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# DIFFERENTIAL EQUATIONS WHICH COME FROM GEOMETRY

by Bernard DWORK (*)

## 1. Introduction.

Let $k$ be an algebraic number field, and $L$ a differential operator in $k(x)[D]$,

$$L = D^n - B_{n-1} D^{n-1} - B_{n-2} D^{n-2} - \dots - B_0 ,$$

where each $B_j \in k(x)$ , $D = d/dx$ . For each valuation $\mathfrak{p}$ of $k$ we define $r(\mathfrak{p})$ to be the radius of the maximal disk of $\mathfrak{p}$-adic convergence of all the solutions of $L$ at the $\mathfrak{p}$-adic generic point $t$ . This means that if $b$ lies in any residue class (with a finite number of exceptions) in the algebraic closure of the $\mathfrak{p}$-adic completion of $k$ then the solutions of $L$ at $b$ converge in the disk $D(b, r(\mathfrak{p})^-)$.

We recall the conjecture of Grothendieck [Ka 2]. For almost all waluations $\mathfrak{p}$ we may reduce the coefficients of $L$ modulo $\mathfrak{p}$ and obtain an operator $L_{\mathfrak{p}}$ with coefficients in $\bar{k}_{\mathfrak{p}}(x)$ , $\bar{k}_{\mathfrak{p}}$ being the residue class field of $k$ at $\mathfrak{p}$ . For such $\mathfrak{p}$ we view $L_{\mathfrak{p}}$ as $\bar{k}_{\mathfrak{p}}(x^p)$ linear operator on $\bar{k}_{\mathfrak{p}}(x)$ and let $V_{\mathfrak{p}}$ denote the kernel.

Conjecture of Grothendieck. - If

$$\dim_{\bar{k}_{\mathfrak{p}}(x^p)} V_{\mathfrak{p}} = n \quad \text{for almost all valuations } \mathfrak{p} \text{ of } k$$

then all the solutions of the differential equation $Ly = 0$ are algebraic functions.

A weaker form of Grothendieck's conjecture, adequate for the present work, may be stated :

Conjecture G' . - If for almost all $\mathfrak{p}$ there exists a residue class $C_{\mathfrak{p}}$ such that there are n-solutions $u_1 , u_2 , \dots , u_n$ of $L$ which converge and are bounded by unity on $C_{\mathfrak{p}}$ and such that the wronskian, $\det(u_j^{(i)})$ , assumes only unit values on $C_{\mathfrak{p}}$ then all the solutions of $L$ are algebraic functions.

In our application we will take $C_{\mathfrak{p}}$ to be the generic $\mathfrak{p}$-adic residue class.

Guided by our own results [Ka 1], and those of Katz [Ka 2], we say that $L$ is DFG ("derived from geometry") if

(1) $r(\mathfrak{p}) = 1$ for almost all $\mathfrak{p}$ .

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(2) Grothendieck's conjecture is known to be true.

In particular, it is known that the hypergeometric differential operator

$$L = x(1-x)\, D^2 + (c-(a+b+1)x)\, D - ab$$

with $a, b, c \in \mathbb{Q}$ is DFG. Property 2 being known from [Ka 2] while Property 1 may be deduced for either [Ka 1] or [Dw] or from [D-R 2] (Theorem 8.6).

The object of this note is to present some evidence in support of the following conjecture :

Conjecture $D_n$. - Let $L$ be a DFG differential operator of order $n$, irreducible over $k(x)$ with a solution $w$ such that $w, w', \dots, w^{(n-1)}$ are algebraically dependent over $k(x)$. Then all the solutions of $L$ are algebraic functions.

We prove the conjecture for the case $n = 2$.

This work is based upon correspondence with Fritz BEUKERS who considered this question from a different point of view. The proof of $D_2$ in § 3 has been simplified with the help of N. KATZ.

## 2. Inhomogeneous relations.

Let $\mathcal{E}$ be a differential field of characteristic zero with $D = d/dx$ as differential operator. Let $n \geq 2$, $L \in \mathcal{E}[D]$.

2.1 $$L = D^n - \sum_{i=0}^{n-1} B_i D^i, \quad B_i \in \mathcal{E}, \quad 0 \leq i < n.$$

We generalize a result stated by SIEGEL [S] (page 60) for the case $n = 2$.

2.2. LEMMA. - Let $\hat{\mathcal{E}}$ be a differential extension field of $\mathcal{E}$ with algebraically closed field of constants $C$ and let $K$ be the kernel of $L$ in $\hat{\mathcal{E}}$. We assume

2.2.1 $$\dim_C K \geq 2.$$

2.2.2. - There exists a non-trivial $w \in K$ such that $w, w', \dots, w^{(n-1)}$ are algebraically dependent over $\mathcal{E}$.

