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ON THE REDUCTION OF ABELIAN VARIETIES

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Let A be an abelian variety of dimension n over a **complete non-Archimedean** field k. Viewing A as a rigid analytic group variety over k, we say that an open analytic subgroup $N \subset A$ has semi-abelian reduction if it is smooth over the valuation ring $\mathring{k}$ of k (see below) and if the analytic reduction $\tilde{N}$ of N is an extension of an abelian variety $\tilde{B}$ by an affine torus $\tilde{T}$ (everything defined over the residue field $\tilde{k}$ of k), i. e., if there is an exact sequence

$$0 \longrightarrow \tilde{T} \longrightarrow \tilde{N} \longrightarrow \tilde{B} \longrightarrow 0 .$$

The existence of a subgroup $N \subset A$ having semi-abelian reduction is of indispensible value for the construction of the universal covering $\hat{A}$ of A, see [2] and [3].

If the valuation on k is discrete, one can obtain a subgroup $N \subset A$ of the above type as follows. One considers the formal completion $\hat{\mathcal{N}}$ of the Néron model $\mathcal{N}$ of A. Then $\hat{\mathcal{N}}$ can be viewed as an open analytic subgroup of A, and its identity component $N := \hat{\mathcal{N}}_0$ has potential semi-abelian reduction (meaning that $N \otimes k'$ has semi-abelian reduction for some finite extension field k' of k). It is not yet known how to construct such a group N by analytic means. In this article, we want to discuss this question. In particular, we will characterize the semi-abelian reduction in terms of analytic properties.

Let H be an analytic group variety over k. Then H is called <u>formal</u> if it carries a formal analytic structure (given by some formal affinoid covering) (see [1], § 1), such that the structure is compatible with the group operations. The formal structure of H is unique if it exists. In particular, it gives rise to a well-defined analytic reduction $\tilde{H}$ of H. (The reduction is a scheme of locally finite type over $\tilde{k}$.) A formal analytic group H is called <u>smooth over</u> $\mathring{k}$ if the reduction $\tilde{H}$ is geometrically regular and if H is <u>distinguished</u>. The latter means that, for all formal affinoid parts $\mathrm{Sp}\, C$ of H, the supremum norm on C is a residue norm with respect to some epimorphism $T_m \longrightarrow C$ (where T_m is a free Tate algebra) (see [1], § 2). One knows that $\tilde{H}$ is a group scheme over $\tilde{k}$ if H is smooth over $\mathring{k}$.

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In the following we always assume that k is a p-adic field (i. e., a complete field containing $\underset{\sim}{Q}_p$). The valuation of k can be discrete or dense. We denote by $[p] : A \to A$ the homomorphism of the abelian variety A obtained by multiplying elements with p.

PROPOSITION 1. - Let N be a connected open analytic subgroup of A, smooth over $\overset{\circ}{k}$. Then the following conditions are equivalent :

(i) N is maximal among all connected open formal analytic subgroups of A.

(ii) $[p] : N \to N$ is surjective.

(iii) N has potential semi-abelian reduction.

The difficult part of the proof is to show that condition (iii) of the proposition implies condition (i). One shows more generally that N contains all connected open formal analytic subgroups of A if it has potential semi-abelian reduction. Consequently, a subgroup $N \subset A$ satisfying the equivalent conditions of the proposition is unique.

Each commutative analytic group of dimension n over a field of characteristic 0 is locally isomorphic to the n-dimensional additive group $\underset{\sim}{G}_a^n$. Therefore one can find an open analytic subgroup $I \subset A$ which is isomorphic to the unit ball in $\underset{\sim}{G}_a^n$. For $\nu \geq 1$, we denote by I_ν the identity component of the affinoid group $[p^\nu]^{-1}(I)$. Let $A_+ := \bigcup_{\nu=1}^{\infty} I_\nu$.

PROPOSITION 2. - Let N be a connected open analytic subgroup of A, smooth over $\overset{\circ}{k}$. Let N_+ be the kernel of the reduction map $N \to \tilde{N}$. Then N has potential semi-abelian reduction if, and only if $N_+ = A_+$.

This result suggests how to proceed with an analytic construction of subgroups $N \subset A$ having semi-abelian reduction. All one has to do is to construct some "quasi-compact closure" of the Stein group $A_+ \subset A$. Two steps are necessary. The first one is established by the following result :

THEOREM. - Modulo extension of the ground field k, the analytic variety A_+ is isomorphic to the "open" unit ball in affine n-space.

The second step is still in the stage of a conjecture.

PROBLEM. - Find a distinguished open affinoid subspace $U \subset A$ containing the unit element $e \in A$ such that $U_+(e) = A_+$ (where $U_+(e) := \pi^{-1}(\pi(e))$ with $\pi : U \to \tilde{U}$ denoting the canonical reduction map).

It is expected that the problem can be solved, at least if the ground field k is replaced by a finite extension. If U is an open affinoid subspace of A satisfying the stated properties, then it follows from the theorem that U is smooth over

$\overset{\circ}{k}$ in some formal neighborhood U' of e . We may assume U' is connected. Let N be the subgroup of A generated by U' . Then N is a connected open analytic subgroup, smooth over $\overset{\circ}{k}$, such that $N_+ = U'_+(e) = U_+(e) = A_+$. Hence by proposition 2, N has potential semi-abelian reduction. Thereby we see that, modulo finite field extension, an open subgroup $N \subset A$ having semi-abelian reduction can be obtained by an analytic construction, provided the problem stated above can be solved.

REFERENCES

[1] BOSCH (S.). - Zur Kohomologietheorie rigid analytischer Raüme, Manuscripta math., t. 20, 1977, p. 1-27.

[2] BOSCH (S.). - On p-adic uniformization (to appear).

[3] RAYNAUD (M.). - Variétés abéliennes et géométrie rigide, "Actes du Congrès international des Mathématiciens", [1970. Nice], p. 473-477. - Paris, Gauthier-Villars, 1971.