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NON-ARCHIMEDEAN NUCLEARITY

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Introduction. - The purpose of this paper is to collect the results on non-archimedean nuclear spaces which are up to now scattered over different articles under different names.

We also (§ 1 and § 2) state all the properties on compactoid sets and compactoid linear mappings needed for the study of non-archimiedean nuclearity.

The following notations are used :

K is a complete non-archimedean (n. a.) field with a non-trivial valuation.

If E is a locally K-convex space, then we denote by $\mathfrak{U}_E$ a fundamental system of K-convex closed zero-neighbourhoods of 0 in E , and by $\mathcal{P}_E$ the corresponding set of n. a. semi-norms on E .

We always assume that E is Hausdorff.

For all the other basic notions appearing in this paper without reference, we refer to [15].

1. Compactoid sets and c-compact sets

1.1. Definition. - Let E be a locally K-convex space. A subset A of E is called compactoid if

$$\forall\ U \in \mathfrak{U}_E\ ,\quad \exists\ S \subset E\ ,\quad S \text{ finite, such that } A \subset C(S) + U\ ,$$

where $C(S)$ is the K-convex hull of S .

1.2. Properties of compactoid sets.

(i) If A is compactoid and $B \subset A$, then B is compactoid.

(ii) If A and B are compactoid, then $A + B$ is compactoid.

(iii) A compactoid set is bounded.

(iv) If $A_i \subset E_i$, $\forall\ i \in I$, then A_i is compactoid in E_i , $\forall\ i \in I$ if, and only if, $\prod_{i\in I} A_i$ is compactoid in $\prod_{i\in I} E_i$.

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(v) Let F be a subspace of E and $A \subset F$. Then A is compactoid in F if, and only if, A is compactoid in E.

(vi) Let $X \subset K$ and $C(X)$, $\|\cdot\|_\infty$ the n. a. Banach space of the continuous functions from X to K. If $A \subset C(X)$ is equicontinuous and pointwise bounded, then A is compactoid in $C(X)$, $\|\cdot\|_\infty$.

(vii) Let $A \subset E$ be compactoid and complete, and let τ_1 be a locally K-convex topology which is coarser than the topology τ on E. Then $\tau_1 = \tau$ on A.

(viii) Let E, F be locally K-convex spaces and $f : E \longrightarrow F$ a continuous linear map. If $A \subset E$ is compactoid, then $f(A)$ is compactoid in F.

(ix) Let E, F be locally K-convex and (f_n) an equicontinuous sequence of linear maps from E to F, converging pointwise to f. If A is compactoid in E then (f_n) converges to f uniformly on A.

(x) If the unit ball of a n. a. Banach space E is compactoid, then $\dim E < \infty$.

(xi) Every compactoid subset of a n. a. Banach space is of countable type.

<u>Proof.</u>

- (i), (ii), (iii) and (viii) follow immediately from the definition.
- (iv), (v) and (vi) are proved in [7].
- (vii) is proved in [8].
- For (ix), see [2].
- For (x) and (xi), see [14].

1.3. <u>Definition</u>. - A subset A of a locally K-convex space E is called c-compact if every K-convex filter (i. e. a filter generated by sets x + B, B = K-convex) on A has a cluster point on A.

1.4. <u>Remark</u>. - If K is not locally c-compact there are in E no non-trivial K-convex, c-compact sets.

The following are equivalent (see [15]) :

(i) K is locally c-compact.

(ii) K is c-compact.

(iii) K is maximally complete.

1.5. PROPOSITION (see [1]). - <u>If K is maximally complete then every weakly c-compact subset of a locally K-convex space E is c-compact.</u>

1.6. <u>Relation to compactoid sets</u> (see [8]). - If K is maximally complete and A is a K-convex, bounded subset of a locally K-convex space E, then A is c-compact if, and only if, A is compactoid and complete.

2. Compactoid operators.

2.1. Definition. - Let E and F be locally K-convex spaces. A linear mapping $f : E \to F$ is called compactoid if there exists a zero-neighbourhood $U \in \mathcal{U}_E$ such that $f(U)$ is compactoid in F. (If E is a n. a. Banach space then one can take U = unit ball of E).