We assert the existence of non-trivial $u \in K$ such that $u, u', \dots, u^{(n-1)}$ satisfies a non-trivial homogeneous relation over the composition $C\mathcal{E}$ in $\hat{\mathcal{E}}$ of $C$ with $\mathcal{E}$.

Proof. - We may assume that $w$ satisfies no non-trivial homogeneous relation over $\mathcal{E}$ and that $w_1, w_2$ span over $C$ a two dimensional subspace, $K_2$, of $K$. We will find $u \in K_2$ satisfying the conclusion of the lemma.

For $Q \in \mathcal{E}[y_0, \dots, y_{n-1}]$, a polynomial in $n$ variables with coefficients in $\mathcal{E}$, we define

2.3 $$Q^* = Q_x + Q_{y_0} y_1 + \dots + Q_{y_{n-2}} y_{n-1} + Q_{y_{n-1}} \textstyle\sum_{i=0}^{n-1} B_i y_i .$$

Thus for $w \in K$ ;

2.4 $$\frac{d}{dx} Q(v , v' , \dots , v^{(n-1)}) = Q^*(v , v' , \dots , v^{(n-1)}) .$$

Let $\mathfrak{U}$ be the ideal of all $P \in \mathfrak{L}[y_0 , \dots , y_{n-1}]$ such that

$$P(w , w' , \dots , w^{(n-1)}) = 0 .$$

By hypothesis $\mathfrak{U} \neq \{0\}$ . Let $P$ be a non trivial element of $\mathfrak{U}$ which is minimal in the sense that the difference between the degrees of the different homogeneous parts is as small as possible. Explicitly if $P_j$ is the homogeneous part of $P$ of degree $j$ then

$$P = P_a + P_{a+1} + \dots + P_b$$

where $P_a$ and $P_b$ are non-trivial with $b - a$ minimal. By hypothesis $P$ cannot be a form and so $b \neq a$ . Equation 2.4 shows that $P^* \in \mathfrak{U}$ and hence

$$\mathfrak{U} \ni P_a P^* - P_a^* P = \textstyle\sum_{i=a+1} (P_a P_i^* - P_a^* P_i) .$$

Since $Q \longmapsto Q^*$ is a degree preserving mapping of forms, the minimality of $b - a$ shows that $P_a P_i^* - P_a^* P_i = 0$ for $i = a + 1 , \dots , b$ and so in particular

2.5 $$P_a P_b^* - P_a^* P_b = 0 .$$

We may assume that $P_b(v)$ $(\stackrel{def}{=} P_b(v , v' , \dots , v^{(n-1)}) \neq 0$ for each non-trivial $v \in K_2$ and so from 2.5 we obtain

2.6 $$\frac{d}{dx} (P_a(v)/P_b(v)) = 0 .$$

Thus for $(\lambda_1 , \lambda_2) \in C^2 - (0 , 0)$ we conclude that

2.7 $$P_a(\lambda_1 w_1 + \lambda_2 w_2) = f(\lambda_1 , \lambda_2) P_b(\lambda_1 w_1 + \lambda_2 w_2)$$

where $f(\lambda_1 , \lambda_2) \in C$ . Letting $\{w_\alpha\}_{\alpha\in I}$ be a basis of $\hat{\mathfrak{L}}$ over $C$ , we may write

$$P_a(\lambda_1 w_1 + \lambda_2 w_2) = \textstyle\sum_{\alpha\in I} w_\alpha P_{a,\alpha}(\lambda_1 , \lambda_2)$$

$$P_b(\lambda_1 w_1 + \lambda_2 w_2) = \textstyle\sum_{\alpha\in I} w_\alpha P_{b,\alpha}(\lambda_1 , \lambda_2)$$

where $P_{a,\alpha}$ (resp. $P_{b,\alpha}$) is a form in $C[\lambda_1 , \lambda_2]$ of degree $a$ (resp. $b$) it being understood that the zero form has all degrees. We write 2.7 in the form

$$\textstyle\sum_{\alpha\in I} w_\alpha(P_{a,\alpha}(\lambda_1 , \lambda_2) - f(\lambda_1 , \lambda_2) P_{b,\alpha}(\lambda_1 , \lambda_2)) = 0$$

