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AN APPLICATION OF NEWTON ITERATION PROCEDURE TO p-ADIC DIFFERENTIAL EQUATIONS

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This report is based on the author's lectures at Strasbourg, Padova, Grenoble, Groningen and Paris. The motivations of this research were explained in the papers to appear ([3],[5]) and the lecture-notes [4] (joint with S. SPERBER). Therefore, in this paper, we will report only on the technical part.

1. Preliminaries.

Let K be a field of characteristic zero complete with respect to an absolute value $|\ |$ which is non-trivial and ultrametric. The field of rational number, $\mathbb{Q}$, is a subfield of K, and we require that the restriction of $|\ |$ to $\mathbb{Q}$ is a p-adic absolute value for some prime number p. We normalize $|\ |$ so that $|p| = 1/p$.

For $\varphi = \sum_{m=0}^{\infty} a_m x^m \in K[[x]]$, we set

$$|\varphi|_0(r) = \sup_{m\geq 0} |a_m| r^m .$$

If $|\varphi|_0(r_0) < +\infty$ for some positive constant r_0, then φ is convergent for $|x| < r_0$. The following lemma is fundamental throughout this report.

LEMMA 1. - Assume that $\varphi_j = \sum_{m=0}^{\infty} a_{j,m} x^m \in K[[x]]$, $j = 1, 2, \ldots$, with the properties :

(i) $\lim_{j\to+\infty} a_{j,m} = a_m$ exists for every m ;

(ii) $|\varphi_j|_0(r) \leq M(r)$ for $0 \leq r < r_0$, $j = 1, 2, \ldots$, where r_0 is a positive number, and $M(r)$ is a non-negative number which depends only on r. Then, $\varphi = \sum_{m=0}^{\infty} a_m x^m$ is convergent for $|x| < r_0$, and $\lim_{j\to+\infty} |\varphi_j - \varphi|_0(r) = 0$ for $0 \leq r < r_0$. (Cf. B. DWORK [1].)

2. An example (a rough sketch).

Let us consider a non-linear differential equation

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(2.1) $$x\, du/dx + \alpha u = f(x) + u^2 g(x , u) ,$$

where $\alpha \in K$, $f \in K[[x]]$, $g \in K[[x , u]]$, and f and g are convergent. We want to find a convergent power series $\varphi \in K[[x]]$ which satisfies the equation (2.1). To do this, we try to construct φ in the following form

(2.2) $$u = \varphi = \sum_{j=0}^{\infty} \varphi_j , \quad \varphi_j \in K[[x]] .$$

Step 1. - First of all, φ_0 is determined by the linear differential equation

(2.3) $$x\, d\varphi_0/dx + \alpha\varphi_0 = f .$$

Step 2. - Change u by $u = \varphi_0 + v$. Then (2.1) becomes

(2.1') $$x\, dv/dx + \alpha v = \varphi_0(x)^2 g(x , \varphi_0(x)) + \varphi_0(x)\, G(x)\, v + v^2 g_1(x , v) ,$$

where

$$G(x) = 2g(x , \varphi_0(x)) + \varphi_0(x)\, g_u(x , \varphi_0(x)) \quad (g_u = \partial g/\partial u) ,$$

$$v^2 g_1(x , v) = \varphi_0(x)^2\{g(x , \varphi_0(x) + v) - g(x , \varphi_0(x)) - g_u(x , \varphi_0(x))\, v\}$$
$$+ 2\, \varphi_0(x)\, v\{g(x , \varphi_0(x) + v) - g(x , \varphi_0(x))\} + v^2 g(x , \varphi_0(x) + v) .$$

We determine φ_1 by the linear part of (2.1')

(2.4) $$x\, d\varphi_1/dx + \alpha\varphi_1 = \varphi_0^2 g(x , \varphi_0) + \varphi_0\, G(x)\, \varphi_1 .$$

The other φ_j will be determined successively in a similar manner.

This is our Newton iteration procedure.