2.2. General properties.

(i) Every compactoid mapping is continuous, (see [3]).

(ii) If $f : E \to F$ is compactoid, and B is bounded in E, then $f(B)$ is compactoid in F, (see [3]).

(iii) If f is a linear mapping from E into a n. a. Banach space F, then f is compactoid if, and only if, f factors through c_0 as $f = h.g$ with $g : E \to c_0$ compactoid, (see [5]).

2.3. Properties of compactoid maps between n. a. Banach spaces.

(i) The compactoid operators form an operator ideal which is injective and surjective. (For the definitions, see [12]. The proof follows immediately from properties of compactoid sets (see § 1).)

(ii) If $f : E \to F$ is compactoid, then so is its transpose ${}^T f : F' \to E'$, (see [10]).

(iii) $f : E \to F$ is compactoid if, and only if, it can be written as

$$f(x) = \sum_n a_n(x) . y_n ,$$

with $(a_n) \subset E'$, $(y_n) \subset F$ and $\lim_n \|a_n\| . \|y_n\| = 0$, (see [14]).

2.4. Remark. - The characterization stated in (iii) shows that the compactoid linear mappings are in fact the n. a. equivalent of the classical nuclear operators. Therefore the n. a. operators are often called nuclear as well.

3. The definition of a non-archimedean nuclear space.

3.1. Definition. - A locally K-convex space E, $\mathcal{U}_E$, is called nuclear if for each $U \in \mathcal{U}_E$ there exists a $V \in \mathcal{U}_E$ such that the canonical mapping $\varphi_{UV} : \hat{E}_V \to \hat{E}_U$ is compactoid (or nuclear). (The notations have the same meaning as in the classical theory.)

The definition is independent of the choice of $\mathcal{U}_E$.

3.2. Remark. - Nuclear locally K-convex spaces, in this sense, have first been studied in [3] and [5] for K maximally complete. In these papers, they are called Schwartz-spaces (because of the equivalence between nuclear and compactoid mapping in n. a. analysis).

However many properties proved there remain valid when K is not maximally complete

The properties which depend on the maximal completeness of K are mentioned as such in § 5 .

We use the term "nuclear" to point out that we deal with an arbitrary K .

3.3. Remark. - In the classical theory the definition of a nuclear space is equivalent to the following :

" E is nuclear if for every locally convex space F the π-topology and the ε-topology on $E \otimes F$ coincide"

It is proved in [13] that, when K is maximally complete, every locally K-convex space E has this property.

That is the reason why we prefer to work with definition 3.3.

4. Examples of non-archimedean nuclear spaces.

4.1. Non-archimedean Köthe spaces.

4.1.1. Definition. - Let $B = (b_n^k)$ be an infinite matrix consisting of strictly positive real numbers b_n^k with

$$b_n^k \leqslant b_n^{k+1} \ , \quad \forall\ n \ , \quad \forall\ k \ .$$

The n. a. Köthe space $K(B)$, associated with B , is defined by

$$K(B) = \{(\alpha_n) \ ; \ \alpha_n \in K \ , \ \forall\ n \ , \text{ and } \lim_n |a_n|\ b_n^k = 0 \ , \ \forall\ k\} \ ,$$

equipped with the sequence of norms :

$$P_k((\alpha_n)) = \max_n |a_n|\ b_n^k \ , \quad k = 1\ , 2\ , 3\ , \ldots$$

4.1.2. PROPOSITION (see [2] and [6]).

(i) $K(B)$ is a perfect n. a. Fréchet sequence space in which the coordinate vectors (e_n) form a Schauder basis.

(ii) The Köthe dual space $K(B)^x$ of $K(B)$ can be identified with the topological dual space $K(B)$.

(iii) The topology on $K(B)$ is the normal topology $n(K(B)\ , K(B)^x)$ of the dual pair $(K(B)\ , K(B)^x)$.

