Groupe de travail d'analyse ultramétrique

Bernard M. Dwork

Puiseux expansions

Groupe de travail d'analyse ultramétrique, tome 10, nº 2 (1982-1983), exp. nº 14, p. 1-6
<http://www.numdam.org/item?id=GAU_1982-1983__10_2_A1_0>

Groupe d'étude d'Analyse ultramétrique
(Y. AMICE, G. CHRISTOL, P. ROBBA)
10e année, 1982/83, n° 14, 6 p.

11 avril 1983

PUISEUX EXPANSIONS

by Bernard M. DWORK (*)

The object of this note is to discuss p-adic convergence of Puiseux expansions of algebraic functions. We shall review joint work [D-R] with ROBBA on this question and shall discuss the problem of lifting Puiseux expansions in characteristic p.

Notation.

K = field of characteristic zero complete under a discrete nonarchimedean valuation with residue class field of characteristic p.

k = residue class field of K.

$\mathcal{O}$ = ring of integers of K.

$R = \mathcal{O}[x]$, $\bar{R} = k[x]$.

E = completion of $K(x)$ under the Gauss norm.

$\hat{R} = \mathcal{O}[[x]]$, $\hat{\bar{R}} = k[[x]]$.

$\hat{E}$ = quotient field of completion of $\hat{R}$ under the sup norm on $D(0, 1^-)$.

An element $\xi \in K((x^{1/m}))$ will be said to "converge" in $D(x, r^-)$ if for suitable $N \in \mathbb{N}$, $x^N \xi(x^m)$ is a power series converging in $D(0, (r^m)^-)$.

A series $\xi(x) = \sum_{j=-\infty}^{\infty} A_j x^{j/m}$ will be said to be a Puiseux Laurent series "convergent au bord" if $\xi(x^m)$ converges in an annulus $\Delta_{r,1} = \{x ; r < |x| < 1\}$.

Let $f \in R[y]$, $\bar{f}$ its image in $\bar{R}[y]$ under the natural mapping. We say that $\xi \in K((x^{1/m}))$ (resp. $k((x^{1/m}))$) is a Puiseux expansion for f (resp. $\bar{f}$), if $f(x, \xi)$ (resp. $\bar{f}(x, \xi)$) = 0.

We refer to the union of the zeros of the discriminant and the zeros of the leading coefficient as the singular locus of f.

We consider two questions :

Question I : Let ξ be a Puiseux expansion for f. Does ξ converge in $D(0, 1^-)$?

Question II : Let $\bar{\xi}$ be a Puiseux expansion for $\bar{f}$. Can $\bar{\xi}$ be lifted to a Puiseux expansion for f ?

(*) Bernard M. DWORK, Mathematical Department, Fine Hall, Princeton University, PRINCETON, NJ 08540 (États-Unis).

We observe that liftability implies not only convergence on $D(0\ ,\ 1^-)$ of the lifted expansion but also boundedness by unity.

It is clear that if $\deg_y f = n = \deg_y \bar{f}$ and if $\bar{f}$ has n distinct Puiseux expansions, and if the answer to II is affirmative, then the answer to I is also affirmative.

We shall have occasion to consider various conditions :

(H_1): The valuation induced on $K(x)$ by the Gauss norm is at worst tamely ramified in the splitting field of f.

(H_2): The singular locus of f has no element in the punctured disk $D(0,1^-)-\{0\}$.

(H_3): $\bar{f}$ and $\bar{f}_y$ $(=\frac{\partial \bar{f}}{\partial y})$ have no common factor in $\bar{R}[y]$.

(H_4): $\deg_y f = \deg_y \bar{f}$.

THEOREM [D-R]. - Assume (H_1), (H_2), then Question I has an affirmative response.

For proof see [D-R]. The condition (H_2) is clearly necessary.

Example. - $f = y^3 - x(x+p)$.