A closer look at equation (2.3). - If $f = \sum_{m=0}^{\infty} c_m x^m$ $(c_m \in K)$, then φ_0 is given by

(2.5) $$\varphi_0 = \sum_{m=0}^{\infty} \frac{c_m}{m + \alpha} x^m .$$

Assuming that $|f|_0(r) \leq M$ for $0 \leq r < r_0$, where r_0 and M are some positive numbers, we want to derive

(2.6) $$|\varphi_0|_0(r) \leq M \quad \text{for} \quad 0 \leq r < r_0' ,$$

for r_0' a positive number, as large as possible, such that $0 < r_0' \leq r_0$. To do this, we introduce two assumptions

(2.7) $$c_m = 0 \quad \text{for} \quad m < m_0 ,$$

(2.8) $$|m + \alpha|^{-1} \leq C^{m^{1-\delta}} \quad (m \geq m_0) ,$$

where m_0 is a positive integer, C is a positive number greater than one, and δ is a positive number smaller than one, i. e. $C > 1$, $0 < \delta < 1$.

The assumption (2.8) may be called "non-Liouville property" of the exponent α . The condition (2.7) may be written

(2.7') $$f \equiv 0 \pmod{x^{m_0}} .$$

Note that, if equation (2.1) admits a formal power series solution, then, we can change (2.1) so that condition (2.7) may be satisfied for any prescribed m_0. Also note that any algebraic number α satisfies condition (2.8) for any δ if we choose C and m_0 suitably.

Under assumption (2.8), set

(2.9) $$\rho_0 = (1/C)^{m_0^{-\delta}}$$

Then $0 < \rho_0 < 1$, and

$$\rho_0^m = (\rho_0^{m^\delta})^{m^{1-\delta}} = (C^{-(m/m_0)^\delta})^{m^{1-\delta}} \leqslant (1/C)^{m^{1-\delta}} \leqslant |m + \alpha| \quad \text{if} \quad m \geqslant m_0 .$$

Hence, under assumptions (2.7) and (2.8), we have

$$|\varphi_0|_0(r\rho_0) = \sup_{m \geqslant m_0} |m + \alpha|^{-1} \, |c_m| (r\rho_0)^m \leqslant \sup_{m \geqslant m_0} |c_m| \, r^m = |f|_0(r) ,$$

and

(2.6') $$|\varphi_0|_0(r) \leqslant M \quad \text{for} \quad 0 \leqslant r < r_0 \, \rho_0 .$$

<u>Equation</u> (2.4) <u>without</u> $\varphi_0 \, G(t) \, \varphi_1$. – To simplify the explanation, we remove $\varphi_0 \, G(x) \, \varphi_1$ from the right-hand member of equation (2.4); i. e. we consider the equation

(2.10) $$x \, d\varphi_1/dx + \alpha\varphi_1 = \varphi_0^2 \, g(x , \varphi_0) .$$

We know already that

(2.11) $$\varphi_0 \equiv 0 \pmod{x^{m_0}} ,$$

and that φ_0 satisfies (2.6'). First of all, (2.11) implies that

(2.12) $$\varphi_0^2 \, g(\cdot , \varphi_0) \equiv 0 \pmod{x^{2m_0}} .$$

Hence, if we assume that g satisfies the condition

(2.13) $$|\varphi_0^2 \, g(\cdot , \varphi_0)|_0(r) \leqslant M \quad \text{for} \quad 0 \leqslant r < r_0 \, \rho_0 ,$$

we have

(2.14) $$\begin{cases} \varphi_1 \equiv 0 \pmod{x^{2m_0}} , \\ |\varphi_1|_0(r) \leqslant M \quad \text{for} \quad 0 \leqslant r < r_0 \, \rho_0 \, \rho_1 , \end{cases}$$

where $\rho_1 = (1/C)^{(2m_0)^{-\delta}} = \rho_0^{2^{-\delta}}$.