4.1.3. PROPOSITION (see [6]). - The space $K(B)$ is nuclear if, and only if,

$$\forall\ k\ , \ \exists\ k_1 > k \ \text{ such that } \ \lim_n b_n^k / b_n^{k_1} = 0$$

4.1.4. Examples. - Let $a = (a_n)$ be a non-decreasing sequence of positive real numbers with $\lim_n a_n = \infty$.

Let $B_1 = (k^{a_n})$ and $B_2 = ((k/(k+1))^{a_n})$.

Then the Köthe sequence spaces $\Lambda_\infty(a) = K(B_1)$, and $\Lambda_1(a) = K(B_2)$ are nuclear.

4.2. Non-archimedean Fréchet spaces with a Schauder basis.

4.2.1. PROPOSITION (see [4]). - Let E, (P_k) be a n. a. Fréchet space with a Schauder basis (x_n). Then there exists on E an equivalent sequence of semi-norms (P_k^*) such that, $\forall\, x \in E$, $x = \sum_n \alpha_n x_n$: $P_k^*(x) = \max_n |\alpha_n|\, P_k^*$.

4.2.2. PROPOSITION (see [6]). - Every n. a. countably normed Fréchet space E with a Schauder basis (x_n) can be identified with a n. a. Köthe space. Let (P_k^*) be the sequence (of norms) coming out of 4.2.1 then $E = K(B)$ with $B = (P_k^*(x_n))$.

4.2.3. Examples.

(i) The space A_∞, of the entire functions $f : K \to K$,

$A_\infty = \{f : K \to K ;\ f(x) = \sum_n a_n x^n,\ a_n \in K,\ \lim_n |a_n|\, |x|^n = 0,\ \forall\, x \in K\}$,

with norms $P_k(f) = \max_n |a_n|\, k^n$, $k = 1, 2, \dots$,

as well as the space A_1, of functions $f : K \to K$, which are analytic on the unit ball of K,

$A_1 = \{f : K \to K ;\ f(x) = \sum_n a_n x^n,\ a_n \in K,\ \lim_n |a_n|\, |x^n| = 0,\ \forall\, |x| \leqslant 1\}$,

have the Schauder basis $1, x, x^2, \dots, x^n, \dots$

They are nuclear by 4.1.4. (take $a_n = n$).

(ii) Let $\Omega = \underset{\sim}{P}^1 \setminus \{0, \infty\} = K^*$ and

$$\mathcal{O}(\Omega) = \{\textstyle\sum_{n=-\infty}^{+\infty} a_n z^n ;\ a_n \in K,\ \lim_{|n|\to\infty} |a_n|\, R^{|n|} = 0,\ \forall\, R > 1\},$$

with norms $\|\cdot\|_k = \max_n |a_n|\, k^{|n|}$, has the Schauder basis $1, z, z^{-1}, z^2, z^{-2}$,

It follows from 4.2.2 and 4.1.3 that $\mathcal{O}(\Omega)$ is nuclear.

4.3. PROPOSITION (see [6]). - Every locally K-convex space E is nuclear for the weak topology $\sigma(E, E')$.

4.4. The space $C^\infty(X)$.

4.4.1. Definition (see [9]). - Let X be a non-empty subset of K, without isolated points. For $n \geqslant 1$, let

$$\nabla^n X = \{(x_1, x_2, \dots, x_n) \in X^n ;\ x_i \neq x_j \text{ whenever } i \neq j\}.$$

For $f : X \to K$ define $\Phi_n(f) : \nabla^{n+1} X \to K$ by induction as follows : $\Phi_0 f = f$ and, for $n \geqslant 1$,

$$\Phi_n f(x_1, \dots, x_{n+1}) = \frac{\Phi_{n-1}(x_1, x_3, \dots, x_{n+1}) - \Phi_{n-1}(x_2, x_3, \dots, x_{n+1})}{x_1 - x_2}$$

($\Phi_n f$ is called the n-th difference quotient of f).

The function f is said to be n times continuously differentiable, $f \in C^n(X)$, if the function $\Phi_n(f)$ can be extended to a continuous function $\overline{\Phi}_n f$ on X^{n+1}.

The space $C^\infty(X)$ is then defined by $C^\infty(X) = \bigcap_{n=1,2,3,\ldots} C^n(X)$.