A Puiseux expansion $x^{1/3} \sum_{j=0}^{\infty} A_j\, x^{j/3}$ satisfying I would imply

$$\sum A_j\, x^j = (x^3 + p)^{1/3}\ ,$$

and differentiating shows that $x^2/(x^3+p)^{2/3}$ is analytic in $D(0\ ,\ 1^-)$, which is impossible.

Remark. - We attributed to HIRONAKA [Dw] the statement (for $f \in \mathbb{Z}[x\ ,\ y]$).

(F 1) if f is irreducible over $\mathbb{Q}(x)$, and if the discriminant of f and of $\bar{f}$ have the same degrees as polynomials in x, then the Puiseux expansions of $\bar{f}$ lift to Puiseux expansions of f.

This example demonstrates the inaccuracy of (F 1).

LEMMA. - Assume (H_3). We conclude that each Puiseux expansion $\bar{\eta}$ of $\bar{f}$ has a lifting to a Puiseux Laurent series for f "convergent au bord".

Note. - We do not assume (H_2). We do not affirm a positive response to Question II.

Proof. - We may assume that $\bar{\eta} \in K[[x]]$. Let $\hat{\eta}$ be a lifting of $\bar{\eta}$ in $\mathcal{O}[[x]]$. Hence letting (π) be the prime ideal of K,

$$f(x\ ,\ \hat{\eta}) \in \pi\, \mathcal{O}[[x]] \ \text{ while }\ f_y(x\ ,\ \hat{\eta}) \equiv \bar{f}_y(x\ ,\ \bar{\eta}) \neq 0\ ,$$

i. e. $f_y(x, \hat{\eta})$ is an element of $\mathcal{O}[[x]]$ with at least one unit coefficient.

Choose a positive real θ, $1 > \theta > |\Pi|$, then there exists $r < 1$ such that

$$|f_y(x, \hat{\eta})| > (|\Pi|/\theta)^{1/2}, \quad \forall x \in \Delta_{r,1}$$

while

$$|f(x, \hat{\eta})| \leqslant |\Pi|, \quad \forall x \in D(0, 1^-).$$

We now put $y = \hat{\eta} + w$, so

$$f(x, y) = f(x, \hat{\eta}) + wf_y(x, \hat{\eta}) + \sum_{j=2}^{\infty} w^j/j!\, f^{(j)}(\hat{\eta})$$

and then put $w = zf(x, \hat{\eta})/f_y(x, \hat{\eta})$, so that

$$(1) \quad \frac{f(x, y)}{f(x, \hat{\eta})} = 1 + z + \sum_{j=2}^{\infty} z^j \frac{f^{(j)}(\hat{\eta})}{j!} \frac{f(x, \hat{\eta})^{j-1}}{f_y(x, \hat{\eta})^j} = 1 + z + A_2 z^2 + \dots$$

where

$$A_j = \frac{f^{(j)}(\hat{\eta})}{j!} \frac{f(x, \hat{\eta})^{j-1}}{f_y(x, \hat{\eta})^j}, \quad j = 2, 3, \dots$$

Thus on $\Delta_{r,1}$, A_j is bounded by $\theta < 1$ and hence $z \longmapsto -1 - \sum_{j=2} A_j z^j$ is a contractive map on the space of functions analytic and bounded by unity on $\Delta_{r,1}$. It is clear that the unique fixed point z_0 then gives a solution of (1) by setting $\eta = \hat{\eta} + z_0 f(x, \hat{\eta})/f_y(x, \hat{\eta})$, that η converges on $\Delta_{r,1}$ and that the Laurent series $\sum_{j=-\infty}^{\infty} B_j x^j$ representing $z_0 f(x, \hat{\eta})/f_y(x, \hat{\eta})$ is bounded by $(|\Pi|/\theta)^{1/2} < 1$, and hence

$$|B_j| < 1 \quad j \geqslant 0$$

$$|B_j r^j| < 1 \quad j < 0$$

which shows that B_j has zero image in K, i. e. η is a lifting of $\bar{\eta}$ as asserted.

COROLLARY 1. - <u>Assume</u> (H_1), (H_2), (H_3), <u>then question</u> II <u>has an affirmative response</u>.