Suppose that, proceeding inductively as above, we have defined for all $j \geq 0$,

$$\begin{cases} \varphi_j \equiv 0 \pmod{x^{2^j m_0}} , \\ |\varphi_j|_0(r) \leqslant M \quad \text{for} \quad 0 \leqslant r < r_0 \, \rho_0 \, \rho_1 , \cdots , \rho_j . \end{cases}$$

where $\rho_j = \rho_{j-1}^{2^{-\delta}} = \rho_0^{2^{-j\delta}}$; set

$$\psi_j = \sum_{\ell=0}^{j} \varphi_\ell , \quad \rho_\infty = \prod_{\ell=0}^{\infty} \rho_\ell = \rho_0^{(1-2^{-\delta})^{-1}} > 0 .$$

Then $|\psi_j|_0(r) \leq M$ for $0 \leq r < r_0 \rho_\infty$, and ψ_j converges x-adically to

$$\varphi = \sum_{\ell=0}^{\infty} \varphi_\ell .$$

Therefore, by virtue of lemma 1, we conclude that φ is convergent for $|x| < r_0 \rho_\infty$.

The argument of this section is not strictly speaking correct, since we removed $\varphi_0\, G(x)\, \varphi_1$ from the right-hand member of equation (2.4). A correct treatment of equation (2.1) is given in SIBUYA-SPERBER ([2],[4]).

3. Typical results.

In this section, we shall give a rigorous treatment of a problem which is more general than the problem of section 2. We assume that K contains an element π such that

$$|\pi| = \left(\frac{1}{p}\right)^{(p-1)^{-1}} . \tag{3.1}$$

We consider the following situation.

(i) We are given $\alpha_1 , \dots , \alpha_n \in K$ such that

$$|\alpha_j| \leqslant 1 , \quad |m + \alpha_j|^{-1} \leqslant C^{m^{1-\delta}} , \quad |m + \alpha_i - \alpha_j|^{-1} \leqslant C^{m^{1-\delta}} . \tag{3.2}$$

for $m \geqslant 2^k$ and $i , j = 1 , \dots , n$, where k is a non-negative integer, and C and δ are positive numbers such that $C > 1$, $0 < \delta < 1$.

(ii) We are also given $a_1 , \dots , a_n \in K[[x]]$ such that

$$a_j \equiv 0 \pmod{x} , \quad \left|\int_0^x t^{-1} a_j(t)\, dt\right|_0(r) < |\pi| , \tag{3.3}$$

for $0 \leqslant r < r_0$ and $j = 1 , \dots , n$, where r_0 is a positive number, and where, for $a = \sum_{m=1}^{\infty} a_m x^m$, we have denoted $\sum_{m=1}^{\infty}(a_m/m)\, x^m$ by $\int_0^x t^{-1} a(t)\, dt$.

We define two sequences of numbers, $\{\sigma_h\}$ and $\{\tau_h\}$ by

$$\begin{cases} \sigma_1 = 1/C , & \tau_1 = (1/C)^{2(1-2^{-\delta})^{-1}} \\ \sigma_h = \sigma_{h-1}^2 \tau_{h-1} , & \tau_h = (\sigma_1^h \sigma_h)^{(1-2^{-\delta})^{-1}} . \end{cases} \tag{3.4}$$

Note that

$$0 < \tau_h < \sigma_h < \tau_{h-1} < 1 . \tag{3.5}$$

In this section, we shall prove the following two theorems.

THEOREM 1. - Assume that a differential operator $H = \sum_{j=0}^{n-1} b_j(x)\, \partial^j$ $(\partial = xd/dx)$ satisfies the following conditions :

$$(3.6)\qquad \begin{cases} b_j \in K[[x]] \quad \text{and} \quad b_j \equiv 0 \pmod{x^{2^k}}\ , \\ |b_j|_0(r) < |\pi| \quad \text{for} \quad 0 \leq r < r_0\ . \end{cases}$$

Then, there exists $\eta_1, \dots, \eta_n \in K[[x]]$ such that

$$(3.7)\qquad \begin{cases} \eta_j \equiv 0 \pmod{x^{2^k}}\ , \\ |\int_0^{\cdot} t^{-1}\, \eta_j(t)\, dt|_0(r) < |\pi| \quad \text{for} \quad 0 \leq r < r_0\, \sigma_n^{2^{-k\delta}}\ , \quad j = 1, \dots, n\ , \end{cases}$$

and that

$$(3.8)\qquad (\partial + \alpha_1 + a_1) \dots (\partial + \alpha_n + a_n) - H$$
$$= (\partial + \alpha_1 + a_1 - \eta_1) \dots (\partial + \alpha_n + a_n - \eta_1)\ .$$