for all $(\lambda_1 , \lambda_2) \in C^2 - (0 , 0)$ and so

2.8 $$P_{a,\alpha}(\lambda_1 , \lambda_2) - f(\lambda_1 , \lambda_2)\, P_{b,\alpha}(\lambda_1 , \lambda_2) = 0$$

for all $(\lambda_1 , \lambda_2) \in C^2 - (0 , 0)$ . We choose $\alpha$ such that $P_{b,\alpha}$ is not the trivial form and conclude

$$\frac{P_a(\lambda_1\, w_1 + \lambda_2\, w_2)}{P_b(\lambda_1\, w_1 + \lambda_2\, w_2)} = \frac{P_{a,\alpha}(\lambda_1 , \lambda_2)}{P_{b,\alpha}(\lambda_1 , \lambda_2)}$$

for all $(\lambda_1 , \lambda_2) \in C^2$ such that

$$P_{b,\alpha}(\lambda_1 , \lambda_2) \neq 0 .$$

Under this condition we have

2.9 $$P_a(\lambda_1\, w_1 + \lambda_2\, w_2)\, P_{b,\alpha}(\lambda_1 , \lambda_2) - P_b(\lambda_1\, w_1 + \lambda_2\, w_2)\, P_{a,\alpha}(\lambda_1 , \lambda_2) = 0$$

which shows that the left side must be identically zero as element of $\hat{\mathfrak{L}}[\lambda_1 , \lambda_2]$ . Dividing $P_{b,\alpha}$ and $P_{a,\alpha}$ by their greatest common divisor we may assume them to be relatively prime forms in $C[\lambda_1 , \lambda_2]$ . Since $b > a$ , and $C$ is algebraically closed we may choose $(\lambda_1 , \lambda_2)$ in $C^2$ such that

$$P_{b,\alpha}(\lambda_1 , \lambda_2) = 0 , \quad P_{a,\alpha}(\lambda_1 , \lambda_2) \neq 0 .$$

Then putting $u = \lambda_1\, w_1 + \lambda_2\, w_2 \in K_2$ , we see from 2.9 that $P_b(u) = 0$ . This contradiction completes the proof.

## 3. Proof of Conjecture $D_2$ .

THEOREM. - Let $L \in k(x)[D]$ be a second order DFG differential operator irreducible over $k(x)$ with non-trivial solution $w$ such that $w , w'$ are algebraically dependent over $k(x)$ . Then all the solutions of $L$ are algebraic functions.

Proof. - By § 2 there exists a non-trivial solution $u$ of $L$ such that $u , u'$ satisfy a homogeneous relation over $C(x)$ where $C$ is a constant field extension of $k$ .

Hence $u'/u = \eta$ is an algebraic function. Thus $\eta$ is a solution of the Riccati equation associated with $L$ . Since $L$ is irreducible over $k(x)$ , it is also irreducible over $C(x)$ and hence $\eta \notin C(x)$ . Thus there exists a distinct conjugate $\eta_2$ of $\eta = \eta_1$ over $C(x)$ which is again a solution of the Riccati equation of $\mathfrak{L}$ . We extend each valuation $\mathfrak{p}$ of $k$ to $C$ and by [D-R 1] for almost all $\mathfrak{p}$ the branches of $\eta$ at the generic point $t_{\mathfrak{p}}$ are analytic in $D(t_{\mathfrak{p}} , 1^-)$ . Let $u_i$ denote a solution at $t_{\mathfrak{p}}$ of the equation

$$\begin{aligned} u_i' &= u_i \eta_i \\ & \qquad\qquad i = 1 , 2 \\ u_i(t_\wp) &= 1 \end{aligned}$$

$\eta_1$ , $\eta_2$ being two distinct branches of $\eta$ at $t_\wp$ . Since $u_i$ is a solution at $t_\wp$ of $L$ , we know (since $L$ is DFG) that (excluding a finite set of $\wp$ ) $u_i$ converges in $D(t_\wp , 1^-)$ and by the corresponding property of $u_i'/u_i = \eta_i$ , we conclude that $u_i$ is never zero on this disk, and hence for $x$ in this disk we have

$$|u_i(x)| = |u_i(t_\wp)| = 1 .$$

On the other hand the wronskian, $u_1 u_2' - u_1 u_2' = u_1 u_2(\eta_2 - \eta_1)$ assumes only unit values on this disk for almost all $\wp$ since $\eta_2 - \eta_1$ is a branch at $t_\wp$ of an algebraic function defined over $C(x)$ . This shows that $L$ satisfies the hypothesis of Conjecture G' and since $L$ is DFG, we conclude that all solutions of $L$ are algebraic function. This completes the proof.