For $f \in C^n(X)$ and $B \subset X$, B compact, let

$$\|\overline{\Phi}_i f\|_B = \sup_{p \in B^{i+1}} |\overline{\Phi}_i f(p)| , \quad i = 0 , 1 , \ldots$$

and

$$\|f\|_{B,n} = \max_{i=0,1,\ldots,n} \|\overline{\Phi}_i f\|_B$$

On $C^\infty(X)$ a locally K-convex topology is then defined by the family of n. a. semi-norms :

$$\{\|\cdot\|_{B,n} \; ; \; B \subset X \text{ compact}, \; n = 0 , 1 , 2 , \ldots\} .$$

4.4.2. PROPOSITION (see [7]). - The locally K-convex space $C^\infty(X)$ is nuclear.

5. Properties of non-archimedean nuclear spaces.

5.1. PROPOSITION. - Every bounded subset of a n. a. nuclear space is compactoid.

Sketch of proof. - Let $B \subset E$ be bounded and take $U \in \mathcal{U}_E$. Then take $V \in \mathcal{U}_E$, $V \subset U$ as given by the nuclearity.

Now $\varphi_U(B) = \varphi_V \circ \varphi_{UV}(B)$ is compactoid in E_U.

So $\exists\, \hat{x}_1 , \ldots , \hat{x}_n \in E_U$ such that

$$\varphi_U(B) \subset C[\hat{x}_1 , \ldots , \hat{x}_n] + \varphi(U) .$$

It then follows that $B \subset C[x_1 , \ldots , x_n] + U$.

5.2. COROLLARY. - If E is a n. a. normed space which is nuclear then $\dim E < \infty$. (From 5.1 and § 1, 1.2 (x)).

5.3. PROPOSITION (see [3]). - If E is nuclear and F is any n. a. Banach space then every continuous linear map from E to F is compactoid.

5.4. PROPOSITION (see [5]). - A locally K-convex space E is nuclear if, and only if,

(i) E is a subset of some power c_0^I of c_0, and

(ii) Every continuous linear map from E to c_0 is compactoid.

5.5. COROLLARY. - Every n. a. nuclear Fréchet space is of countable type.

5.6. PROPOSITION (see [3]). - Let K be maximally complete. Then a locally K-convex space is nuclear if, and only if, it is semi-reflexive and quasi-normable. (E is quasi-normable if for every K-convex equicontinuous subset A of E' there exists a K-convex equicontinuous set $D \supset A$ such that on A the topologies induced by E', $\beta(E', E)$ and X'_D, $\|\cdot\|_D$ coincide.)

5.7. Stability properties for n. a. nuclear spaces.

(i) A locally K-convex space E is nuclear if, and only if, its completion $\hat{E}$ is nuclear.

(ii) Every subspace of a n. a. nuclear space is a n. a. nuclear space.

(iii) Every quotient, by a closed subspace, of a n. a. nuclear space is a n. a. nuclear space.

(iv) Every product of n. a. nuclear spaces is n. a. nuclear.

(The proof is exactly the same as in the classical case. It is a consequence of § 2, 2.3 (i).) (See e. g. [12].)

5.8. Remark. - In [11], an example is given of a space in which every bounded subset is precompact but which is not a Schwartz-space. The same example, with obvious modifications, gives a n. a. space in which every bounded subset is compactoid but which is not n. a. nuclear.

6. Nuclearity of the dual space.

6.1. PROPOSITION (see [6]). - Let $K(B)$ be any n. a. Köthe space. Then its topological dual space $K(B)^x$ is a n. a. nuclear space for the normal topology $n(K(B)^x, K(B))$.

6.2. PROPOSITION (see [6]). - Suppose K is maximally complete. Then the dual $K(B)^x$ of a n. a. Köthe space $K(B)$ is a n. a. nuclear space for the Mackey topology $\tau(K(B)^x, K(B))$.

6.3. PROPOSITION (see [3]). - Suppose K is maximally complete. Then the strong dual E', $\beta(E', E)$ of a n. a. nuclear Fréchet space is again a n. a. nuclear space.

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