THEOREM 2. - Assume that

$$(3.9)\qquad f \in K[[x]]\ , \quad f \equiv 0 \pmod{x^{2^k}}\ , \quad |f|_0(r) < 1 \quad \text{for} \quad 0 \leq r < r_0\ ,$$

and that

$$(3.10)\qquad \begin{cases} G = \sum_{\substack{\mu_0 + \dots + \mu_{n-1} \geq 2 \\ \mu_j \geq 0}} g_{\mu_0 \dots \mu_{n-1}}(x)\, v_0^{\mu_0} \dots v_{n-1}^{\mu_{n-1}} \in K[[x, v_0, \dots, v_{n-1}]]\ , \\ \qquad\qquad \text{avec} \quad g_{\mu_0 \dots \mu_{n-1}}(x) \in K[[x]]\ , \\ |g_{\mu_0 \dots \mu_{n-1}}|_0(r) \leq |\pi| \quad \text{for} \quad 0 \leq r < r_0\ . \end{cases}$$

Then, there exists a unique $\wp \in K[[x]]$ such that

$$(3.11)\qquad \wp \equiv 0 \pmod{x^{2^k}}\ ,$$

and that

$$(3.12)\qquad (\partial + \alpha_1 + a_1) \dots (\partial + \alpha_n + a_n)(\wp) = f + G(x, \wp, \partial\wp, \dots, \partial^{n-1}\wp)\ .$$

Furthermore, this power series $\wp$ also satisfies the condition

$$(3.13)\qquad |\wp|_0(r) < 1 \quad \text{for} \quad 0 \leq r < r_0\, \tau_n^{2^{-k\delta}}\ .$$

Remark 1. - The power series $\wp$ is a solution of a non-linear differential equation with purely Fuchsian linear part. This is a prototype of the most difficult situations in the study of p-adic non-linear problems. The most important part of theorem 2 is the estimate (3.13), i. e. the r-interval in which $|\wp|_0(r) < 1$ holds.

Remark 2. – Theorem 1 is a Hensel-type lemma. The problem of factorization of a linear differential operator is naturally reduced to a non-linear problem such as that of theorem 2. For example, if the order of the operator is two, the corresponding non-linear problem is a Riccati equation. In general, if the order of the operator is n, the order of the corresponding non-linear problem is $n-1$. Taking advantage of this situation, we can prove theorem 1 and 2 simultaneously by an induction on n. Since the case $n = 1$ was treated in SIBUYA-SPERBER [2], we shall prove these theorems for $n \geq 2$. (Cf. also SIBUYA-SPERBER [4].)

4. Proof of theorem 1 for n.

In this section, assuming theorem 2 for $n-1$, theorem 1 for $n = 1$, and theorem 1 for $n-1$, we shall prove theorem 1 for n. Set

$$(4.1) \qquad \begin{cases} L = (\partial + \alpha_1 + a_1) \cdots (\partial + \alpha_{n-1} + a_{n-1}) , \\ \ell = \partial + \alpha_n + a_n . \end{cases}$$

We want to find $\eta \in K[[x]]$ and $\tilde{L} = \sum_{j=0}^{n-2} Y_j \partial^j$ ($Y_j \in K[[x]]$) such that

$$(4.2) \qquad L\ell - H = (L - \tilde{L})(\ell - \eta) .$$

The relation (4.2) is equivalent to the assertion that

$$(4.2') \qquad L\ell(u) - H(u) = 0$$

for all u belonging to a sufficiently large extension of $K[[x]]$ such that

$$(\ell - \eta)(u) = 0 .$$

Therefore, (4.2) is equivalent to the assertion that

(4.3) $L(u\eta) = H(u)$ for all such u satisfying $\ell(u) = u\eta$.