## 4. Homogeneous solutions.

In § 2 , we showed that under certain conditions we may be sure that a homogeneous relation is satisfied by some solution of 2.1. We now examine this relation more closely.

4.1. LEMMA. - Let $L$ , $\mathfrak{L}$ be as in § 2. Let $F$ be a homogeneous irreducible form in $\mathfrak{L}[y_0 , \ldots , y_{n-1}]$ and $w$ an element in the kernel $K$ of $L$ in a differential extension field $\hat{\mathfrak{L}}$ such that

4.1.1 $$F(w , w' , \ldots , w^{(n-1)}) = 0 ,$$

4.1.2.- $(w , w' , \ldots , w^{(n-1)})$ is projectively algebraically independent over $\mathfrak{L}$ then there exists $\xi$ in some extension of $\hat{\mathfrak{L}}$ such that $\xi'/\xi \in \mathfrak{L}$ and such that

$$\frac{d}{dx} (\xi^{-1} F(v , \ldots , v^{(n-1)}) = 0 \quad \text{for each } v \in K .$$

Proof. - We eliminate $y_{n-1}$ between $F$ and $F^*$ (cf. 2.3) and obtain

$$R(y_0 , y_1 , \ldots , y_{n-2}) = A(y_0 , \ldots , y_{n-1}) F(y) + B(y_0 , \ldots , y_{n-1}) F^*(y)$$

where $R$ , $A$ , $B \in \mathfrak{L}[y]$ , and are indeed homogeneous forms. Specializing

$$(y_0 , \ldots , y_{n-1}) \longmapsto (w , w' , \ldots , w^{(n-1)})$$

we find that $R(w , \ldots , w^{(n-2)}) = 0$ and so $R$ is identically zero.

Thus as polynomial in $y_{n-1}$ with coefficients in the field $\mathfrak{L}(y_0,y_1,\ldots,y_{n-2})$ the polynomials $F$ , $F^*$ have a non-trivial common factor $h(y_{n-1})$ which shows that

$F$ does not lie in $\mathfrak{L}(y_0, y_1, \dots, y_{n-2})$. Since $F$ is irreducible in $[y_0, \dots, y_{n-1}]$, it is also irreducible in $\mathfrak{L}(y_0, \dots, y_{n-2})[y_{n-1}]$, and so $h = F$. We conclude that $F^* = TF$ with $T \in \mathfrak{L}[y_0, \dots, y_{n-1}]$, but $F^*$, if not zero, is a form of the same degree as $F$ and so $T \in \mathfrak{L}$. We choose $\xi$ in a suitable extension field such that $\xi'/\xi \in T$. Thus if $v \in K$ we have

$$\xi^2 \frac{d}{dx} (\xi^{-1} F(v)) = \xi F^*(v) - \xi' F(v) = 0$$

as asserted.

4.2. Application of Lemma 4.1. - Let now $L$ be DFG with coefficients in $k(x)$, and let $\mathfrak{L}$ be a constant field extension of $k(x)$, say $\mathfrak{L} = C(x)$, $C \supset k$. Under the hypothesis of Lemma 4.1, $F \in C(x)[y_0, \dots, y_{n-1}]$ and so $\xi'/\xi \in C(x)$. If $F(v, \dots, v^{(n-1)}) = 0$ for all $v \in K$, then we may put $\xi = 1$. Otherwise for each prime $\mathfrak{p}$ of $k$, we may choose a power series solution $v$ of $Lv = 0$ which is analytic at $t_{\mathfrak{p}}$, the $\mathfrak{p}$ generic point, such that $f(v, \dots, v^{(n-1)}) \neq 0$. Hence there exists a branch of $\xi$ at $t_{\mathfrak{p}}$ such that $\xi/F(v, \dots, v^{(n-1)})$ is a non-zero constant. This shows that for almost all $\mathfrak{p}$, the branch of $\xi$ at $t_{\mathfrak{p}}$ (i. e., the solution at $t_{\mathfrak{p}}$ of $\xi'/\xi = T$) converges in $D(t_{\mathfrak{p}}, 1^-)$. This holds regardless of how we extend the valuation $\mathfrak{p}$ to $C$ and hence we conclude, since the Grothendieck's conjecture is known in the first order case, that $\xi$ is the radical of an element of $C(x)$. Thus replacing $F$ by a power, we obtain $F \in C(x)[y_0, \dots, y_{n-1}]$ such that

$$F(v, v', \dots, v^{(n-1)}) = \text{constant}$$

for each $v \in K$. We believe that it is possible to replace $F$ by a form with coefficients in $k(x)$.

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