows from (1.12.2) and (2.1) that $\lambda^c\, u$ is bounded on $D(0\ ,\ 1^-)$.

We now define the normalized solution matrix

$$(2.2)\qquad Y = \begin{pmatrix} \lambda^{-c} & 0 \\ 0 & 1 \end{pmatrix} V(\lambda)$$

of (1.1) by the condition that V have coefficients in $K[[\lambda]]$, where K is an infinite unramified extension of $\mathcal{Q}_p$, and that

$$(2.2.1)\qquad Y^{\sigma^{\ell}\varphi^{\ell}}\ B(\lambda) = \begin{pmatrix} 1 & 0 \\ 0 & p^{\ell_0} \end{pmatrix} Y$$

and that

(2.2.2) $$\lim_{|\lambda|\to 1} |\det V(\lambda)| = 1$$

(2.2.2) $$\lim_{|\lambda|\to 1} |v_1(\lambda)| = 1 ,$$

v_1 being the first coefficient of the first row of V. Here σ denotes the Frobenius antomorphism of K over $\mathbb{Q}_p$ and (2.2.1) is equivalent to

(2.2.1)' $$V^{\sigma}(\lambda^{p^{\ell}})\, B(\lambda) = \begin{pmatrix} 1 & 0 \\ 0 & p^{\ell_0} \end{pmatrix} \begin{pmatrix} \lambda^{c(p^{\ell}-1)} & 0 \\ 0 & 1 \end{pmatrix} V(\lambda) .$$

As in [4], Chapter 9, we write

$$Y = \begin{pmatrix} 1 & 0 \\ \tau & 1 \end{pmatrix}\begin{pmatrix} u & 0 \\ 0 & \hat{u} \end{pmatrix}\begin{pmatrix} 1 & \eta \\ 0 & 1 \end{pmatrix}$$

so that

$$\tau = k\,\lambda^{c}\, \frac{F(a\ ,\ b\ ,\ 1+c\ ,\ \lambda)}{F(a-c\ ,\ b-c\ ,\ 1-c\ ,\ \lambda)}$$

$$u = k_1\, \lambda^{-c}\, F(a-c\ ,\ b-c\ ,\ 1-c\ ,\ \lambda)$$

$$\hat{u} = \hat{k}(1-\lambda)^{c-a-b}/F(a-c\ ,\ b-c\ ,\ 1-c\ ,\ \lambda)$$

where k , k_1 , $\hat{k} \in K$. More explicitly,

$$k_1^{1-\sigma^{\ell}} = e_4 .$$

$$k^{1-\sigma^{\ell}} = (e_1/p^{\ell_0})/e_4 .$$

Following the proof of [4], Theorem 9.6, we deduce :

2.3. THEOREM.

(2.3.1) $$u^{1-\varphi^{\ell}\sigma^{\ell}} = B_1 + B_3\, \eta^{\varphi^{\ell}} \in H_0(S_H)$$

(2.3.2) $$p^{\ell_0}\,\tau - \tau^{\sigma^{\ell}\varphi^{\ell}} = \frac{\hat{u}^{\sigma^{\ell}\varphi^{\ell}}}{u}\, B_3 .$$

<u>Using</u> (1.12.2) <u>and putting</u>

$$\eta = \lambda\hat{\eta}\ ,\quad u = \lambda^{-c}\, v_1\ ,\quad \tau = \lambda^{c}\, \tau_1\ ,$$

<u>we obtain</u>

$$(2.3.3) \qquad v_1^{1-\varphi^\ell\sigma^\ell} = K_1 + pK_3\, \lambda^{(p^\ell-1)(1-c)}\, \lambda\hat{u}^{\varphi^\ell}$$

$$(2.3.4) \qquad p^{\ell_0}\, \tau_1 - \lambda^{(p^\ell-1)c}\, \tau_1^{\sigma^\ell\varphi^\ell} = p\, \frac{\hat{u}^{\sigma^\ell\varphi^\ell}}{v_1}\, K_3 \ .$$

This congruence is the generalization of (1). There is a similar formula for the type 2 situation.

Note. - The evaluation at $\lambda = 1$ of the right side of (2.3.3) has been studied by KOBLITZ [6] and DIAMOND [1].