Observe that

$$(\partial + \alpha_j + a_j)(uv) = u(\partial + (\alpha_j - \alpha_n) + (a_j - a_n) + \eta)(v) ,$$

if $\ell(u) = u\eta$. Hence

$$(4.4) \qquad L(u\eta) = u(\partial + (\alpha_1 - \alpha_n) + (a_1 - a_n) + \eta) \cdots (\partial + (\alpha_{n-1} - \alpha_n) + (a_{n-1} - a_n) + \eta)(\eta) ,$$

if $\ell(u) = u\eta$. We can write

$$(4.4') \qquad (\partial + (\alpha_1 - \alpha_n) + (a_1 - a_n) + \eta) \cdots (\partial + (\alpha_{n-1} - \alpha_n) + (a_{n-1} - a_n) + \eta)(\eta)$$
$$= (\partial + (\alpha_1 - \alpha_n) + (a_1 - a_n)) \cdots (\partial + (\alpha_{n-1} - \alpha_n) + (a_{n-1} - a_n))(\eta) - \tilde{F}(x, \eta, \ldots, \partial^{n-2}\eta) ,$$

where

$$\tilde{F} = \sum_{\substack{\mu_0+\cdots+\mu_{n-2}\geq 2 \\ \mu_j \geq 0}} \tilde{F}_{\mu_0 \cdots \mu_{n-2}}(x)\, v_0^{\mu_0} \cdots v_{n-2}^{\mu_{n-2}} \in K[[x]][v_0, \ldots, v_{n-2}] ,$$

$$\tilde{F}_{\mu_0 \cdots \mu_{n-2}} \in K[[x]] \quad \text{and} \quad |\tilde{F}_{\mu_0 \cdots \mu_{n-2}}|_0(r) \leq 1 \quad \text{for} \quad 0 \leq r < r_0 .$$

On the other hand, if $\phi(u) = u\eta$, we have

$$\partial u = u(-\alpha_n - a_n + \eta) , \quad \partial^2 u = u\{(-\alpha_n - a_n + \eta)^2 + \partial(-\alpha_n - a_n + \eta)\} , \text{ etc.}$$

Hence, $H(u)$ has the following form

(4.5) $$H(u) = uF(x, \eta, \ldots, \partial^{n-2}\eta) ,$$

where

$$F = \sum_{\substack{\mu_0+\cdots+\mu_{n-2}\geq 0 \\ \mu_j \geq 0}} F_{\mu_0 \cdots \mu_{n-2}}(x)\, v_0^{\mu_0} \cdots v_{n-2}^{\mu_{n-2}} \in K[[x]][v_0, \ldots, v_{n-2}] ,$$

$$F_{\mu_0 \cdots \mu_{n-2}} \in K[[x]] , \quad F_{\mu_0 \cdots \mu_{n-2}} \equiv 0 \pmod{x^{2^k}} ,$$

and

$$|F_{\mu_0 \cdots \mu_{n-2}}|_0(r) < |\pi| \quad \text{for} \quad 0 \leq r < r_0 .$$

Thus, we derive from (4.3) the equation for η :

$$(\partial + (\alpha_1 - \alpha_n) + (a_1 - a_n)) \cdots (\partial + (\alpha_{n-1} - \alpha_n) + (a_{n-1} - a_n))(\eta) = F + \tilde{F} .$$

Set $\eta = \pi w$, and $\tilde{f}(x) = F_{0\cdots 0}(x)$, $\tilde{H} = \sum_{j=0}^{n-2} \tilde{b}_j(x)\,\partial^j$, where

$$\sum_{j=0}^{n-2} \tilde{b}_j(x)\, v_j = \sum_{\substack{\mu_0+\cdots+\mu_{n-2}=1 \\ \mu_j \geq 0}} F_{\mu_0 \cdots \mu_{n-2}}(x)\, v_0^{\mu_0} \cdots v_{n-2}^{\mu_{n-2}} ,$$

and

$$\tilde{G}(x, v_0, \ldots, v_{n-2}) = \sum_{\substack{\mu_0+\cdots+\mu_{n-2}\geq 2 \\ \mu_j \geq 0}} \{F_{\mu_0 \cdots \mu_{n-2}}(x) + \tilde{F}_{\mu_0 \cdots \mu_{n-2}}(x)\}\, v_0^{\mu_0} \cdots v_{n-2}^{\mu_{n-2}} .$$