COROLLARY 2. - <u>Assume</u> (H_2), (H_3), (H_4), <u>then question</u> II <u>has an affirmative response</u>.

<u>Proof</u>. - Assumptions (H_3), (H_4) imply (H_1) and hence the first corollary implies the second. The theorem shows that f has a full set of Puiseux expansions converging in $D(0, 1^-)$. If $\bar{\xi}$ is a Puiseux expansion of $\bar{f}$ then by the lemma $\bar{\xi}$ has a Laurent series lifting, ξ, "convergent au bord". This ξ must coincide with one of the previously mentioned solutions and so converges in $D(0, 1^-)$.

We now disprove :

(F_2) <u>Assumptions</u> (H_2), (H_3) <u>imply question</u> II <u>has an affirmative solution</u>.

To construct a counter-example, it seems useful to consider a polynomial not satisfying (H_1). For this reason, we consider $y^{p+1} + xy + p$ which over E has factors $f_1 \equiv y^p + x \bmod p$, and f_2 of degree 1 in y. It is more convenient to write $y = pz$ and so consider

$$p^p\, z^{p+1} + xz + 1 = f(z)\ .$$

Mod p, we have the solution $z = - x^{-1}$ which clearly cannot lift to a Puiseux expansion at $x = 0$ in characteristic zero since in that characteristic $x = 0$ is not a singularity. Trivially (H_3) is satisfied. To check (H_2), we must compute the discriminant. We recall that for $y^N + A_y + B = g(y)$, the discriminant is

$$(-1)^{\binom{N+1}{2}}\, [(-1)^N\, N^N\, B^{N-1} - (N-1)^{N-1}\, A^N]\ .$$

Thus, for $z^{p+1} + (x/p^p)\, z + (1/p^p)$, the discriminant is

$$\pm\, [(-1)^{p+1}\, (p+1)^{p+1}\, (\tfrac{1}{p^p})^p - p^p(\tfrac{x}{p^p})^{p+1}] = \pm\, p^{-p^2}\, [(-1)^{p+1}\, (p+1)^{p+1} - x^{p+1}]\ ,$$

i. e. the zeros are outside of $D(0\ ,\ 1^-)$.

We now discuss in detail a well known example.

(E)
$$f(y) = y^p - y - \frac{1}{x}\ .$$

Here the discriminant is given by our previous formula to be

$$(-1)^{p(p+3)/2}\, [p^p\, (-\tfrac{1}{x})^{p-1} - (p-1)^{p-1}]\ .$$

Hence the singular locus consists of

$$\{p^{p/(p-1)}\, \frac{\omega}{1-p}\}_{\omega^p=\omega}$$

a set of p points.

(E 1) <u>There are no Puiseux expansions at</u> $x = 0$ <u>in characteristic</u> p .

<u>Proof</u>. - If $\mathfrak{p}$ is the prime above $x = 0$, then $\mathrm{ord}_{\mathfrak{p}}\, y = - 1/p$.

Hence a Puiseux expansion, if it exists, must be of the form

(1.1)
$$y = A_{-1}\, z^{-1} + A_0 + A_1\, z + \dots\ ,\quad z = x^{1/p}\ ,\ \text{with all } A_j \text{ in } \underset{\sim}{F}_p\ ,$$

but then

$$y = y^p - \frac{1}{x} = \Sigma_{j=-1}^{\infty}\, A_j^p\, z^{pj} - \frac{1}{x} \in k((x))$$

a contradiction as is well known [Ch], (p. 64).

(E 2) No Puiseux expansion at zero in characteristic zero converges for $|x| \geqslant |p|^{p/p-1} = r_0$.

Proof. - A Puiseux expansion convergent for $x \in D(0, r^-) - \{0\}$ means that we obtain an element of $K((z))$ with $z = x^{1/p}$ which converges for $0 < |x| < r$. For $r \lesssim 1$, we then have point wise

$$|y^p - y| = |x|^{-1} > 1 \text{ , and so } |y^p| > |y| > 1 \text{ .}$$

Thus $|y| = |x|^{-1/p}$ point wise, and so if $r > r_0$, then y^{p-1} will assume the value $1/p$ at suitable values of x such that $|x| = r_0$.