3. Relation between type 1 and type 3.

It follows from the symplectic relation ([4] (2.5.2)), that if Y is a solution matrix of $(1.1)_{(a,b,c)}$ then

$$(3.1) \qquad \tilde{Y} = Y^* \begin{pmatrix} (a-c)\lambda & 0 \\ 0 & (b-c)(1-\lambda) \end{pmatrix}^*$$

is a solution matrix of $(1.1)_{(1-a,1-b,1-c)}$. Here Y^* is the transpose of the inverse. It follows from [4], Theorem 4.7, that if we use $\tilde{B}$ to denote the matrix of § 1 with (a,b,c) replaced by $(1-a,1-b,1-c)$, and if

$$(3.2) \qquad Y^{\sigma^\ell\varphi^\ell}\, B = CY \qquad (C = \text{constant matrix})$$

then

$$(3.3) \qquad \tilde{Y}^{\sigma^\ell\varphi^\ell}\, \tilde{B} = p^{\ell_0}\, C^*\, \tilde{Y} \ .$$

Thus if

$$C = \begin{pmatrix} p^{\ell_1} & 0 \\ 0 & p^{\ell_2} \end{pmatrix} \quad \text{then} \quad p^{\ell_0}\, C^* = \begin{pmatrix} p^{\ell_2} & 0 \\ 0 & p^{\ell_1} \end{pmatrix} .$$

If (a,b,c) is of type 1 then $(1-a,1-b,1-c)$ is of type 2 and this shows how we may pass from one situation ot the other.

4. Supersingular - Singular disk.

We justify our definition of ordinary.

LEMMA. - If $\alpha^*_{\ell-1}$ is a split step but $D(0\ ,\ 1^-)$ is not ordinary, then $(1.1)_{(a,b,c)}$ has no solution bounded on $D(0\ ,\ 1^-)$.

Proof. - By hypothesis $c \neq 1$. It is enough to show that each row of the matrix U is unbounded on $D(0\ ,\ 1^-)$. We use Lemma 1.6 and, to fix ideas, we let $\alpha^*_{\ell-1}$ be of type 1, and let $(u_1\ ,\ u_2)$ be the first row. We assume $u_1\ ,\ u_2$ bounded.

We have

$$(4.1) \qquad e_1(u_1\ ,\ u_2) = (u_1\ ,\ u_2)^{\varphi^{\ell}} \begin{pmatrix} \lambda^{c(p^{\ell}-1)}\ K_1\ , & \lambda^{1+c(p^{\ell}-1)}\ K_2 \\ pK_3\ , & pK_4 \end{pmatrix}.$$

The important point is that $|e_1| \lesssim |p|$, while $K_1(\lambda)$ takes on unit values. Thus for $|\lambda|$ very close to 1 , $e_1\ u_1$ is dominated by $\lambda^{c(p^{\ell}-1)}\ u_1^{\varphi^{\ell}}$ and hence we must have

$$(4.2) \qquad |u_1^{\varphi^{\ell}}\ \lambda^{c(p^{\ell}-1)}\ K_1| = |pu_2^{\varphi^{\ell}}\ K_3|$$

for such values of λ . This shows that, in the boundary norm,

$$|u_1|_{\mathrm{bdy}} < |p|\ \ |u_2|_{\mathrm{bdy}}\ .$$

Putting $\eta = u_2/u_1$, we obtain a solution of the Riccati equation.

$$(4.3) \qquad \frac{d\eta}{d\lambda} = (1\ ,\ \eta) \begin{pmatrix} \dfrac{-c}{\lambda} & \dfrac{c-a}{1-\lambda} \\ \dfrac{c-b}{\lambda} & \dfrac{a+b-c}{1-\lambda} \end{pmatrix} \begin{pmatrix} 1 \\ \eta \end{pmatrix}$$

where η is meromorphic, bounded on $D(0\ ,\ 1^-)$ with boundary norm greater than $1/|p|$. Clearly the term $\dfrac{a+b-c}{1-\lambda}\ \eta^2$ dominates all the other terms and this contradicts the assertion.

The same argument is valid for the second row of U as e_4 is not a unit.

5. Canonical lifting.

We again assume that $D(0\ ,\ 1^-)$ is ordinary and that $(1.1)_{(a,b,c)}$ is type 1. We then know [4], Chapter 13, that there exists a canonical lifting of Frobenius,

$$\varphi_{\ell} = \varphi^{\ell} + q$$

such that equation (2.3.4) takes the form

$$(5.1) \qquad p^{\ell_0}\ \lambda^c\ \tau_1 = (\varphi_{\ell}(\lambda))^c\ \ \tau_1^{\sigma^{\ell}}(\varphi_{\ell}(\lambda))\ .$$

The important point here is that we know that φ_{ℓ} is defined on an annulus $\epsilon < |\lambda| < 1$, but do not know if φ_{ℓ} extends to the disk $D(0\ ,\ 1^-)$.