Then the equation for w is given by

(4.6) $$(\partial + (\alpha_1 - \alpha_n) + (a_1 - a_n)) \cdots (\partial + (\alpha_{n-1} - \alpha_n) + (a_{n-1} - a_n))(w)$$
$$= (1/\pi)\,\tilde{f} + \tilde{H}(w) + (1/\pi)\,\tilde{G}(x, \pi w, \pi\partial w, \ldots, \pi\partial^{n-2} w)$$

Utilizing theorem 1 for $n-1$, we find $\tilde{\eta}_1, \ldots, \tilde{\eta}_{n-1} \in K[[x]]$ such that

$$\tilde{\eta}_j \equiv 0 \pmod{x^{2^k}}, \quad \left|\int_0^{\cdot} t^{-1}\,\tilde{\eta}_j(t)\,dt\right|_0(r) < |\pi| \quad \text{for} \quad 0 \leq r < r_0\,\sigma_{n-1}^{2^{-k\delta}},$$

and that

$$(\partial + (\alpha_1 - \alpha_n) + (a_1 - a_n)) \cdots (\partial + (\alpha_{n-1} - \alpha_n) + (a_{n-1} - a_n)) - \tilde{H}$$
$$= (\partial + (\alpha_1 - \alpha_n) + (a_1 - a_n) - \tilde{\eta}_1) \cdots (\partial + (\alpha_{n-1} - \alpha_n) + (a_{n-1} - a_n) - \tilde{\eta}_{n-1}) .$$

Then, applying to (4.6) theorem 2 for $n - 1$, we find a unique solution $w(x)$ such that

$$\begin{cases} w \equiv 0 \pmod{x^{2^k}}, \\ |w|_0(r) < 1 \quad \text{for} \quad 0 \leq r < r_0(\sigma_{n-1}\,\tau_{n-1})^{2^{-k\delta}} . \end{cases}$$

Thus, we constructed η so that (4.3) is satisfied and

$$\begin{cases} \eta \equiv 0 \pmod{x^{2^k}}, \\ |\eta|_0(r) < |\pi| \quad \text{for} \quad 0 \leq r < r_0(\sigma_{n-1}\,\tau_{n-1})^{2^{-k\delta}} . \end{cases}$$

To compute $\tilde{L}$, we derive $\tilde{L}(\ell - \eta) = H - L\eta$. Putting

$$H - L\eta = \sum_{j=0}^{n-1} \hat{b}_j(x)\,\partial^j, \quad \hat{b}_j \in K[[x]],$$

we get

$$\begin{cases} \hat{b}_j \equiv 0 \pmod{x^{2^k}}, \\ |\hat{b}_j|_0(r) < |\pi| \quad \text{for} \quad 0 \leq r < r_0(\sigma_{n-1}\,\tau_{n-1})^{2^{-k\delta}} ; \end{cases}$$

furthermore,

(4.3) $\quad Y_{n-2} = \hat{b}_{n-1}, \quad Y_\mu = \hat{b}_{\mu+1} - \sum_{j=\mu+1}^{n-2} f_{j,\mu+1}\,Y_j, \quad \mu = 0, \ldots, n-3,$

where $f_{j,\mu} \in K[[x]]$, and $|f_{j,\mu}|_0(r) \leq 1$ for $0 \leq r < r_0(\sigma_{n-1}\,\tau_{n-1})^{2^{-k\delta}}$.

Finally, applying to $L - \tilde{L}$ theorem 1 for $n - 1$, and to $\ell - \eta$ theorem 1 for $n = 1$, and utilizing the inequality $\sigma_{n-1} < \sigma_1$, we complete the proof.