Since $dy/dx(py^{p-1} - 1) = - 1/x^2$, and since

$$dy/dx = (dy/dz)/(dx/dz) = (1/pz^{p-1})\, dy/dz$$

is analytic as function of z for $x \in D(0, r^-) - \{0\}$, we obtain a contradiction if $r > r_0$. The same analysis shows that convergence for $|x| = r_0$ is also impossible.

(E 3) There are p distinct Puiseux expansions at infinity in characteristic p .

Proof. - Let $\bar{y}_0 = -\frac{1}{z} - (\frac{1}{z})^p - (\frac{1}{z})^{p^2} - \dots$, then $\bar{y}_0^p - \bar{y}_0 = 1/z$.

The p solutions are $\{\bar{y}_0 + a\}_{0 \leqslant a < p}$.

(E 4) The p distincts Puiseux expansions at infinity (in characteristic zero) converge and are bounded by unity for $|z| > 1$, but do not converge for $|z| = 1$.

Proof. - At $x = \infty$ condition (H_2) is satisfied. The global conditions (H_3), (H_4) are also satisfied. Hence by the corollary the Puiseux expansions in characteristic p may be lifted. These then are $\{y_a\}_{a=0,1,\dots,p-1}$ where $y_a \pmod p = a + \bar{y}_0$.

This shows that $y_a \in Z_p[[1/z]]$, but if we write

$$y_a = \sum_{j=0}^{\infty} B_j \frac{1}{z^j} \text{ ,}$$

$|B_j| = 1$ for an infinite set of j . This shows that the domain of convergence is precisely $|z| > 1$.

This concludes our discussion of the example.

Generalizations.

1°. Let $f \in \hat{R}[y]$. The theorem and the lemma generalize replacing (H_2) by H_2' , (H_3) by (H_3') as indicated below.

(H_2') : The valuation induced on the quotient field of $\mathcal{O}[[x]]$ by the Gauss norm is at worst tamely ramified in the splitting field of f

(H_3') : $\bar{f}$ <u>and</u> $\bar{f}_y$ <u>have no common factor in</u> $\hat{R}[y]$.

2°. Let G be an $n \times n$ matrix with coefficients in the quotient field of $\hat{R}$. We assume that the differential equation $dy/dx = Gy$ has no singularity in $D(0\ ,\ 1^-)$ except for a regular singularity at $x = 0$ with rational exponents. We assume that at the generic point t (in the sense of ROBBA [Ro 1]) the equation has n independent solutions bounded and analytic on $D(t\ ,\ 1^-)$. We conclude that the solution matrix at the origin is of the form $D\,x^H$ where H is a constant diagonal matrix and D is a bounded matrix converging on $D(0\ ,\ 1^-)$.

The proof is omitted since it is so close to that of [D-R]. The key point is that the argument of ROBBA [Ro 2] shows that the hypothesis of boundedness on the generic disk implies the semi-simplicity of H .

REFERENCES

[D-R] DWORK (B.). - On natural radii of p-adic convergence, Trans. Amer. math. Soc., t. 256, 1979, p. 199-213.

[Dw] DWORK (B.). - Arithmetic theory of differential equations, Symposia Mathematica, vol. 24, p. 225-243. - Roma, Istituto nazionale di Alta Matematica ; London, Academic Press, 1981.

[Ch] CHEVALLEY (C.). - Introduction to the theory of algebraic functions of one variable. - New York, American mathematical Society, 1951 (Mathematical Surveys, 6).

[Ro 1] ROBBA (P.). - Index of p-adic differential operators, II, Pacific J. of Math., t. 96, 1982, p. 393-418.

[Ro 2] ROBBA (P.). - Solutions bornées des systèmes différentiels linéaires, Groupe d'étude d'Analyse ultramétrique, 3e année, 1975/76, n° 5, 16 p.