Our basic relation [4], equation (13.3.21"), does not help here as the matrices M_s have poles at $\lambda = 0$. We may eliminate this pole by a change in variable. If $c = n/m$, $(n\ ,\ m) = 1$, and we put $\lambda = z^m$

$$(5.2) \qquad v = u \begin{pmatrix} z^n & 0 \\ 0 & 1 \end{pmatrix}$$

then $(1.1)_{(a,b,c)}$ takes the form

$$(5.3)_{(a,b,c)} \qquad \frac{dv}{dz} = v \begin{pmatrix} 0 & (c-a)\, mz^{m-n-1}/(1-z^m) \\ (c-b)\, mz^{n-1} & mz^{m-1}(a+b-c)/(1-z^m) \end{pmatrix}$$

which shows that the pole at $z = 0$ has been removed. The Frobenius matrix for (5.3) is given by

$$(5.4) \qquad \hat{B} = \begin{pmatrix} z^{-np^{\ell}} & 0 \\ 0 & 1 \end{pmatrix} B \begin{pmatrix} z^n & 0 \\ 0 & 1 \end{pmatrix}$$

and using equation (1.12.2) we obtain

$$(5.5) \qquad \hat{B} = \begin{pmatrix} K_1 & z^{m-n} K_2 \\ p z^n K_3 & pK_4 \end{pmatrix}.$$

The canonical lifting $\hat{\varphi}_{\ell}$ for z is then given by

$$(5.6) \quad 0 = pz^n \frac{K_3}{K_1} + \hat{q}\,(-(c-b)\, mz^{p^{\ell}(n-1)}) + \sum_{s=2}^{\infty} \hat{q}^s \, (- \hat{M}^{\varphi^{\ell}}_{s,3} + pz^n \frac{K_3}{K_1} \hat{M}^{\varphi^{\ell}}_{s,1}) ,$$

where

$$\hat{M}_s = \begin{pmatrix} \hat{M}_{s,1} & \hat{M}_{s,2} \\ \hat{M}_{s,3} & \hat{M}_{s,4} \end{pmatrix}$$

is defined starting with (5.3) by the equation

$$(5.7) \qquad \frac{1}{s\,!} \frac{d^s}{dz^s} v = v \hat{M}_s .$$

We see no reason to believe that $\hat{\varphi}$ is defined by (5.6) on the punctured disk.

If however $\ell = 1$, $c = 1$ then equation (2.3.4) takes the form

$$(5.8) \qquad \tau = \log \lambda + \tau_1 , \quad \tau_1 \in K[[\lambda]] .$$

$$p\tau_1 \equiv \tau_1^{\sigma}(\lambda^p) \mod p\, 0_K[[\lambda]] ,$$

$0_K =$ ring of integers of K .

We deduce that

$$(5.9) \qquad e^{\tau_1(\lambda)-\tau_1(0)} \in \lambda\, 0_k[[\lambda]]$$

and that $e^{\tau_1(0)} \in 1 + p\, O_K$. This shows that

$$e^{\tau(\lambda)} = \lambda(\rho_0 + \rho_1 \lambda + \dots) \tag{5.10}$$

where ρ_0 is a unit and each $\rho_j \in O_k$.

Letting $y(\lambda) = \rho_0 \lambda + \rho_1 \lambda^2 + \dots$ the condition for the canonical lifting φ_1 now take the form

$$y(\varphi_1(\lambda)) = y(\lambda)^p \ . \tag{5.11}$$

It is clear that this relation defines φ_1 on the disk $D(0\ ,\ 1^-)$.

REFERENCES

[1] DIAMOND (J.). - Hypergeometric series with a p-adic variable, Pacific J. of Math., t. 94, 1981, p. 265-276.

[2] DWORK (B.). - p-adic cycles. - Paris, Presses universitaires de France, 1969 (Institut des hautes Etudes scientifiques, Publications mathématiques, 37, p. 27-116.

[3] DWORK (B.). - On p-adic differential equations, IV : Generalized hypergeometric functions on p-adic analytic functions in one variable, Ann. scient. Ec. Norm. Sup., 4e série, t. 6, 1973, p. 295-316.

[4] DWORK (B.). - Lectures on p-adic differential equations. - New York, Heidelberg, Berlin, Springer-Verlag, 1982 (Grundlehren der mathematischen Wissenschaften, 253).

[5] DWORK (B.). - On the Boyarsky principle, Amer J. of Math., t. 105, 1983, p. 115-156.

[6] KOBLITZ (N.). - The hypergeometric function with p-adic parameters, "Proceedings of the Queen's number theory conference", 1979, p. 319-328. - Kingston, Queen University, 1980 (Queen's paper in pure and applied Mathematics, 54).