5. Proof of theorem 2 for n.

In this section, assuming theorem 1 for n, and theorem 2 for $n = 1$, we shall prove theorem 2 for n. Setting

(5.1) $$\psi_j = \sum_{\ell=0}^{j} \varphi_\ell = \psi_{j-1} + \varphi_j ,$$

we determine $\psi_j \in K[[x]]$ by

(5.2) $(\partial + \alpha_1 + a_1) \cdots (\partial + \alpha_n + a_n)(\psi_j)$

$$= f + G(x, \psi_{j-1}, \partial\psi_{j-1}, \cdots, \partial^{n-1}\psi_{j-1}) + \sum_{i=0}^{n-1} G_{v_i}(x, \psi_{j-1}, \cdots, \partial^{n-1}\psi_{j-1})\, \partial^i \varphi_j ,$$

where $G_{v_i} = \partial G/\partial v_i$. This means that the φ_j are determined by linear differential equations :

(5.3) $$L_j(\varphi_j) = f_j \quad (j = 0, 1, \cdots) ,$$

where

(5.4) $$\begin{cases} L_0 = (\partial + \alpha_1 + a_1) \cdots (\partial + \alpha_n + a_n) , \\ L_j = L_0 - \sum_{i=0}^{n-1} G_{v_i}(x, \psi_{j-1}, \cdots, \partial^{n-1}\psi_{j-1})\, \partial^i \quad (j \geq 1) ; \end{cases}$$

(5.5) $$\begin{cases} f_0 = f \\ f_j = G(x, \psi_{j-1}, \cdots, \partial^{n-1}\psi_{j-1}) - G(x, \psi_{j-2}, \cdots, \partial^{n-1}\psi_{j-2}) \\ \qquad - \sum_{i=0}^{n-1} G_{v_i}(x, \psi_{j-2}, \cdots, \partial^{n-1}\psi_{j-2})\, \partial^i \varphi_{j-1} , \quad (j \geq 1) \end{cases}$$

where $\psi_\ell = 0$ if $\ell < 0$.

We want to construct the φ_j so that

(5.6) $$\begin{cases} \varphi_j \equiv 0 \pmod{x^{2^{k+j}}} , \\ |\varphi_j|_0(r) < 1 \quad \text{for} \quad 0 \leq r < r_0\, \sigma_1^{n2^{-(k+j)\delta}} \prod_{\ell=0}^{j-1} (\sigma_1^n \sigma_n)^{2^{-(k+\ell)\delta}} . \end{cases}$$

To do this, set

(5.7) $$L_j = L_{j-1} - H_j \qquad (j \geq 1) ,$$

where by (5.4)

(5.8) $$H_j = \sum_{i=0}^{n-1} \{ G_{v_i}(x, \psi_{j-1}, \cdots, \partial^{n-1}\psi_{j-1}) - G_{v_i}(x, \psi_{j-2}, \cdots, \partial^{n-1}\psi_{j-2}) \}\, \partial^i .$$

Using an induction on j , we can achieve a factorization of L_j into linear factors, by virtue of theorem 1 for n , for

$$|x| < r_0 \prod_{\ell=0}^{j-1} (\sigma_1^n \sigma_n)^{2^{-(k+\ell)\delta}} .$$

Then, by using theorem 2 for $n = 1$ (n-times) , we can achieve (5.6) .

Thus, we get

$$|\psi_j|_0(r) < 1 \quad \text{for} \quad 0 \leq r < r_0 \, \tau_n^{2^{-k\delta}} \quad , \quad j = 0 , 1 , \ldots ,$$

and ψ_j converges x-adically to $\rho = \sum_{\ell=0}^{\infty} \rho_\ell$. Hence, by lemma 1 of section 1,

$$|\rho|_0(r) < 1 \quad \text{for} \quad 0 \leq r < r_0 \, \tau_n^{2^{-k\delta}} \quad .$$

Finally, letting j tend to infinity on the both sides of (5.2), we complete the proof.

Results for more general cases, applications, and treatments of systems of differential equations were given in SIBUYA-SPERBER ([3],[4]).

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